Tripartite Version of the Corrádi-Hajnal Theorem

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Abstract

Let G be a tripartite graph with N vertices in each vertex class. If each vertex is adjacent to at least $(2/3)N$ vertices in each of the other classes, then either G contains a subgraph that consists of N vertex-disjoint triangles or G is a specific graph in which each vertex is adjacent to exactly $(2/3)N$ vertices in each of the other classes.

1 Introduction

A central question in extremal graph theory is the determination of the minimum density of edges in a graph G which guarantees a monotone property $\mathcal{P}$. If the property is the inclusion of a fixed size subgraph H, the answer is given by the classic theorems of Turán [10] (when H is a complete graph) and Erdős and Stone [4].

However, in the case when a graph G is required to contain a spanning subgraph H; that is, H has the same number of vertices as G, an important parameter is a lower bound on the minimum degree that guarantees H is a subgraph of G. Perhaps the most well-known result of this type is a theorem of Dirac [3] which asserts that every n vertex graph with minimum degree at least $\frac{n}{2}$ contains a Hamiltonian cycle. Another theorem of this type is the so-called Hajnal-Szemerédi theorem, with the case $k = 3$ proven first by Corrádi and Hajnal [2].

Theorem 1.1 (Hajnal-Szemerédi [5]) *Let G be graph on n vertices with minimum degree $\frac{k-1}{k}n$. If k divides n, then G has a subgraph that consists of $\frac{n}{k}$ vertex-disjoint cliques of size k.*

A tripartite graph is said to be balanced if it contains the same number of vertices in each class. Theorem 1.2 is a tripartite version of the Corrádi-Hajnal result.

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Theorem 1.2 *Let $G = (V_1, V_2, V_3; E)$ be a balanced tripartite graph on $3N$ vertices such that each vertex is adjacent to at least $(2/3)N$ vertices in each of the other classes. If $N \geq N_0$ for some absolute constant N_0, then G has a subgraph consisting of N disjoint triangles or $G = \Gamma_3(N/3)$ for $N/3$ an odd integer.*

The graph $\Gamma_3(N/3)$ is defined in Section 1.2. The proof is in two parts. Theorem 2.1 in Chapter 2 states that if the degree condition is relaxed, then all graphs, except a specific class, have the spanning subgraph of disjoint triangles. We will then show how find the spanning subgraph for that excluded class of graphs by proving Theorem 3.1 in Chapter 3. Assume that N is divisible by 3. If not, Section 4 shows that the case where N is not divisible by 3 comes as a corollary.

1.1 The Regularity and Blow-up Lemmas

Throughout this paper, we will try to keep much of the notation and definitions in [6]. The symbol $+$ will sometimes be used to denote the disjoint union of sets. $V(G)$ and $E(G)$ denote the vertex-set and edge-set of the graph G, respectively. The triple $(A, B; E)$ denotes a bipartite graph $G = (V, E)$, where $V = A + B$ and $E \subset A \times B$. $N(v)$ denotes the set of neighbors of $v \in V$. For $U \subset V \setminus \{v\}$, $N_U(v)$ denotes the set of neighbors of v intersected with U. The **degree of** v is $\deg(v) = |N(v)|$. The **degree of** v **in** U is $\deg_U(v) = |N_U(v)|$. If H is a subgraph of G, then we relax notation so that $\deg_H(v) = \deg_{V(H)}(v)$. For $U \subset V$, $G|_U$ denotes the graph G induced by the vertices U.

The graph K_3 is the complete graph on 3 vertices, the "**triangle**." We say edges and triangles are **disjoint** if their common vertex set is empty. A balanced tripartite graph on $3N$ vertices is **covered with triangles** if it contains a subgraph of N disjoint triangles. The tripartite version of the Corrádi-Hajnal result is Theorem 1.2. When A and B are subsets of $V(G)$, we define

$$e(A, B) = |\{(x, y) : x \in A, y \in B, \{x, y\} \in E(G)\}| .$$

For nonempty A and B,

$$d(A, B) = \frac{e(A, B)}{|A||B|}$$

is the **density** of the subgraph of edges that contain one endpoint in A and one in B.

Definition 1.3 *The bipartite graph $G = (A, B, E)$ is **ϵ-regular** if*

$$X \subset A, Y \subset B, |X| > \epsilon|A|, |Y| > \epsilon|B|$$

*imply $|d(X, Y) - d(A, B)| < \epsilon$, otherwise we say G is **ϵ-irregular**.*

We will also need a stronger version.

Definition 1.4 *$G = (A, B, E)$ is* **(ϵ, δ)-super-regular** *if*

$$X \subset A, Y \subset B, |X| > \epsilon|A|, |Y| > \epsilon|B|$$

imply $d(X, Y) > \delta$ and

$$\deg(a) > \delta|B|, \quad \forall a \in A \qquad \text{and} \qquad \deg(b) > \delta|A|, \quad \forall b \in B.$$

One of our main tools will be the Regularity Lemma [9], but more specifically, a corollary known as the Degree Form:

Lemma 1.5 (Degree Form of the Regularity Lemma) *For every positive ϵ there is an $M = M(\epsilon)$ such that if $G = (V, E)$ is any graph and $d \in [0, 1]$ is any real number, then there is a partition of the vertex set V into $\ell + 1$ clusters $V_0, V_1, \ldots, V_\ell$ and there is a subgraph $G' = (V, E')$ with the following properties:*

- *$\ell \leq M$,*
- *$|V_0| \leq \epsilon|V|$,*
- *all clusters V_i, $i \geq 1$, are of the same size $L \leq \lceil \epsilon|V| \rceil$,*
- *$\deg_{G'}(v) > \deg_G(v) - (d + \epsilon)|V|$, $\forall v \in V$,*
- *$G'|_{V_i} = \emptyset$ (V_i are independent in G'),*
- *all pairs $G'|_{V_i \times V_j}$, $1 \leq i < j \leq l$, are ϵ-regular, each with density either 0 or exceeding d.*

The above definition is the traditional statement of the Degree Form. In fact, we can guarantee that each cluster that is not V_0 has that all vertices belong to the same vertex class. The Degree Form is derived from the original Regularity Lemma (see [8]) which shows that any partition can be refined so that it is in the form of the Regularity Lemma. The reduced graph G_r, has a vertex set $V_1, \ldots, V_\ell$ with $V_i \sim V_j$ if and only if $G'|_{V_i \times V_j}$ is ϵ-regular with density exceeding d.

We will also make use of the so-called Blow-up Lemma. The graph H can **be embedded into** graph G if G contains a subgraph isomorphic to H.

Lemma 1.6 (Blow-up Lemma [7]) *Given a graph R of order r and positive parameters δ, Δ, there exists an $\epsilon > 0$ such that the following holds: Let N be an arbitrary positive integer, and let us replace the vertices of R with pairwise disjoint N-sets $V_1, V_2, \ldots, V_r$ (blowing up). We construct two graphs on the same vertex-set $V = \cup V_i$. The graph $R(N)$ is obtained by replacing all edges of R with copies of the complete bipartite graph $K_{N,N}$ and a sparser graph G is constructed by replacing the edges of R with some (ϵ, δ)-super-regular pairs. If a graph H with maximum degree $\Delta(H) \leq \Delta$ can be embedded into $R(N)$, then it can be embedded into G.*

1.2 Further Definitions

We will frequently refer to the well-known König-Hall condition, which states that if $G = (A, B; E)$ is a bipartite graph, then there is a matching in G that involves all the vertices of A unless there exists an $X \subseteq A$ such that, $|N(X)| < |X|$. Specifically, we often use the immediate corollary that if $|A| = |B|$ and each vertex in A has degree at least $|B|/2$ and each vertex in B has degree at least $|A|/2$, then G must have a perfect matching.

With G a k-partite graph, $V(G) = V_1 + \cdots + V_k$, each V_i being a partition, we refer to each V_i as a **vertex class**. We refer to the graph defined by the Regularity Lemma, denoted G_r, as the **reduced graph** of G. G itself is the **real graph**. Any triangle in G_r or in a similar reduced graph is referred to as a **super-triangle**. A triangle in G is often called a **real triangle** to avoid confusion.

The notation $a \ll b$ means that the constant a is small enough relative to b. This has become standard notation in these kinds of proofs. A set is of size **γ-approximately** M if its size is $(1 \pm \gamma)M$. Let us also define two classes of graphs. The first is $\Theta_{m\times n}$. The vertices of $\Theta_{m\times n}$ are $\{h_{i,j} : i = 1, \ldots, m; j = 1, \ldots, n\}$ and $h_{i,j} \sim h_{i',j'}$ iff $i \neq i'$ and $j \neq j'$. Note that $\Theta_{3\times 2}$ contains no triangle. The second graph is the graph Γ_k. The vertices are $\{h_{i,j} : i = 1, \ldots, k; j = 1, \ldots, k\}$ and the adjacency rules are as follows: $h_{i,j} \sim h_{i',j'}$ if $i \neq i'$ and $j \neq j'$ and either j or j' is in $\{1, \ldots, k-2\}$. Also, $h_{i,k-1} \sim h_{i',k-1}$ and $h_{i,k} \sim h_{i',k}$ for $i \neq i'$. No other edges exist. If k is even, then Γ_k can be covered by K_k's, but it cannot if k is odd.

