Polychromatic cliques and related questions

Tom Bohman* Alan Frieze† Ryan Martin‡
Miklós Ruszinkó§ Clifford Smyth ¶

Department of Mathematical Sciences,
Carnegie Mellon University
Pittsburgh PA 15213.

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Abstract

Let the edges of a graph G be coloured so that no colour is used more than k times. We refer to this as a *k-bounded colouring.* We say that a subset of the edges of G is *polychromatic* if each edge is of a different colour. In this paper we address the problem of finding the minimum number m such that every k-bounded colouring of K_m contains a polychromatic copy of K_n. We then generalise this to some related problems.

1 Introduction

Let the edges of a graph G be coloured so that no colour is used more than k times. We refer to this as a *k-bounded colouring.* We say that a subset of the edges of G is *polychromatic* if each edge is of a different colour. In this setting, the following question arises: What is the required relationship between k, G, H such that every k-bounded colouring of G contains a polychromatic copy of some other (smaller) graph H. For example, the existence of polychromatic Hamilton cycles in edge coloured copies of complete graphs was studied in [4], [8], [7],

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‡Supported in part by NSF VIGRE grant DMS-9819950.
§Permanent address: Computer and Automation Research Institute of the Hungarian Academy of Sciences, Budapest, P.O.Box 63, Hungary-1518. Supported in part by OTKA Grants T 030059 and T 038198 and KFKI ISYS Datamining Lab.
¶Zeev Nehari Assistant Professor

[1], [3]. The existence of polychromatic stars was studied in Hahn [9], [10] and Fraisse, Hahn and Sotteau [6]. The complexity of finding polychromatic sub-graphs was studied by Fenner and Frieze [5].

To further fix ideas, let us consider $G = K_m$ and $H = K_n$ where $m \leq n$ and define the *Anti-Ramsey* numbers

$$ar(n,k) = \min\{m : \text{ every } k\text{-bounded colouring of } K_m \text{ contains a polychromatic } K_n\}.$$

Alspach, Gerson, Hahn and Hell [2] proved that $ar(n,k) = O(kn^3)$ and $\Omega(kn)$. This was improved by Hell and Montellano [11] to $ar(n,k) \leq (2n-3)(n-2)(k-1)+3$ and $ar(n,k) = \Omega(n^{3/2})$ for $k \geq 15$ and $ar(n,k) = \Omega(n^{4/3})$ for $3 \leq k < 15$. Our first result tightens the lower bound to within a logarithmic factor of the upper bound.

Theorem 1.

$$ar(n,k) = \Omega(kn^2/\ln n).$$

$\diamondsuit$

We then consider *proper* colourings. An edge colouring is proper if the colour classes are matchings. We let

$$ar^*(n,k) = \min\{m : \text{ every proper } k\text{-bounded colouring of } K_m \text{ contains a polychromatic } K_n\}.$$

Theorem 2.

$$ar^*(n,k) = \Omega(k^{1/2}n^{3/2}/(\ln n)^{1/2}).$$

$\diamondsuit$

We will also give some quite simple proofs of upper bounds:

Theorem 3.

(a) $ar(n,k) = O(kn^2)$.

(b) $ar^*(n,k) = O(k^{1/2}n^{3/2})$.

$\diamondsuit$

Part (a) of the above theorem is already known from [11] and part (b) can easily be derived by a simple modification of the proof therein. However, our proof is different and we venture, somewhat simpler.

We then abstract the proof technique of Theorem 1 and prove a general theorem. Let G be an m-vertex graph G which is d-regular. H is an n-vertex graph which is d'-regular. G contains N copies of H. We edge colour G and ask for a polychromatic copy of H. Thus we define

$$\theta(k,G,H) = \begin{cases} 1 & \text{Every } k\text{-bounded colouring of } G \text{ contains a polychromatic } H \\ 0 & \text{Otherwise} \end{cases}$$

$\theta^*(k, G, H)$ is defined analogously for proper k-bounded colourings: ($A \ll B$ will mean A/B is sufficiently small.)