For a graph G, define $G(t)$ to be the graph formed by replacing each vertex with a cluster of t vertices and each edge with the complete bipartite graph $K_{t,t}$. For $\epsilon \geq 0$ and $\Delta \geq 0$, a graph H is **(ϵ, Δ)-approximately** $G(t)$ if each vertex of G is replaced with a cluster of size ϵ-approximately t and each non-edge is replaced by a bipartite graph of density at most Δ. For brevity, we will say a graph is **Δ-approximately** $G(t)$ if it is $(0, \Delta)$-approximately $G(t)$. Note that if $\Delta < \Delta'$ and $\epsilon \ll \Delta' - \Delta$, then (if we are allowed to add or subtract vertices to guarantee that clusters are the same size) a graph that is (ϵ, Δ)-approximately $G(t)$ is also Δ'-approximately $G(t)$.

1.3 An Easy Result

Let G be a balanced tripartite graph on $3M$ vertices such that each vertex in G is adjacent to at least $(3/4)M$ vertices in each of the other classes. Proposition 1.7 shows that this graph can be covered with triangles. Proposition 1.7 is used repeatedly in Section 3.

Proposition 1.7 *Let $G = (V_1, V_2, V_3; E)$ be a balanced tripartite graph on $3M$ vertices such that each vertex is adjacent to at least $(3/4)M$ vertices in each of the other classes. Then, we can cover G with M vertex-disjoint triangles.*

Proof. Let H be the graph induced by (V_2, V_3). Each vertex in H is adjacent to at least $(3/4)M > (1/2)M$ vertices in each of the other classes. Therefore, H can be covered by M disjoint edges. Each of these edges is adjacent to at least $\left(1 - 2 \times \frac{1}{4}\right) M = M/2$ vertices in V_1 and each vertex in V_1 is adjacent to at least $M/2$ of the disjoint edges. So, by König-Hall, there exists a 1-factor between V_1 and the M disjoint edges – giving us our M disjoint triangles. □

1.4 A Useful Proposition

Proposition 1.8 is quite valuable and is used in both Section 2 and Section 3.

Proposition 1.8 *For a Δ small enough, there exists $\epsilon > 0$ such that if H is a tripartite graph with at least $2(1-\epsilon)t$ vertices in each vertex class and each vertex is nonadjacent to at most $(1+\epsilon)t$ vertices in each of the other classes. Furthermore, let H contain no triangles. Then, each vertex class is of size at most $2(1+\epsilon)t$ and H is (ϵ, Δ)-approximately $\Theta_{3\times 2}(t)$.*

Proof. Let $\epsilon \ll \delta \ll \delta' \ll \Delta$. First we bound the sizes of the V_i. Choose vertices v_1 and v_2 from $V(G) \setminus V_i$ such that they form an edge. These vertices can have no common neighbor, giving that $|V_i| \leq 2(1+\epsilon)t$.

Now choose $w \in V_3$. Let $N(w) \cap V_i$ be written as $A_{i,1}$, for $i = 1, 2$, such that each vertex in $A_{i,1}$ is adjacent to no vertices in $A_{3-i,1}$. Furthermore, define $A_{3,1}$ to be those vertices in V_3 that are adjacent to less than δt vertices in each of $A_{1,1}$ and $A_{2,1}$. The set $A_{3,1}$ cannot be of size larger than $(1+\epsilon)t$. If it were, then there exists an edge in $(A_{2,1}, A_{3,1})$. By the degree condition, if δ is small enough, this edge must have a common neighbor in $V_1 \setminus A_{1,1}$.

For all $i \in [3]$, remove vertices (if necessary) from the sets $A_{i,1}$ to create $A'_{i,1}$ so that each vertex in $A'_{i,1}$ is adjacent to less than δt vertices in each $A_{i',1}$ for $i' \neq i$. By the same arguments given before, $|A'_{i,1}| \leq (1+\epsilon)t$, for $i = 1, 2, 3$. As a result, each vertex in $A'_{i,1}$ is adjacent to less than $\delta' t$ vertices in each $A'_{i',1}$, for $i' \neq i$. Let $A'_{i,2} = V_i \setminus A'_{i,1}$ for $i = 1, 2, 3$.

We now want to show that each pair of the form $\left(A'_{i,2}, A'_{i',2}\right)$ is sparse. Let $v \in A'_{1,2}$. If $N(v) \cap A'_{2,2} \neq \emptyset$, then $|A_{3,1} \setminus N(v)| \leq \delta t$ which implies $|A_{2,1} \setminus N(v)| \leq \delta t$. As a result, $|N(v) \cup A_{2,1}|, |N(v) \cup A_{3,1}| \leq (1+\epsilon+\delta)t$, implying $\left|N(v) \cap A'_{2,2}\right|, \left|N(v) \cap A'_{3,2}\right| \leq \delta' t$. Similar results occur for $w \in A'_{2,2} \cup A'_{3,2}$. Once again, it must be true that each $\left|A'_{i,2}\right| \leq (1+\epsilon)t$.

Note that each set $A'_{i,j}$ is of size at least $(1-3\epsilon)t$ because the others are of size at most $(1+\epsilon)t$. Therefore, vertices can be moved from the sets larger than $(1-\epsilon)t$ to the smaller sets to create sets $A''_{i,j}$ of size in $((1-\epsilon)t, (1+\epsilon)t)$ with pairwise density at most Δ. □

2 The Fuzzy Tripartite Theorem

2.1 Statement of the Theorem

Theorem 2.1 allows us, with an exceptional case, to cover G with triangles, even if the minimum degree is a bit less than $(2/3)N$.

Theorem 2.1 *Given $\epsilon \ll \Delta \ll 1$, let $G = (V_1, V_2, V_3; E)$ be a balanced tripartite graph on $3N$ vertices such that each vertex is adjacent to at least $(2/3 - \epsilon)N$ vertices in each of the other classes. Then, if N is large enough, either G can be covered with triangles, or G has three sets of size $N/3$, each in a different vertex class, with pairwise density at most Δ.*

2.2 Proof of the Theorem

As usual, there is a sequence of constants:

$$\epsilon \ll \epsilon_1 \ll \epsilon_5 \ll \epsilon_3 \ll \alpha \ll \delta_4 \ll \delta_3 \ll d_3 \ll d_1 \ll \epsilon_2 \ll \Delta_0 \ll \Delta$$

Begin with $G = (V_1, V_2, V_3; E)$, a balanced tripartite graph on $3N$ vertices with each vertex adjacent to at least $(2/3 - \epsilon)N$ vertices in each of the other classes. Define the extreme case to be the case where G has three sets of size $N/3$ with pairwise density at most Δ. Apply the Degree Form of the Regularity Lemma (Lemma 1.5), with d_1 and ϵ_1, to partition each of the vertex classes into $\ell + 4$ clusters. Let us define G'_r to be the reduced graph defined by the Lemma. It may be necessary to place clusters into the exceptional sets (the sets of vertices in each vertex class that make up the V_0 in Lemma 1.5) to ensure that ℓ is divisible by 3. It is important to observe that in the proof, the exceptional sets will increase in size, but will always remain of size $O(\epsilon_1)N$.

For $i = 1, 2, 3$, there exist $V_i = V_i^{(0)} + V_i^{(1)} + \cdots + V_i^{(\ell+3)}$ and $\left|V_i^{(j)}\right| = L \leq \lceil \epsilon_1 N \rceil$, $\forall i$, $\forall j \geq 1$. The reduced graph G'_r has the condition that every cluster is adjacent to at least $(2/3 - \epsilon_2)(\ell + 3)$ clusters in each of the other vertex classes. Apply Lemma 2.2 repeatedly to G'_r with $M = \ell + 3$ to get a decomposition of G'_r into ℓ vertex-disjoint triangles. If this is not possible, then Lemma 2.2 and Proposition 2.3 imply that G is in the extreme case.

Lemma 2.2 (Almost-covering Lemma) *Let $\epsilon' \ll \Delta_0 \ll 1$, and let $G = (V_1, V_2, V_3; E)$ be a balanced tripartite graph on $3M$ vertices so that each vertex is adjacent to at least $(2/3 - \epsilon')\, M$ vertices in each of the other classes. If $\mathcal{T}_0$ is a partial cover by disjoint triangles with $|\mathcal{T}_0| < M - 3$, then we can find another partial cover by disjoint triangles, $\mathcal{T}$ with $|\mathcal{T}| > |\mathcal{T}_0|$ and $|\mathcal{T} \setminus \mathcal{T}_0| \leq 15$, unless G contains three sets of size $M/3$ and pairwise have density less than Δ_0.*

Proposition 2.3 *If a reduced graph G_r has two sets of size $\ell/3$ and have density less than Δ_0, then some vertices can be added to the underlying graph induced* *by those clusters so that it is two sets of size $\lfloor N/3 \rfloor$ and have density less than Δ.*

Call these super-triangles $S(1), S(2), \ldots, S(\ell)$. We put the vertices in the remaining clusters into the appropriate leftover set. Let the reduced graph involving the clusters of $S(1), S(2), \ldots, S(\ell)$ be denoted G_r. By Proposition 2.4, at most $2\epsilon_1 L'$ vertices can be removed from each cluster to obtain (ϵ_3, δ_3)-super-regular pairs in the vertex-disjoint triangular decomposition of G_r. Furthermore, Proposition 2.5 guarantees that any edge in G_r must still correspond to an ϵ_3-regular pair of density at least d_3.

Proposition 2.4 *Given $\epsilon < 1/4$, let (S'_i, S'_j) for $\{i,j\} \in \binom{[3]}{2}$, be three ϵ-regular pairs with density at least d and $|S'_i| = L'$ for $i = 1,2,3$. Some vertices can be removed from each S'_i to create S_1, S_2 and S_3 that form three pairwise $(2\epsilon, d-3\epsilon)$-super-regular sets of size $L \geq (1-2\epsilon)L'$.*

Proposition 2.5 *Let $|X| = |Y|$, $X' \subseteq X$, $Y' \subseteq Y$, $|X'| = |Y'|$ with $|X'| > \epsilon|X|$. If (X,Y) is ϵ-regular, then (X',Y') is $\max\left\{\left(\frac{|X|}{|X'|}\epsilon\right), 2\epsilon\right\}$-regular.*

One cluster y is **reachable** from another, x, if there is a chain of super-triangles, $T_1, \ldots, T_{2k}$ ($k \in \{1,2\}$) with x an endpoint of T_1, and y an endpoint of T_{2k} with the added condition that T_{2i+1} and T_{2i+2} ($i = 0, \ldots, k-1$) share a common edge and T_{2i} and T_{2i+1} ($i = 1, \ldots, k-1$) share only one common vertex. Fix one super-regular super-triangle, $S(1)$. The set of all such triangles that connects some cluster to a cluster of $S(1)$ is a **structure**. We would like to show that each cluster in G_r and V_i is reachable from the cluster that is $S(1) \cap V_i$. If this is not possible, then Lemma 2.6 and Proposition 2.3 imply that G must be in the extreme case.

Lemma 2.6 (Reachability Lemma) *In the reduced graph G_r, all clusters are reachable from other clusters in the same class, unless some edges can be deleted from G_r so that the resulting graph obeys the minimum degree condition, but is Δ_0-approximately $\Theta_{3\times 3}(\ell/3)$.*

So, suppose that every cluster is reachable from the appropriate cluster of $S(1)$. Consider some cluster y and the structure that connects it to x. This structure contains clusters from at most 8 of the $S(i)$, not including $S(1)$ itself. For any such structure, $T_1, \ldots, T_{2k}$, find 3 real triangles in each of the T_i, for i odd. Note that if some T is in more than one structure, then there exist 3 real triangles for each time that T occurs in a structure. Do this for all possible structures, ensuring that these real triangles are mutually disjoint and color these real triangles red. No cluster can possibly contain more than $r = 9\ell$ red vertices. Thus, there are still $L - r$ uncolored vertices in each cluster, but $L \geq [1 - O(\epsilon_1)]\frac{N}{\ell}$, which goes to infinity as $N \to \infty$. Proposition 2.7 gives that finding these red triangles is easy.

Proposition 2.7 *Let (X_1, X_2, X_3) be a triple with $|X_i| = L$ for $i = 1, 2, 3$ and each pair is ϵ-regular with density $d > 3\epsilon$. Then, there exist $(1 - 2\epsilon)L$ disjoint real triangles in the graph induced by (X_1, X_2, X_3).*

This process of creating red triangles may result in an unequal number of red vertices in the clusters of some of the $S(i)$'s. Let s_i denote the maximum number of red vertices in any one class of $S(i)$. Pick a set of uncolored vertices of size $L - s_i$ in each class of $S(i)$. Proposition 2.5 gives that the pairs of $S(i)$ are (ϵ', δ')-super-regular for some ϵ' and δ'. Then, apply the Blow-up Lemma (Lemma 1.6) to get $L - s_i$ disjoint triangles among the uncolored vertices of $S(i)$. Color these triangles blue.

Now, place the remaining uncovered vertices into the leftover sets. Apply the Almost-covering Lemma (Lemma 2.2) to the non-red vertices of G. Each time this is applied, we may end up destroying at most 15 of the blue triangles in order to create our larger covering. So, suppose that, at some point, there are less than $(1 - \delta_4)L + 18$ vertices remaining in some $S(i)$, then we still apply the Almost-covering Lemma, but this time exclude vertices in the blue triangles of $S(i)$ as well as red vertices. There are at most $\epsilon_5 \ell$ of the $S(i)$'s that we may have to exclude in this manner.

Color green any new triangles formed by using the Almost-covering Lemma (Lemma 2.2). There are at most 9 uncolored vertices that remain after we are finished. Let $x_1 \in V_1$ be an uncolored vertex. We will show how to insert this vertex; inserting the other vertices is similar.

The cluster containing x_1 has degree at least $2\delta_4 L$ in at least $(2/3 - \alpha)\ell$ of the clusters in V_2 and V_3. So, choose some $S(i)$ where x_1 is adjacent to at least $2\delta_4 L$ vertices in the V_2 and V_3 clusters of $S(i)$. Color x_1 blue. Now look at the structure that connects $S(i)$ to $S(1)$, and call the triangles in this structure $T_1, \ldots, T_{2k}$. Find a triangle between the blue vertices of T_{2k}. Color the edges and vertices of this triangle red. Next take one of the red triangles from T_{2k-1}, uncolor its edges and color its vertices blue. Continue in the same manner, adding a red triangle to $T_{2\xi}$ and removing one from $T_{2\xi-1}$ for $\xi = k, k-1, \ldots, 1$. At the end of this process, the same number of blue vertices are in each cluster of each $S(j)$, except for one extra in $V(S(1)) \cap V_1$.

Apply the same procedure to uncolored vertices in V_2 and V_3. Now, the same number of blue vertices are in each $S(j)$, including $S(1)$, which now has 9 more blue vertices in each class than before inserting the extra vertices. Finally, apply the Blow-up Lemma (the pairs are $(2\epsilon_3, \delta_4)$-super-regular) to the blue vertices in each of the $S(j)$'s to create vertex-disjoint blue triangles that involve all of the blue vertices. So, the red, green and blue triangles are vertex-disjoint and cover all vertices of G. $\square$

2.3 Proofs of Propositions

Proof of Proposition 2.3. This is immediate from the fact that the density of any pair of clusters nonadjacent in G_r is at most $d_1 + 2\epsilon_1$ and from the fact that $\Delta_0 \ll \Delta$. $\square$

Proof of Proposition 2.4. Let T be the subset of S_1' consisting of vertices with degree at most $(d-\epsilon)L'$ in S_2'. Clearly $d(T, S_2') \leq d - \epsilon$. But, if $|T| > \epsilon L'$, then $d(T, S_2') > d - \epsilon$, a contradiction. So, $|T| \leq \epsilon L'$. We then have at least $(1-2\epsilon)L'$ vertices in S_1' that have degree at least $(d-\epsilon)L'$ in both S_2' and S_3'. Call that set S_1 and similarly define S_2 and S_3. Proposition 2.5 gives that these sets are pairwise 2ϵ-regular if $\epsilon < 1/4$, then the proposition is proven. $\square$

A proof of Proposition 2.5 is straightforward and left to the reader.

Proof of Proposition 2.7. We apply Proposition 2.4 to the triple (X_1, X_2, X_3) to get a triple (X_1', X_2', X_3') such that each pair is $(2\epsilon, d-3\epsilon)$ super-regular each on $L^* \geq (1-2\epsilon)L$ vertices. We then apply the Blow-up Lemma (Lemma 1.6) to (X_1', X_2', X_3') getting our L^* vertex-disjoint triangles. $\square$.

2.4 Proof of the Almost-covering Lemma (Lemma 2.2)

Given the constants $\epsilon' \ll \Delta' \ll \Delta_0$, let $\mathcal{T}_0$ be as in the statement of the lemma. Denote U_1, U_2 and U_3 as the portions of V_1, V_2, and V_3, respectively, left uncovered by $\mathcal{T}_0$. Let $U = U_1 + U_2 + U_3$. We want to show that if $|U| > 9$ then the covering can be expanded unless G contains three sets of size $M/3$ which pairwise have density less than Δ_0. (We always assume that M is divisible by 3.)

Thus, assume that U contains at least four vertices in each class. We want to show that there are at least three disjoint edges, one between each class. Let $x_1 \in U_1$ and $x_2 \in U_2$ with $x_1 \not\sim x_2$ then it will be possible to exchange these vertices with the vertices of $\mathcal{T}$ that maintains or increases the number of disjoint triangles, uses no other vertices in U and places an edge between U_1 and U_2.

By assumption, both $\left|N_{V_2\setminus U_2}(x_1)\right| \geq (2/3-\epsilon')\,M$ and $\left|N_{V_3\setminus U_3}(x_1)\right| \geq (2/3-\epsilon')M$. This implies that there are at least $(1/3-2\epsilon')M$ triangles, T, in $\mathcal{T}_0$ so that x_1 is adjacent to both the V_2 and V_3 vertices in T.