Theorem 4.

(a)

$$\theta(k, G, H) = 0 \text{ } \textit{if (i)} \log m \ll \frac{d}{k} \text{ } \textit{and (ii)} \log N \ll \frac{kn(d')^2}{d}.$$

(b)

$$\theta^*(k, G, H) = 0 \text{ } \textit{if (i)} \log m \ll \frac{d}{k} \text{ } \textit{and (ii)} \log N \ll \frac{kn^2(d')^2}{md}.$$

$\diamondsuit$

The following special cases will then be proved: Let

$$ar_B(n, k) = \min\{m : \text{ every } k\text{-bounded colouring of } K_{m,m} \text{ contains a polychromatic } K_{n,n}\}.$$

Let Q_n denote the n-cube i.e. the graph with vertex set $\{0, 1\}^n$ and with edges joining two sequences at Hamming distance 1. Let

$$ar_Q(n, k) = \min\{m : \text{ every } k\text{-bounded colouring of } Q_m \text{ contains a polychromatic } Q_n\}.$$

Let $\Gamma_n = K_n \oplus K_n$ be the graph with vertex set $[n]^2$ and an edge $((i, j), (i', j'))$ whenever $i = i'$ or $j = j'$. Let

$$ar_\Gamma(n, k) = \min\{m : \text{ every } k\text{-bounded colouring of } \Gamma_m \text{ contains a polychromatic } \Gamma_n\}.$$

Theorem 5.

(a) $ar_B(n, k) = \Omega(kn^2/\ln n)$ *and* $ar_B(n, k) = O(kn^2)$.

(b) $ar_\Gamma(n, k) = \Omega(kn^2/\ln n)$ *and* $ar_\Gamma(n, k) = O(kn^2)$.

(c) $ar_Q(n, k) = \Omega(2^{n/2}n)$ *and* $ar_Q(n, k) = O(kn^2 2^n)$.

$\diamondsuit$

Finally, we see that the proof techniques we use are robust enough to extend to hypergraphs. Let $K_m^{(r)}$ denote the complete r-uniform hypergraph on vertex set $[m]$. Then let

$$ar(n, k, r) = \min\{m : \text{ every } k\text{-bounded colouring of } K_m^{(r)} \text{ contains a polychromatic } K_n^{(r)}\}.$$

Theorem 6. *Assume that r is fixed, independent of k, n.*

(a) $ar(n, k, r) = \Omega(kn^r/\log n)$.

(b) $ar(n, k, r) = O(kn^r)$.

2 Lower Bounds: Proofs of Theorems 1 and 2

We start with Theorem 1. Let $m = \lfloor \frac{\varepsilon_0 k n^2}{\ln n} \rfloor$ where $\varepsilon_0 \ll 1$. We colour K_m randomly: Initially $i = 1$ and $G_1 = K_m$ and we obtain G_{i+1} from G_i in the following manner.

(a) Choose a random vertex x of degree at least k.

(b) Choose a random copy H_i of $K_{1,k}$ which has centre x and uses only edges of G_i.

(c) Give the edges of H_i a new colour in K_m and then delete the edges of H_i from G_i to produce G_{i+1}.

This process continues until $\Delta(G_i) < k$. Then we give each of the uncoloured edges of K_m a new distinct colour.

The above colouring is clearly k-bounded. We prove that **whp**[1] there is no K_n which is polychromatic. This will prove Theorem 1.

Now let $t = \lfloor \frac{\varepsilon_1 m^2}{k} \rfloor$ for some $\varepsilon_1 \ll 1$. We claim that **whp** the first t rounds of the above colouring process are sufficient to ensure that every K_n will contain at least 2 edges of the same colour.

Let δ_i denote the minimum degree of G_i and let $m_0 = (1 - 3\varepsilon_1)m$. Let $\mathcal{D}_i$ be the event: $\{\delta_i \geq m_0\}$.