Let

$$\begin{aligned}
A_1 &:= \{x \in V_1 : T \in \mathcal{T}_0, V(T) = \{x,y,z\}, x_1 \sim y, \text{ and } x_1 \sim z\} \\
A_2 &:= \{y \in V_2 : T \in \mathcal{T}_0, V(T) = \{x,y,z\}, x_1 \sim y, \text{ and } x_1 \sim z\} \\
A_3 &:= \{z \in V_3 : T \in \mathcal{T}_0, V(T) = \{x,y,z\}, x_1 \sim y, \text{ and } x_1 \sim z\}
\end{aligned}$$

Simply, A_1 is the set of all vertices so that x_1 can be exchanged with such a vertex so as to leave the number of triangles in $\mathcal{T}_0$ unchanged. The sets A_2 and A_3 are the vertices in the other classes that correspond to the triangles in $\mathcal{T}_0$ with vertices in A_1. Clearly, $|A_1| = |A_2| = |A_3| \geq (1/3-2\epsilon')\,M$.

Consider x_2. Define B_1, B_2 and B_3 in a similar manner so that x_2 can be exchanged with each of the vertices of B_2. We will show that the intersection of

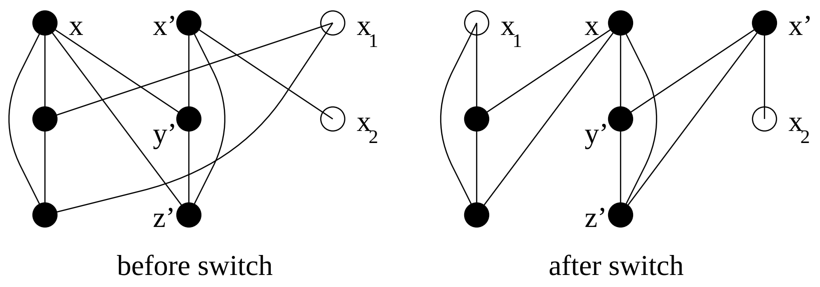

Figure 1: (A_1, C_3) not sparse

A_1 and B_1 is empty. If there is a triangle $\{x, y, z\} \in \mathcal{T}_0$ such that $x \in A_1 \cap B_1$, then x and x_1 can be exchanged in order to obtain a covering with the same number of triangles but with an edge in (U_1, U_2).

The pair (A_1, B_2) is void of edges. If not, then both x_1 and x_2 can be exchanged with the endvertices of that edge. The number of triangles does not change, but there will be an edge between U_1 and U_2. Now let $C_i = V_i(\mathcal{T}_0) \setminus (A_i \cup B_i)$, for $i = 1, 2, 3$. Clearly, $|C_1| = |C_2| = |C_3| \leq (1/3 + 4\epsilon')M$. But, since no vertex in A_1 can be adjacent to a vertex in U_2 and (A_1, B_2) is void, then $|C_2| \geq (1/3 - 2\epsilon')M$.

Since (A_1, B_2) is void, each vertex in A_1 must be adjacent to at least $(1/3 - 3\epsilon')M$ vertices in C_2. Therefore, if there exists some vertex $x \in A_1$ adjacent to more than $7\epsilon' M$ vertices in C_3, then there is a triangle $\{x', y', z'\}$ such that $x \sim y', z'$. Thus, according to Figure 1, x, x' and x_1 could be moved so that there exists a $\mathcal{T}_0$ of the same size with an edge in (U_1, U_2).

Similarly, (B_2, C_3) must be sparse. Therefore, the triple (A_1, B_2, C_3) has sets of size $4\epsilon'$-approximately $M/3$ with pairwise density at most $7\epsilon' M/|C_3| < \Delta'$. The same procedure can be applied to (U_1, U_3) and then (U_2, U_3) to create 6 edges, $e_1, e_2 \in (U_1, U_2)$, $f_1, f_3 \in (U_1, U_3)$ and $g_2, g_3 \in (U_2, U_3)$ that are disjoint. Let this new partial triangular cover be $\mathcal{T}_2$. Note that $|\mathcal{T}_0 \setminus \mathcal{T}_2| \leq 12$ but $|\mathcal{T}_2| = |\mathcal{T}_0|$.

Given the edges e_1, e_2, f_1, f_3, g_2, g_3 in U, redefine the "A", "B", and "C" sets. Let

$$\begin{aligned} A_i &= \{x_i \in V_i : \{x_1, x_2, x_3\} \in \mathcal{T}_2 \text{ and } x_1 \sim g_i\} & i = 2, 3 \\ B_i &= \{x_i \in V_i : \{x_1, x_2, x_3\} \in \mathcal{T}_2 \text{ and } x_2 \sim f_i\} & i = 1, 3 \\ C_i &= \{x_i \in V_i : \{x_1, x_2, x_3\} \in \mathcal{T}_2 \text{ and } x_3 \sim e_i\} & i = 1, 2 \end{aligned}$$

The sets $|A_i|, |B_i|, |C_i| \geq (1/3 - 2\epsilon')M$ for all relevant i. This is the case because the neighborhood of each edge is of this size and these neighborhoods must be entirely within $V(\mathcal{T}_2)$, otherwise $\mathcal{T}_2$ is not maximal.

We wish to show that these sets are disjoint. Suppose, without loss of generality, $x_1 \in B_1 \cap C_1$ so that $\{x_1, x_2, x_3\} \in \mathcal{T}_2$. Then x_2 and f_1 form a

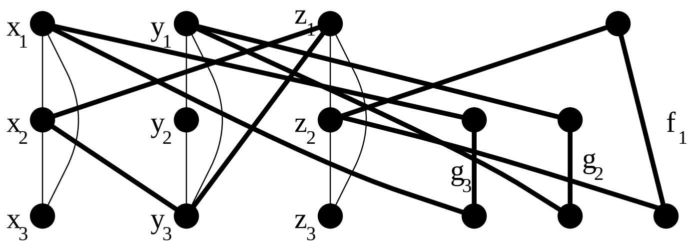

Figure 2: A triangle in (B_1, A_2, A_3)

triangle and x_3 and e_1 form a triangle – giving that there exists a $\mathcal{T}$ of size larger than $\mathcal{T}_2$. As a result, $|B_1 \cup C_1|, |A_2 \cup C_2|, |A_3 \cup B_3| \geq (2/3 - 4\epsilon')M$.

Further, there can be no triangle in the triple $(B_1 \cup C_1, A_2 \cup C_2, A_3 \cup B_3)$. We will just show one example; suppose there is a triangle T in (B_1, A_2, A_3). Then there are $\{x_1, x_2, x_3\}, \{y_1, y_2, y_3\}, \{z_1, z_2, z_3\} \in \mathcal{T}_2$ such that $x_1 \sim g_2$, $y_1 \sim g_3$, $z_2 \sim f_1$ and $T = \{z_1, x_2, y_3\}$. If $x_1 = y_1$, then $\{x_1, x_2, x_3\}$ and $\{z_1, z_2, z_3\}$ can be replaced with the triangle formed by x_1 and g_2, the triangle formed by z_2 and f_1 and T itself. If $x_1 \neq y_1$, then we can replace the x, y and z triangles with the triangles formed by x_1 and g_2, by y_1 and g_3 and by z_2 and f_1 as well as T. See Figure 2. Thus, if there were a triangle in $(B_1 \cup C_1, A_2 \cup C_2, A_3 \cup B_3)$, then a $\mathcal{T}$ could be found such that $|\mathcal{T}| > |\mathcal{T}_0|$ and $|\mathcal{T}_0 \setminus \mathcal{T}| \leq 15$. If there is no triangle in $(B_1 \cup C_1, A_2 \cup C_2, A_3 \cup B_3)$, then Proposition 1.8 gives that this subgraph is Δ_0-approximately $\Theta_{3\times 2}(M/3)$, which means G contains 3 sets of size $M/3$ with pairwise density at most Δ_0. $\Box$

2.5 Proof of the Reachability Lemma (Lemma 2.6)

Let us be given constants

$$\epsilon_2 \ll \Delta' \ll \Delta'' \ll \Delta''' \ll \Delta_0.$$

In order to prove the lemma, we distinguish two triangles, call them $S(1) = \{x_1(1), x_2(1), x_3(1), \}$ and $S(\ell) = \{x_1(\ell), x_2(\ell), x_3(\ell)\}$ and suppose $x_1(\ell)$ is not reachable from $x_1(1)$. We will show that edges can be deleted from G_r so that the minimum degree condition holds and the resulting graph is Δ_0-approximately $\Theta_{3\times 3}(\ell/3)$. Every cluster is adjacent to at least $(2/3 - \epsilon_2)\ell$ clusters in each of the other classes. Let

$$\begin{array}{rcll} A_{i,1} & := & [N(x_1(\ell)) \setminus N(x_1(1))] \cap V_i & \\ A_{i,3} & := & [N(x_1(1)) \setminus N(x_1(\ell))] \cap V_i & \qquad i = 2, 3 \\ A_{i,2} & := & V_i \setminus (A_{i,1} \cup A_{i,3}) & \end{array}$$

Observe that $(1/3 - 2\epsilon_2)\ell \leq |A_{i,1}|, |A_{i,3}| \leq (1/3 + \epsilon_2)\ell$ for $i = 2, 3$.

If there is an edge in $(A_{2,2}, A_{3,2})$, then $x_1(\ell)$ must be reachable from $x_1(1)$. Thus, it must be that $d(A_{2,2}, A_{3,2}) = 0$. Combining the information, it must be true that $|A_{i,j}| \in ((1/3 - 4\epsilon_2)\ell, (1/3 + 4\epsilon_2)\ell)$ for $i \in \{2,3\}$ and $j \in \{1,2,3\}$. Define the sets $A_{1,1}$ and $A_{1,3}$ by first letting

$$A_{1,1} \cup A_{1,3} := \left\{ v \in V_1 : \exists i \in \{2,3\} \text{ s.t. } \deg_{A_{i,2}}(v) \geq 2\Delta'\ell \right\} .$$

Suppose $v \in A_{1,1} \cup A_{1,3}$ with $\deg_{A_{i,2}}(v) \geq 2\Delta'\ell$ and $\deg_{A_{i',1}}(v), \deg_{A_{i',3}}(v) \geq \Delta''\ell$, where $\{i'\} = \{2,3\} \setminus \{i\}$. Then there exists an edge in $(A_{i,2}, A_{i',3})$ that is adjacent to both $x_1(1)$ and v. Also, there exists another edge in $(A_{i,2}, A_{i',1})$ that is adjacent to both v and $x_1(\ell)$. This makes $x_1(\ell)$ reachable from $x_1(1)$ by a chain of 4 triangles.

Suppose $v \in A_{1,1} \cup A_{1,3}$ and v is adjacent to less than $\Delta''\ell$ vertices in $A_{i',1}$ but is adjacent to more than $\Delta''\ell$ vertices in $A_{i,1}$. In this case, there exists an edge in $(A_{i,2}, A_{i',3})$ that is adjacent to both $x_1(1)$ and v. There also exists an edge in $(A_{i,1}, A_{i',2})$ that is adjacent to both v and $x_1(\ell)$. With the above suppositions about the degree of v, $x_1(\ell)$ is reachable from $x_1(1)$ by a chain of 4 triangles. Therefore, each vertex either is adjacent to less than $\Delta''\ell$ vertices in both $A_{2,4}$ and $A_{3,4}$ (call these vertices $A_{1,1}$) or is adjacent to less than $\Delta''\ell$ vertices in both $A_{2,1}$ and $A_{3,1}$ (call these vertices $A_{1,3}$). This gives that $d(A_{1,1}, A_{i,3}) < \Delta'''$ and $d(A_{1,3}, A_{i,1}) < \Delta'''$ for $i = 2, 3$.

Because of the minimum degree condition, $|A_{1,1}|, |A_{1,3}| < (1/3 + 2\epsilon_2)\ell$. Define $A_{1,2}$ to be those vertices adjacent to less than $2\Delta'\ell$ vertices in both $A_{2,2}$ and $A_{3,2}$. It must be true that $V_1 = A_{1,1} \cup A_{1,2} \cup A_{1,3}$ with all sets being disjoint, because the definition of $A_{1,1} \cup A_{1,3}$ gives that all vertices not in those sets must be in $A_{1,2}$. From before, $|A_{1,2}| < (1/3 + 2\epsilon_2)\ell$ and $d(A_{1,2}, A_{i,2}) < \Delta'''$ for $i = 2, 3$. Summarizing, $|A_{i,j}| \in ((1/3 - O(\epsilon_2))\ell, (1/3 + O(\epsilon_2))\ell)$ for all i and j. Furthermore, $d(A_{1,1}, A_{i,3}) < \Delta'''$ and $d(A_{1,3}, A_{i,1}) < \Delta'''$ for $i = 2, 3$ and, as we just showed, $d(A_{1,2}, A_{i,2}) < \Delta'''$ for $i = 2, 3$.

What remains is to show that one of the pairs $(A_{2,1}, A_{3,1})$ or $(A_{2,3}, A_{3,3})$ is pairwise sparse. Note that if one is pairwise sparse, we might as well allow the other to be pairwise sparse, since extra edges only help. If it is not the case, then $d(A_{2,1}, A_{3,1}) \geq \Delta'''$ and $d(A_{2,3}, A_{3,3}) \geq \Delta'''$ are both not sparse. There exists an edge e in $(A_{2,1}, A_{3,1})$ that is adjacent to many vertices in $A_{1,2}$ as well as an edge f in $(A_{2,3}, A_{3,3})$ that is adjacent to many vertices in $A_{1,2}$. Thus, there exists a vertex v that is adjacent to both edges. Since f is adjacent to both $x_1(1)$ and v and e is adjacent to both v and $x_1(\ell)$ we have that $x_1(\ell)$ is reachable from $x_1(1)$. Therefore, if $x_1(\ell)$ is not reachable from $x_1(1)$, we must have that $d(A_{2,1}, A_{3,1}), d(A_{2,3}, A_{3,3}) \ll \Delta_0$. $\square$

3 The Extreme Tripartite Theorem

3.1 Statement of the Theorem

Theorem 2.1 leaves the extreme case, which we consider in Theorem 3.1.

Theorem 3.1 *Given $\Delta \ll 1$, let $G = (V_1, V_2, V_3; E)$ be a balanced tripartite graph on $3N$ vertices such that each vertex is adjacent to at least $(2/3)N$ vertices in each of the other classes. Furthermore, let G have three sets with size $N/3$ and pairwise density at most Δ. Then, if N is large enough, either G can be covered with triangles or G is $\Gamma_3(N/3)$.*

3.2 Proof of the Theorem

Assume that G is minimal. That is, no edge of G can be deleted so that the minimum degree condition still holds. We will prove that for minimal G, either G can be covered with triangles or $G = \Gamma_3(N/3)$. With that proven, it suffices to show that adding any edge to $\Gamma_3(N/3)$ will allow the resultant graph to be covered with triangles – this will be discussed in Section 3.3. Begin with the usual sequence of constants:

$$\Delta \ll \Delta_1 \ll \Delta_2 \ll \eta \ll \theta - \frac{3}{4}$$

for some θ, $3/4 < \theta < 1$. Let $t := N/3$ with N divisible by 3.

Let the sets of size t mentioned in the theorem be designated A_i, with $A_i \subset V_i$ for $i = 1, 2, 3$. Let $B_i := V_i \setminus A_i$ for $i = 1, 2, 3$. For each $i \in \{1, 2, 3\}$, let A'_i be the vertices that are adjacent to at least $(1 + \theta)t$ vertices in B_j for each $j \neq i$. Let B'_i be the vertices that are adjacent to at least $(1/2)(1 + \theta)t$ vertices in A_j for each $j \neq i$. Furthermore, let $C'_i = V_i \setminus (A'_i \cup B'_i)$. The key feature of each $c \in C'_i$ is that there is a $j \neq i$ such that c is adjacent to at least $(1 - \theta)t$ vertices in A_j. Let us compute $|A'_i|$ and $|B'_i|$ for $i = 1, 2, 3$. Proposition 3.2 restricts the sizes of these sets.

Proposition 3.2 *If $\Delta \ll \Delta_1$, then for all $i \in \{1, 2, 3\}$,*

$$\begin{aligned} |A'_i| &\in ((1 - \Delta_1)t, (1 + \Delta_1)t) \\ |B'_i| &\in ((2 - \Delta_1)t, (2 + \Delta_1)t) \end{aligned}$$

Furthermore, for $i = 1, 2, 3$, $|A_i \setminus A'_i|, |B_i \setminus B'_i| \leq \Delta_1 t$.

The key lemma for this proof is Lemma 3.3.

Lemma 3.3 *Let $\Delta_1 \ll \Delta'_1 \ll \Delta''_1 \ll \Delta'''_1 \ll \Delta'_2 \ll \Delta_2 \ll \theta - \frac{3}{4}$ for some constant $\theta > 3/4$. Let $G = (V_1, V_2, V_3; E)$ be a balanced tripartite graph on $9t'$ vertices with each vertex adjacent to at least $(2 - \Delta'_1)t'$ vertices in each*

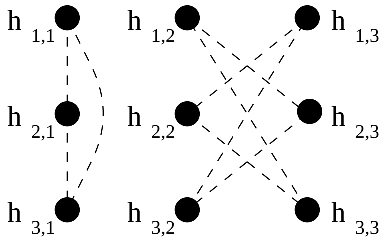

Figure 3: Diagram of Γ_3. The dotted lines correspond to non-edges.

of the other vertex classes. Suppose further that we have sets A_i' of size Δ_1''-approximately t' such that for all $a \in A_i'$, $\deg_{V_j \setminus A_j'}(a) \geq (1+\theta)t'$ for all $j \neq i$. Furthermore, let $d(A_i', A_j') < \Delta_1'''$, $\forall \{i,j\} \in \binom{[3]}{2}$ and let each $v \in V_i \setminus A_i'$ have the property that there is a $j \neq i$ such that $\deg_{A_j'}(v) \geq (1-\theta-\Delta_1'')t'$. If G is minimal and cannot be covered with triangles then either

1. $|A_1'| + |A_2'| + |A_3'| > 3t$,
2. *G is Δ_2'-approximately $\Gamma_3(t')$, or*
3. *G is Δ_2'-approximately $\Theta_{3\times 3}(t')$.*

We make adjustments according to whether or not $|A_1'|+|A_2'|+|A_3'| \leq 3t$. It is true that, $3(1-\Delta_1)t \leq |A_1'|+|A_2'|+|A_3'| \leq 3(1+\Delta_1)t$. If $|A_1'|+|A_2'|+|A_3'| \leq 3t$, then apply Lemma 3.3 to G. Thus, G can be covered with triangles unless G is Δ_2-approximately $\Gamma_3(t)$ or G is Δ_2-approximately $\Theta_{3\times 3}(t)$.