Lemma 1.

$$\Pr(\overline{\mathcal{D}_{t+1}}) = o(1).$$

Proof Let X_v denote the number of times that vertex v is chosen as x in Step (a) of the procedure. We count up to time t or the occurrence of $\overline{\mathcal{D}_i}$, $i \leq t$. More precisely we let $X_v = \sum_{i=1}^{t} 1_{x=v} 1_{\mathcal{D}_i}$. Now $\Pr(x = v \mid \mathcal{D}_i) = m^{-1}$ and so X_v is dominated by $Bin(t, m^{-1})$. Using a Chernoff bound[2] we see that

$$\Pr(X_v \geq 3\varepsilon_1 m/(2k)) \leq \exp\{-\varepsilon_1 m/(13k)\}.$$

Next let Y_v be the number of times that v occurs as a neighbour of x in H_i. Then Y_v is dominated by $Bin(t, km_0^{-1})$ and

$$\Pr(Y_v \geq 3\varepsilon_1 m/2) \leq \exp\{-\varepsilon_1 m/13\}.$$

Note that v loses $kX_v + Y_v$ incident edges. Inflating the RHS of the above inequalities by m to account for all vertices v, we see that **whp** $kX_v + Y_v < 3\varepsilon_1 m$ for all v and this implies the occurrence of $\mathcal{D}_{t+1}$. □

Fix a copy K of K_n. Let $\mathcal{A}_i$ be the event: $\{$Exactly one edge of K gets coloured in round $i\}$. Let $Y_i = 1_{\mathcal{A}_i}$ and let $\mathcal{E}_1 = \mathcal{E}_1(K)$ be the event: $\left\{\sum_{i=1}^{t} Y_i < 3\varepsilon_1 n^2\right\}$.

[1] A sequence of events $\mathcal{A}_n$ is said to occur *with high probability* **whp**, if $\lim_{n\to\infty} \Pr(\mathcal{A}_n) = 1$

[2] $\Pr(Bin(n,p) \geq (1+\varepsilon)np) \leq e^{-\varepsilon^2 np/3}$ for $0 \leq \varepsilon \leq 1$

Lemma 2.

$$\Pr(\overline{\mathcal{E}_1} \cap \mathcal{D}_t) \leq e^{-\varepsilon_1 n^2/4}.$$

Proof Let $W_i = Y_i \cdot 1_{\delta_i \geq m_0}$ for $1 \leq i \leq t$. Then for any $u \geq 0$ we have

$$\Pr(Y_1 + \cdots + Y_t \geq u \text{ and } \mathcal{D}_t) \leq \Pr(W_1 + \cdots + W_t \geq u). \tag{1}$$

Now

$$\Pr(W_i = 1 \mid G_i) \leq \Pr(Y_i = 1 \mid G_i, \delta_i \geq m_0) \leq \frac{n}{m} \cdot \frac{kn}{(1-3\varepsilon_1)m} = \frac{kn^2}{(1-3\varepsilon_1)m^2}.$$

So $W_1 + \cdots + W_t$ is stochastically dominated by $Bin\left(t, \frac{kn^2}{(1-3\varepsilon_1)m^2}\right)$ and

$$\Pr\left(W_1 + \cdots + W_t \geq \frac{2\varepsilon_1}{(1-3\varepsilon_1)}n^2\right) \leq \exp\left\{-\frac{\varepsilon_1}{3(1-3\varepsilon_1)}n^2\right\}.$$

It follows from this and (1) that

$$\Pr\left(\left[\sum_{i=1}^{t} Y_i \geq \frac{2\varepsilon_1}{(1-3\varepsilon_1)}n^2\right] \cap \mathcal{D}_t\right) \leq \exp\left\{-\frac{\varepsilon_1}{3(1-3\varepsilon_1)}n^2\right\} \tag{2}$$

and the lemma follows. □

Now let $\mathcal{B}_i = \mathcal{B}_i(K)$ be the event: {At most one edge of K gets coloured in round i} and let $\mathcal{C}_i = \left\{\sum_{j=1}^{i} Y_j < 3\varepsilon_1 n^2\right\}$. If K is to be polychromatic then the event $\bigcap_{i=1}^{t} \mathcal{B}_i$ must occur.

Lemma 3.

$$\Pr\left(\bigcap_{i=1}^{t} \mathcal{B}_i \cap \mathcal{E}_1 \cap \mathcal{D}_t\right) \leq \exp\left\{-\frac{\varepsilon_1 k n^3}{5m}\right\}$$

Proof We write $\mathcal{J}_i = \bigcap_{j=1}^{i-1} \mathcal{B}_j \cap \mathcal{C}_j \cap \mathcal{D}_j$ and then

$$\begin{aligned}\Pr\left(\bigcap_{i=1}^{t} \mathcal{B}_i \cap \mathcal{E}_1 \cap \mathcal{D}_t\right) &= \prod_{i=1}^{t} \Pr\left(\mathcal{B}_i \cap \mathcal{C}_i \cap \mathcal{D}_i \mid \mathcal{J}_i\right) \\ &\leq \prod_{i=1}^{t} \Pr\left(\mathcal{B}_i \mid \mathcal{J}_i\right).\end{aligned} \tag{3}$$

Now let $\mathcal{G}_i = \{G_i : \mathcal{J}_i \text{ occurs}\}$. Then

$$\Pr\left(\mathcal{B}_i \mid \mathcal{J}_i\right) = \sum_{G \in \mathcal{G}_i} \Pr(\mathcal{B}_i \mid G_i = G) \Pr(G_i = G \mid \mathcal{J}_i). \tag{4}$$

Fix $G \in \mathcal{G}_i$ and then for $z \in K$ define d_z to be the number of K-neighbours that z has in the graph G. Then

$$\Pr(\overline{\mathcal{B}_i} \mid G_i = G) \geq \binom{k}{2} \frac{1}{m} \sum_{z \in K} \frac{d_z}{m-1} \cdot \frac{d_z - 1}{m-2} \left(1 - \frac{n}{m(1-3\varepsilon_1)}\right)^k \geq \frac{k^2}{4} \frac{n^3(1-7\varepsilon_1)^2}{m^3}. \tag{5}$$

Explanation: We estimate the probability that $x = z$ in Step (a) and then $\binom{k}{2}$ enumerates pairs $i_1 < i_2$ such that i_1, i_2 are the first two times that vertices of K are chosen as part of H_i. Given i_1, i_2, the probability that the corresponding choices are in K is at least $\frac{d_z}{m-1} \cdot \frac{d_z-1}{m-2}$ and the term $\left(1 - \frac{n}{m(1-3\varepsilon_1)}\right)^k$ bounds the probability of the remaining first $i_2 - 2$ choices being outside K. This latter term is at least $1 - \frac{kn}{m(1-3\varepsilon_1)} = 1 - o(1)$. Finally, using Jensen's inequality $\sum_{z \in K} d_z(d_z - 1)$ is at least $n^3(1-7\varepsilon_1)^2$ since K contains at least $\frac{1}{2}\sum_z d_z \geq \binom{n}{2} - 3\varepsilon_1 n^2$ edges in G_i.

Plugging this into (4) gives

$$\Pr\left(\mathcal{B}_i \mid \mathcal{J}_i\right) \leq 1 - \frac{k^2}{4} \frac{n^3(1-7\varepsilon_1)^2}{m^3}$$

and then plugging this into (3) gives

$$\Pr\left(\bigcap_{i=1}^{t} \mathcal{B}_i \cap \mathcal{E}_1 \cap \mathcal{D}_t\right) \leq \left(1 - \frac{k^2}{4} \frac{n^3(1-7\varepsilon_1)^2}{m^3}\right)^t$$

and the lemma follows. □

We can now finish the proof of Theorem 1. Let $\mathcal{E}_2(K) = \bigcap_{i=1}^t \mathcal{B}_i(K)$ and $\mathcal{F} = \{\exists \text{polychromatic } K\}$. Then