If $|A_1'| + |A_2'| + |A_3'| > 3t$, then we want to create a matching of size $|A_1'|+|A_2'|+|A_3'|-3t$ in (A_1', A_2', A_3'). After finding the matching, find common neighbors in $(B_1' \cup C_1', B_2' \cup C_2', B_3' \cup C_3')$ and remove those disjoint triangles so that Lemma 3.3 can be applied to the remaining graph. Each vertex in A_i' is adjacent to at least $\max\{|A_j'|-t, 0\}$ vertices in A_j', for all distinct i and j. Thus, we can create matchings sequentially in (A_i', A_j'), for all pairs (i,j) so that they do not coincide and together they exclude exactly t vertices in each of A_1', A_2' and A_3' that are larger than t. The details are left to the reader.

Thus, G can be covered with triangles unless G is either Δ_2-approximately $\Gamma_3(t)$ (Section 3.3) or Δ_2-approximately $\Theta_{3\times 3}(t)$ (Section 3.4).

3.3 G is Δ_2-approximately $\Gamma_3(t)$ (Case (2) of Lemma 3.3)

Let the sets $A_{i,j}$, $i,j = 1,2,3$ be as the $h_{i,j}$ in Figure 3. Note that the figure depicts the non-edges of this graph. Each row of vertices corresponds to a vertex class and the dotted lines correspond to non-edges. Given $\Delta_2 \ll \Delta_3 \ll \Delta_4 \ll \eta$, our goal is to modify the sets $A_{i,j}$ to form sets $\tilde{A}_{i,j}$. The triangles will come from each of the following:

$$(\tilde{A}_{1,1}, \tilde{A}_{2,2}, \tilde{A}_{3,2}) \quad (\tilde{A}_{2,1}, \tilde{A}_{1,2}, \tilde{A}_{3,2}) \quad (\tilde{A}_{3,1}, \tilde{A}_{1,2}, \tilde{A}_{2,2})$$
$$(\tilde{A}_{1,1}, \tilde{A}_{2,3}, \tilde{A}_{3,3}) \quad (\tilde{A}_{2,1}, \tilde{A}_{1,3}, \tilde{A}_{3,3}) \quad (\tilde{A}_{3,1}, \tilde{A}_{1,3}, \tilde{A}_{2,3}).$$

The triangles will receive one of 6 labels $(i; j)$, for $i \in \{1,2,3\}$ and $j \in \{2,3\}$. A triangle with the label $(i; j)$ will be in the triple $(\tilde{A}_{i,1}, \tilde{A}_{i_2,j}, \tilde{A}_{i_3,j})$, where i_2, i_3 are distinct indices in $\{1,2,3\} \setminus \{i\}$.

Define $A'_{i,j}$ to be the set of "typical" vertices in $A_{i,j}$. That is, if $\{h_{i_1,j_1}, h_{i_2,j_2}\}$ is a non-edge in Γ_3, then each vertex in A'_{i_1,j_1} is adjacent to less than ηt vertices in A_{i_2,j_2}. Let $C_i = V_i \setminus \left(A'_{i,1} \cup A'_{i,2} \cup A'_{i,3}\right)$, for $i = 1,2,3$. Since $\Delta_2 \ll \Delta_3$, $|A_{i,j} \setminus A'_{i,j}| < \Delta_3 t$. We will make the sets $A'_{i,1}$ into sets $A''_{i,1}$ of size t, for $i = 1,2,3$. If there is some $|A'_{i,1}| > t$, then we find a matching in $(A'_{1,1}, A'_{2,1}, A'_{3,1})$ of size $\sum_{i=1}^{3} \max\{|A'_{i,1}| - t, 0\}$ similar to the one we constructed above. Color this matching red and for $|A'_{i,1}| > t$, take $|A'_{i,1}| - t$ red edges and remove the V_i endvertices that are in $A'_{i,1}$ and add them to one of $A'_{i,2}$ or $A'_{i,3}$, whichever has size smaller than t. This creates sets $A''_{i,1}$ of size at most t, for $i = 1,2,3$. The endvertices of this red edge will receive label $(i; j)$ if one of its vertices is added to $A'_{i,j}$.

Suppose that $|A'_{i,1}| < t$. Then find vertices in either $A'_{i,2}$ or $A'_{i,3}$, color them green and add them to $A'_{i,1}$ to form $A''_{i,1}$. To show that these green vertices will act as $A'_{i,1}$ vertices, suppose, without loss of generality, v is a green vertex added to $A'_{1,1}$. Observe that v must be adjacent to at least $(1-\eta)t$ vertices in either $A_{2,2}$ and $A_{2,3}$ or $A_{2,3}$ and $A_{3,3}$. Thus, if we move a green vertex from $A'_{i,j}$, it will receive the label $(i; j)$. The resulting sets $A''_{i,1}$ are of size exactly t, so let them be renamed $\tilde{A}_{i,1}$, $i = 1,2,3$.

Now we want to show that vertices in C'_i behave like vertices in either $A'_{i,2}$ or $A'_{i,3}$. Let $c \in C'_1$ and, without loss of generality, show that c can be added to $A'_{1,2}$. There exists an $i \in \{2,3\}$ such that c is adjacent to at least ηt vertices in $A_{i,1}$. If c is adjacent to at least ηt vertices in $A_{5-i,2}$, then c can receive the label $(i; 2)$. Otherwise c can receive the label $(5-i; 2)$. Color the C'_i vertices green and add them to either $A'_{i,2}$ or $A'_{i,3}$ (the smaller of the two) to form $A''_{i,2}$ and $A''_{i,3}$.

Unfortunately, one of the sets $A''_{i,2}$ or $A''_{i,3}$ might be of size more than t. In order to create sets of size t, let us suppose without loss of generality, that both $|A''_{1,2}| > t$ and $A''_{1,2}$ is the largest from among $A''_{1,2}$, $A''_{1,3}$, $A''_{2,2}$ and $A''_{2,3}$. Let $\tau = |A''_{1,2}| - t$ and observe that $A''_{1,2} = A'_{1,2}$. Let $q = |A''_{1,2}| - |A''_{2,3}| \leq 2\tau$ because $|A''_{1,2}| \geq |A''_{2,2}|$. Let $W \subset A''_{2,3} \cap A'_{2,3}$,

$$\left(|A''_{1,2}| - t\right)|W| \leq e(W, A''_{1,2}) \leq (\gamma + 2\Delta_2)t\left|N_{A''_{1,2}}(W)\right|.$$

So, $|N_{A''_{1,2}}(W)| \gg 2\tau$, provided $|W|$ is not too small, and there exists a matching of size q in $(A''_{1,2}, A''_{2,3} \cap A'_{2,3})$. Color this matching blue.

If $|A''_{2,2}| > t$, take $|A''_{1,2}| - |A''_{2,2}|$ blue vertices from $A''_{1,2}$ and add them to $A''_{1,3}$. Also, take $|A''_{2,2}| - t$ edges in $(A''_{1,2}, A''_{2,2})$, color them blue and add their vertices to $A''_{1,3}$ and $A''_{2,3}$. Such blue edges will be in triangles with label $(3;3)$. If $|A''_{2,2}| \leq t$, then take $t - |A''_{2,2}|$ blue vertices from $A''_{2,3}$ and add them to $A''_{2,2}$. The endvertices of these blue edges will be in triangles labeled $(3;2)$. For the remaining blue edges, take $\left|A''_{1,2}\right| - t$ of the vertices from $A''_{1,2}$ and add them to $A''_{1,3}$. The endvertices of these blue edges will be in triangles labeled $(3;3)$.

It may be necessary to find a similar matching in (V_2, V_3) if either $|A''_{3,2}| > t$ or $|A''_{3,3}| > t$. It is easy to see that we can do so without using any of the other colored vertices by choosing a W that excludes blue vertices. The sets that result from moving the vertices of blue edges are of size exactly t, so denote them $\tilde{A}_{i,j}$, for $i = 1, 2, 3$ and $j = 2, 3$.

Recall that $\Gamma_3(t)$ cannot be covered with triangles if t is odd. A similar dilemma must also be resolved in this case. Suppose t is odd. Our goal is to find three triangles in G such that each vertex is from a different $\tilde{A}_{i,j}$. Call these **parity triangles**. To find them, we look for an edge in $(\tilde{A}_{1,2} \cup \tilde{A}_{2,2} \cup \tilde{A}_{3,2}, \tilde{A}_{1,3} \cup \tilde{A}_{2,3} \cup \tilde{A}_{3,3})$ with a common neighbor in $\tilde{A}_{1,1} \cup \tilde{A}_{2,1} \cup \tilde{A}_{3,1}$.

If there is such an edge among uncolored vertices, then there are many neighbors in $\tilde{A}_{1,1} \cup \tilde{A}_{2,1} \cup \tilde{A}_{3,1}$. If any vertex was colored, then by re-examining the process by which it was constructed we see that it is possible to create such a triangle. For example, if there is a red edge in $(\tilde{A}_{2,1}, \tilde{A}_{1,2})$, then we can find a common neighbor in $\tilde{A}_{3,3}$. In any case, if the desired triangle is found, remove it, along with two other triangles so that the resulting "$\tilde{A}$" sets are of size $t - 1$. If a parity triangle cannot be created in this way, then G contains no colored vertices. In that case, if there is an edge in the graph that is induced by $\tilde{A}_{1,1} \cup \tilde{A}_{2,1} \cup \tilde{A}_{3,1}$, then it is easy to create the parity triangles. Otherwise, $G = \Gamma_3(t)$ and the theorem would be proven.

Therefore, suppose that the remaining $\tilde{A}_{i,j}$ sets are of the same even cardinality. Partition each $\tilde{A}_{i,j}$ uniformly at random into two equally-sized pieces. Each piece will receive one of six labels $(i;j)$, for $i = 1, 2, 3$ and $j = 2, 3$. For $i \in \{1, 2, 3\}$, $\tilde{A}_{i,1}$ will be partitioned into one set labeled $(i;2)$ and the other labeled $(i;3)$. For $i \in \{1, 2, 3\}$ and $j \in \{2, 3\}$, $\tilde{A}_{i,j}$ will be partitioned into one set labeled $(i_1;j)$ and the other labeled $(i_2;j)$, where i, i_2, i_3 are distinct members of $\{1, 2, 3\}$.

The triangle cover will only consist of triangles with vertices in pieces with the same label. Each of the colored vertices corresponds to at least one of the two labels, but not necessarily both. For example, if there is a red edge in $(\tilde{A}_{2,1}, \tilde{A}_{1,2})$, then we want to ensure that each of its endvertices are in pieces labeled $(2;2)$. So, it may be necessary to exchange colored vertices in one piece with uncolored ones in the other piece. A total of at most $\Delta_4 t$ vertices will be so exchanged in any $\tilde{A}_{i,j}$.

The covering by triangles can be completed by taking each piece that has the same label and covering the corresponding triple with triangles. Consider,

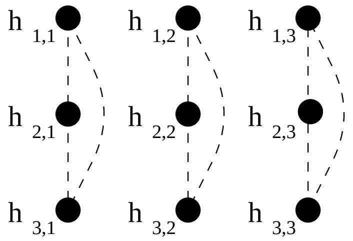

Figure 4: Diagram of $\Theta_{3\times 3}$. The dotted lines correspond to non-edges.

for example, the vertices in $\tilde{A}_{1,1}$, $\tilde{A}_{2,2}$ and $\tilde{A}_{3,2}$ that carry the label $(1;2)$. For simplicity, call them S_1, S_2 and S_3, respectively.

Any green vertex $v \in S_1$ is adjacent to at least ηt vertices in both $A_{2,2}$ and $A_{3,2}$. Thus, it is adjacent to at least $(\eta - 2\Delta_3)t$ vertices in both $\tilde{A}_{2,2}$ and $\tilde{A}_{3,2}$. Since the S_i were chosen at random, Stirling's inequality (see Corollary 3.5 in Section 3.7) gives that v is adjacent to at least $(\eta - O(\Delta_4))(t/2)$ uncolored and unexchanged vertices in both S_2 and S_3. Since all but $O(\Delta_2)t$ of the vertices in $A_{2,2}$ have degree at least $(1-2\Delta_2)t$ in $A_{3,3}$, Stirling's inequality again gives that there exists an edge among the uncolored and unexchanged vertices of $(N(v)\cap S_2, N(v)\cap S_3)$. Do this for all the green vertices in order to get disjoint green triangles.

The red and blue edges are even easier. For example, since each endvertex of a red edge in (S_1, S_2) is adjacent to at least $(1-\eta)t$ vertices in $A_{3,2}$, we can find a common vertex among the uncolored and unexchanged vertices of S_3. So, extend the colored edges to find red and blue triangles disjoint from each other and from the green triangles. Finally each uncolored vertex in S_i that was "exchanged" has degree at least $(1-\eta-O(\Delta_3))(t/2)$ in each of the S_j, $\forall j \neq i$. Put these in black triangles disjoint from each other and from other colored triangles. Let there be $t' \geq (1-O(\Delta_4))(t/2)$ uncolored vertices remaining in each class. Call them $S_i' \subset S_i$, for $i = 1,2,3$. Since $\Delta_4 \ll \eta$, each vertex in S_i' is adjacent to at least $(3/4)t'$ vertices in each of the S_j, $\forall j \neq i$. Proposition 1.7 finishes the covering and the proof of this case.

3.4 G is Δ_2-approximately $\Theta_{3\times 3}(N/3)$ (Case (3) of Lemma 3.3)

Let the sets $A_{i,j}$, $i,j = 1,2,3$ be as the $h_{i,j}$ in Figure 4. Note that the figure depicts the non-edges of this graph. Each row of vertices corresponds to a vertex class and the dotted lines correspond to non-edges. Our goal is again to modify the sets $A_{i,j}$ to form sets $\tilde{A}_{i,j}$. Our triangles will come from $\left(\tilde{A}_{i_1,1}, \tilde{A}_{i_2,2}, \tilde{A}_{i_3,3}\right)$ for distinct i_1, i_2, i_3. Triangles that come from this triple will receive the label (i_1, i_2).

The method is very similar to that in Section 3.3. The sets $A'_{i,j}$, $A''_{i,j}$, $\tilde{A}_{i,j}$ and C'_i will be created similarly to before. But this case is easier not only because each $c \in C_i$ can be added to any set $A'_{i,j}$ for any $j \in \{1,2,3\}$ and it is easy to find the parity triangles. Parity triangles can be found in uncolored vertices of $\left(\tilde{A}_{1,1}, \tilde{A}_{2,2}, \tilde{A}_{3,3}\right)$, $\left(\tilde{A}_{2,1}, \tilde{A}_{3,2}, \tilde{A}_{1,3}\right)$ and $\left(\tilde{A}_{3,1}, \tilde{A}_{1,2}, \tilde{A}_{2,3}\right)$. Partitioning the $\tilde{A}_{i,j}$ sets in half uniformly at random, exchanging the colored vertices and applying Proposition 1.7 finishes the proof. $\square$

3.5 Proof of Proposition 3.2

Let X be the set of vertices in A_1 that are adjacent to less than $(1+\theta)t$ vertices in B_2. Computing the densities,

$$\Delta|A_1||A_2| \geq e\left(A_1, A_2\right) \geq 2t|A_1| - |A_1||B_2| + |X|\left[|B_2| - (1+\theta)t\right].$$

So, it must be true that

$$|X| \leq |A_1| \frac{|B_2| - 2t + \Delta|A_2|}{|B_2| - (1+\theta)t} \leq \frac{\Delta}{1-\theta} t.$$

As a result, $|A_i \setminus A'_i| \leq \frac{2\Delta}{1-\theta} t$. Similarly, $|B_i \setminus B'_i| \leq \frac{4\Delta}{1-\theta} t$. With $\Delta_1 \geq \frac{4\Delta}{1-\theta}$ the proposition is proven. $\square$

3.6 Proof of Lemma 3.3

Again, there are a sequence of constants:

$$\Delta'''_1 \ll \delta_1 \ll \delta_2 \ll \delta_3 \ll \delta_4 \ll \delta_5 \ll \delta_6 \ll \delta_7 \ll \delta_8 \ll \Delta'_2.$$

Begin by defining

$$B'_i = \left\{v \in V_i : \deg_{A'_j}(v) \geq (1/2)(1+\theta)t', \ \forall j \neq i\right\}, \quad \text{for } i = 1,2,3.$$

Define $C'_i = V_i \setminus (A'_i \cup B'_i)$. Again, using Proposition 3.2, we see that $|C'_i| \leq \delta_1 t'$. Now, we find $3t' - |A'_1| - |A'_2| - |A'_3|$ disjoint triangles in $(B'_1 \cup C'_1, B'_2 \cup C'_2, B'_3 \cup C'_3)$. If this is not possible, Proposition 1.8 gives that G must be Δ'_2-approximately $\Theta_{3\times 3}(t')$.

If such disjoint triangles exist, then remove them from the graph to create B''_i and C''_i for $i = 1,2,3$. All that remains to prove is that there exists a matching, M, in $(B''_1 \cup C''_1, B''_2 \cup C''_2, B''_3 \cup C''_3)$ such that for any triple $\{i_1, i_2, i_3\}$, there is a matching in $(B''_{i_1} \cup C''_{i_1}, B''_{i_2} \cup C''_{i_2})$ of size $|A'_{i_3}|$ with each $c \in C''_{i_1}$ is adjacent to at least $(1-\theta-\delta_1)t'$ vertices in A'_{i_3}. What we will do is first form triangles that involve the c vertices and then, because $\delta_1 \ll \theta - 3/4$, we can see that each remaining vertex in, say A'_3, is adjacent to at least half of the edges in the portion of the matching that is in $(B''_1 \cup C''_1, B''_2 \cup C''_2)$ and each edge of this portion of the matching is adjacent to at least half of the remaining vertices in A_3'. König-Hall gives that there must be a covering by triangles.