$$\begin{aligned}
\Pr(\mathcal{F}) &\leq \Pr(\exists K :\ \mathcal{E}_2(K)) \\
&\leq \Pr([\exists K :\ \mathcal{E}_2(K) \cap \mathcal{E}_1(K)] \cap \mathcal{D}_t) + \Pr([\exists K :\ \overline{\mathcal{E}_1(K)}] \cap \mathcal{D}_t) + \Pr(\overline{\mathcal{D}_t}) \\
&\leq \binom{m}{n} \left(\exp\left\{-\frac{\varepsilon_1 k n^3}{5m}\right\} + \exp\left\{-\frac{\varepsilon_1 n^2}{4}\right\}\right) + o(1) \qquad (6) \\
&= o(1)
\end{aligned}$$

if ε_0 is small enough. This completes the proof of Theorem 1. □

We turn to the proof of Theorem 2. We follow a similar strategy.

Let $m = \lfloor \frac{\varepsilon_0 k^{1/2} n^{3/2}}{(\ln n)^{1/2}} \rfloor$ for some $\varepsilon_0 \ll 1$. Once again, we colour K_m randomly:
Initially $G_1 = K_m$ and in round i we obtain G_{i+1} from G_i in the following manner.

(a) Sequentially choose k random, disjoint uncoloured edges A_i of K_n

(b) Give the edges of A_i a new colour in K_m and then delete the edges of A_i from G_i to produce G_{i+1}.

This process continues until G_i contains no matching of size k. Then we give each of the uncoloured edges of K_m a new distinct colour.

The above colouring is clearly k-bounded and proper. We next prove that **whp** there is no K_n which is polychromatic.

Again let $t = \lfloor \frac{\varepsilon_1 m^2}{k} \rfloor$ for $\varepsilon_1 \ll 1$. We claim that **whp** the first t rounds are sufficient to ensure that every K_n will contain at least 2 edges of the same colour.

We can follow the proof of Theorem 1 making minor changes. Then when we come to the proof of Lemma 3 we replace the expression in (5) by

$$\begin{aligned}
\Pr(\overline{\mathcal{B}_i} \mid G_i = G) &\geq \binom{k}{2} \sum_{z_1 \neq z_2 \in K} \frac{d_{z_1} - 1}{m(m-1)} \cdot \frac{d_{z_2} - 2}{(m-2)(m-3)} \left(1 - \frac{n^2}{m^2(1-3\varepsilon_1)^2}\right)^k \\
&\geq \frac{k^2 n^4 (1 - 7\varepsilon_1)^2}{4m^4}.
\end{aligned}$$

Then the RHS of (6) is replaced by

$$\binom{m}{n} \left(\exp\left\{ -\frac{\varepsilon_1 k n^4}{5m^2} \right\} + \exp\left\{ -\frac{\varepsilon_1 n^2}{4} \right\} \right) + o(1)$$

and this will be $o(1)$ if ε_0 is small enough. This completes the proof of Theorem 2. □

3 Upper Bounds: Proof of Theorem 3

We can prove upper bounds in a simple manner by using the FKG inequality. Since part (a) is already in [11], we concentrate on (b).

Suppose we are given a k-bounded proper colouring of K_m where $m > 2k^{1/2}n^{3/2}$ and we choose a random subset S by including each vertex with probability $p = \frac{2n}{m}$. We claim that with positive probability the set of vertices selected will contain a polychromatic K_n, thus proving part (b) of the theorem. Let the colour classes be $C_1, C_2, \ldots, C_M$ and let $k_i = |C_i|$. Define the event $\mathcal{A}_i$: $\{S$ contains two edges from $C_i\}$.

Now for each i, $\Pr(\mathcal{A}_i) \leq k_i^2 p^4/2$. These events are monotone increasing in S and so by the

FKG inequality,

$$\begin{aligned}
\Pr\left(\bigcap_{i=1}^{M}\overline{\mathcal{A}_i}\right) &\geq \prod_{i=1}^{M}(1-k_i^2p^4/2)\\
&\geq \exp\left\{-\sum_{i=1}^{M}(k_i^2p^4/2+k_i^4p^8)\right\} \qquad \text{since } k^2p^4=o(1)\\
&= (1-o(1))\exp\left\{-\sum_{i=1}^{M}k_i^2p^4/2\right\}\\
&\geq (1-o(1))e^{-4kn^4/m^2}.
\end{aligned}$$

Here we have used $\sum_{i=1}^{M}k_i^2 \leq k\sum_{i=1}^{M}k_i \leq m^2k/2$ and $\sum_{i=1}^{M}k_i^4 \leq k^3\sum_{i=1}^{M}k_i \leq m^2k^3/2$. The latter implies $\sum_{i=1}^{M}k_i^4p^8=o(1)$.

Now let $\mathcal{B}$ denote the event $|S|\geq n$ then we have

$$\begin{aligned}
\Pr(\exists\text{polychromatic } K_n) &\geq \Pr\left(\mathcal{B}\cap\bigcap_{i=1}^{M}\overline{\mathcal{A}_i}\right)\\
&\geq \Pr\left(\bigcap_{i=1}^{M}\overline{\mathcal{A}_i}\right)-\Pr(\overline{\mathcal{B}})\\
&\geq (1-o(1))e^{-4kn^4/m^2}-e^{-n} \qquad (7)\\
&> 0
\end{aligned}$$

if $m>2k^{1/2}n^{3/2}$.

The proof of part (a) is almost identical. The main difference is that we bound $\Pr(\mathcal{A}_i)\leq k_i^2p^3/2$ and then arrive at $(1-o(1))e^{-2kn^3/m}-e^{-n}$ in place of the RHS of (7). □

4 Generalities: Proof of Theorem 4

(a) We follow the argument of Theorem 1. We consider the same colouring algorithm. We put $t=\lfloor\frac{\varepsilon_1 md}{k}\rfloor$. Now let $d_0=(1-3\varepsilon_1)d$ and $\mathcal{D}_i=\{\delta_i\geq d_0\}$. Following the argument of Lemma 1 we get

$$\Pr(\overline{\mathcal{D}_t})\leq m(e^{-\varepsilon_1 d/(13k)}+e^{-\varepsilon_1 d/13})=o(1). \qquad (8)$$

Now fix a copy K of H in G and define $\mathcal{A}_i$ as before (prior to Lemma 2). Now define $\mathcal{E}_1=\left\{\sum_{i=1}^{t}Y_i<\frac{2\varepsilon_1 nd'}{1-3\varepsilon_1}\right\}$. Following the argument of Lemma 2 to equation (2) we arrive at

$$\Pr(\overline{\mathcal{E}_1}\cap\mathcal{D}_t)\leq\Pr\left(\left[\sum_{i=1}^{t}Y_i\geq\frac{2\varepsilon_1 nd'}{(1-3\varepsilon_1)}\right]\cap\mathcal{D}_t\right)\leq\exp\left\{-\frac{\varepsilon_1 nd'}{3(1-3\varepsilon_1)}\right\} \qquad (9)$$

Continuing, we define the events $\mathcal{B}_i$ as before and amend $\mathcal{C}_i$ to $\left\{\sum_{j=1}^{i} Y_j < \frac{2\varepsilon_1 nd'}{1-3\varepsilon_1}\right\}$. The proof of Lemma 3 now yields

$$\Pr\left(\bigcap_{i=1}^{t} \mathcal{B}_i \cap \mathcal{E}_1 \cap \mathcal{D}_t\right) \leq \exp\left\{-\frac{\varepsilon_1 kn(d')^2}{5d}\right\} \tag{10}$$

and Theorem 4 follows.