In order to find this matching, we will randomly partition the sets $B_i'' \cup C_i''$. Let $B_i'' \cup C_i'' = S_i(j) \cup S_i(k)$, where $\{j,k\} = \{1,2,3\} \setminus \{i\}$ and $|S_i(j)| = |A_j'|$ for all distinct i and j. It is important to take note that with probability $1 - o(1)$, and for all vertices v in the graph,

$$\deg_{S_i(j)}(v) - \left(\frac{|A_j'|}{|B_i'' \cup C_i''|}\right) \deg_{B_i'' \cup C_i''}(v) \in (-o(t'), +o(t')) \ .$$

This is a result of Stirling's inequality [1] (see Section 3.7).

Once the "S" sets are randomly chosen, it may be necessary to move the "C''" vertices. Let us suppose that $c \in C_i'' \cap S_i(j)$ is not adjacent to at least $(1 - \theta - \delta_1)t'$ vertices in A_j'. Then, we will exchange c with a vertex in $B_i'' \cap S_i(k)$ (where $k = \{1,2,3\} \setminus \{i,j\}$). Do this for all i and all $c \in C_i''$ and then match each moved vertex in $S_i(j)$ with an arbitrary neighbor in $S_k(j)$ (for $j \neq k$). Color these edges red. There are at most $\delta_2 t'$ red edges in any pair $(S_i(j), S_k(j))$. Then, finish by finding a matching between the uncolored vertices of $(S_i(j), S_k(j))$. If this is not possible, then Proposition 3.4, a simple consequence of König-Hall, gives that edges can be removed so that the minimum degree condition holds, but the pairs must be δ_3-approximately $\Theta_{2\times 2}(|A_j'|/2)$.

Proposition 3.4 *Let $\epsilon \ll \Delta$ and $G = (V_1, V_2; E)$ be a balanced bipartite graph on $2M$ vertices such that each vertex is adjacent to at least $\left(\frac{1}{2} - \epsilon\right) M$ vertices in the other class. If G has no perfect matching, then some edges can be deleted so that the minimum degree condition is maintained and G is Δ-approximately $\Theta_{2\times 2}(M/2)$.*

If, with probability at least $2/3$, the pair $(S_i(j), S_k(j))$ has such a matching, then we complete the triangle cover via König-Hall and the proof is complete. Otherwise, with probability at least $1/3$, the pair $(S_i(j), S_k(j))$ is δ_3-approximately $\Theta_{2\times 2}(|A_j'|/2)$. Since this is true and $\delta_3 \ll \delta_4 \ll \delta_5$, $(B_i'' \cup C_i'', B_k'' \cup C_k'')$ itself is (δ_4, δ_5)-approximately $\Theta_{2\times 2}(t')$.

We want to show that, unless all three pairs are (δ_6, δ_7)-approximately $\Theta_{2\times 2}(t')$, the matching M exists. Without loss of generality, suppose that $(B_2'' \cup C_2'', B_3'' \cup C_3'')$ is not (δ_6, δ_7)-approximately $\Theta_{2\times 2}(t')$. Then choose "S" sets as before and move the "C" vertices as before. A matching exists among the uncolored vertices of $(S_1(3), S_2(3))$ that involves all but $O(\delta_5)t'$ vertices. But then exchange vertices – outside of this matching – in $S_2(3)$ with vertices in $S_2(1)$ so that M can be completed. If necessary, do the same with $S_3(2)$ and $S_3(1)$. Color the edges formed by the switching red. If there does not exist a matching among the uncolored vertices in $(S_2(1), S_3(1))$, then, as before, we must have that $(B_2'' \cup C_2'', B_3'' \cup C_3'')$ is (δ_6, δ_7)-approximately $\Theta_{2\times 2}(t')$, a contradiction. So, each pair $(B_i'' \cup C_i'', B_k'' \cup C_k'')$ must be (δ_6, δ_7)-approximately $\Theta_{2\times 2}(t')$.

The objective is to show that the subsets of vertices involved in forming the (δ_6, δ_7)-approximately $\Theta_{2\times 2}(t')$ must have a trivial intersection. Write $(B_i'' \cup C_i'', B_j'' \cup C_j'')$ as $(P_{i\to j}(a) \cup P_{i\to j}(b), (P_{j\to i}(a) \cup P_{j\to i}(b))$ where each of the "P" sets are of size δ_6-approximately t' and

$$d\left(P_{i\to j}(a), P_{j\to i}(b)\right), d\left(P_{i\to j}(b), P_{j\to i}(a)\right)) < \delta_7.$$

Suppose, without loss of generality, that

$$\begin{aligned} &\left|P_{3\to 1}(a) \cap P_{3\to 2}(a)\right|, \left|P_{3\to 1}(a) \cap P_{3\to 2}(b)\right|, \\ &\left|P_{3\to 1}(b) \cap P_{3\to 2}(a)\right|, \left|P_{3\to 1}(b) \cap P_{3\to 2}(b)\right| \quad \geq \quad \delta_8 t'. \end{aligned}$$

Then, as in the paragraph above, we can simply choose "S" sets of appropriate size at random. Exchange vertices so as to force a matching in $(S_1(3), S_2(3))$ and then, since the intersections of the "P" sets are so large, it is easy to exchange vertices in $B_3'' \cup C_3''$ so that matchings are forced in both $(S_1(2), S_3(2))$ and $(S_2(1), S_3(1))$. Using König-Hall to complete the covering by triangles gives us a contradiction.

Since, within a tolerance of $\delta_8 t'$, the "P" sets coincide, we may assume that $P_{2\to 1}(a)$ and $P_{2\to 3}(a)$ coincide and that $P_{3\to 2}(a)$ and $P_{3\to 1}(a)$ coincide. Therefore, the issue is whether $P_{1\to 2}(a)$ and $P_{1\to 3}(a)$ coincide or whether they are virtually disjoint. If they coincide, then G is Δ_2'-approximately $\Gamma_3(t')$. If they are disjoint, then G is Δ_2'-approximately $\Theta_{3\times 3}(t')$. $\square$

3.7 Stirling's Inequality

Stirling's inequality (see, for example, [1]) is a well-known result that gives

$$b(n,k) \exp\left[-\frac{1}{12}\left(\frac{1}{n-k} + \frac{1}{k}\right)\right] \leq \binom{n}{k} \leq b(n,k) \exp\left[\frac{1}{12}\left(\frac{1}{n}\right)\right].$$

if $b(n,k) = \frac{n^n}{(n-k)^{(n-k)}k^k}\sqrt{\frac{n}{2\pi k(n-k)}}$. The proofs of Theorem 3.1 will use the following corollary:

Corollary 3.5 *If G is a graph on n vertices and X is a set on $\Omega(n)$ vertices then, with $\epsilon \ll p$ and n large enough, if X' is chosen uniformly from $\binom{X}{p|X|}$,*

$$\Pr\left\{\left|\deg_{X'}(v) - p\deg_X(v)\right| \leq \epsilon n, \forall v \in V(G) \setminus X\right\} \to 1$$

as $n \to \infty$.

4 N is Not a Multiple of 3

We have proven the theorem for the case where $N/3$ is an integer. The other cases come as a corollary.

Let t be an integer so that $N = 3t + 1$ and let $N_0 = 3t$ be large enough so that Theorem 1.2 is true for all multiples of 3 larger than N_0. Remove any triangle from G to form the graph G'. Then, since every vertex in G is adjacent to at least $\lceil 2t + 2/3 \rceil = 2t + 1$ vertices in each of the other classes of G, every vertex in G' is adjacent to at least $2t$ vertices in the other classes of G'. If G' can be covered with triangles, then clearly G can also. If $G' = \Gamma_3(t)$, for t odd, then each vertex in G' must be adjacent to both the vertices in the other vertex classes of $G \setminus G'$. Therefore, by removing a triangle in $(A_{1,3}, A_{2,3}, A_{3,3})$ (with vertex clusters of G' labeled similarly to the diagram in Figure 4) and removing triangles formed by the vertices of $V(G) \setminus V(G')$ and edges that span the remaining A sets, the resulting graph, G'' is $\Gamma_3(t-1)$, which can be covered with triangles, by the earlier proof.

The case for $N = 3t+2$ is similar. Remove 2 disjoint triangles. The resulting graph can either be covered by disjoint triangles or it is $\Gamma_3(t)$, for t odd. In that case we do the same as in the previous paragraph, forming 5 disjoint triangles and what remains is the graph $\Gamma_3(t-1)$.

5 An Open Problem

An interesting question is that of how to eliminate the phrase "if N is large enough." It may not be necessary to write another proof. In fact, Conjecture 5.1 would give us a proof of Theorem 1.2 with $N_0 = 1$.

Conjecture 5.1 *Let G be a graph and t be a positive integer so that both of the blow-up graphs $G(t)$ and $G(t+1)$ can be covered with triangles. Then G itself can be covered with triangles.*

To see that this implies $N_0 = 1$, suppose that Conjecture 5.1 is true. Furthermore, suppose there is a balanced tripartite G on $3N$ vertices with the minimum degree condition, but G cannot be covered with triangles, and $G \neq \Gamma_3(N/3)$ for $N/3$ odd. We know by Theorem 1.2 that, for $t \geq N/N_0$, both $G(t)$ and $G(t+1)$ can be covered with triangles. This would contradict Conjecture 5.1.

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