The proof of (b) is developed analogously. □

Theorems 1, 2 and the lower bounds in Theorem 5(a),(b) are immediate corollaries of this theorem. The upper bounds in Theorem 5(a),(b) can be proved by the method of Theorem 3. Theorem 5(c) needs a slightly different proof.

5 Polychromatic Sub-Cubes: Proof of Theorem 5(c)

For the lower bound we use $ar_Q(n,k) \geq ar_Q(n,2)$ and assume $k=2$. Let $m = n2^{n/2}$. We follow the proof of Theorem 3. We randomly colour the edges of Q_m as follows: Initially $G_1 = Q_m$ and we obtain G_{i+1} from G_i in the following manner.

(a) Choose a random path P_i of length 2 in G_i (as opposed to choosing a vertex x and then 2 random neighbours).

(b) Give the edges of P_i a new colour in Q_m and then delete the edges of P_i from G_i to produce G_{i+1}.

This process continues until $\Delta(G_i) \leq 1$. Then we give each of the uncoloured edges of Q_m a new distinct colour. The above colouring is clearly 2-bounded. Now let $t = \lfloor \varepsilon_1 m2^{m-1} \rfloor$ for some $\varepsilon_1 \ll 1$. (Recall that Q_m has $m2^{m-1}$ edges.) We claim that **whp** the first t rounds of the above colouring process are sufficient to ensure that every Q_n will contain at least 2 edges of the same colour.

Observation 1: A graph G with ν vertices and average degree ρ has at least $\nu\rho^2/2 - \nu\rho/2$ distinct paths of length 2.

Now some computations:
(i) The probability that Q_m contains a Q_n which has more than a fraction $4\varepsilon_1$ of its edges coloured before round t is at most

$$\binom{m}{n} 2^{m-n} \exp\{-\varepsilon_1 n2^{n-1}/28\} = o(1).$$

Explanation: Q_m contains $\binom{m}{n}2^{m-n}$ copies of Q_n. Then observe that for a fixed copy Q of Q_n, the number of paths in the first t paths which contain an edge of Q is dominated by

$Bin(t,p)$ where, crudely, $p = \frac{3mn2^n}{m^2 2^m}$ bounds the ratio of the number of paths of length 2 which meet Q and the total number of paths of length 2 that remain in Q_t. Here we use Observation 1 to bound the latter quantity from below.

(ii) The probability that there is a Q_n which does not get coloured by a whole path is at most

$$\binom{m}{n} 2^{m-n} \left(1 - \frac{2^n(1-4\varepsilon_1)^2 n^2}{2^m m^2}\right)^t \le \exp\left\{-\left(\frac{\varepsilon_1 n^2 2^n}{4m} - n\log m - (m-n)\log 2\right)\right\} = o(1).$$

Observation 1 is used again to bound from below the number of paths contained in K.

The above calculation is a heuristic and a proper argument must be made as in the proof of Lemma 3 to avoid conditioning on the future. We leave this to the reader.

The proof of the upper bound uses the first moment method. Let $m = kn^2 2^n$ and let a k-bounded colouring of Q_m be given. Now choose a random n-cube $Q \subseteq Q_m$ and consider the probability that it is not polychromatic. Fix a colour class C_i of size k_i, $i = 1, 2, \ldots, M$. Since there are at most $\binom{m-2}{n-2}$ cubes which contain 2 given edges of Q_m we see that the probability that the cube Q contains 2 edges of C_i is at most

$$\frac{k_i^2}{2} \cdot \frac{\binom{m-2}{n-2}}{\binom{m}{n}} \cdot \frac{1}{2^{m-n}}.$$

Thus the probability that Q is not polychromatic is bounded by

$$\begin{aligned} \frac{\binom{m-2}{n-2}}{\binom{m}{n}} \cdot \frac{1}{2^{m-n+1}} \sum_{i=1}^{M} k_i^2 &\le \frac{n^2}{m^2} \cdot \frac{1}{2^{m-n+1}} \cdot \frac{m 2^{m-1}}{k} \cdot k^2 \\ &< 1. \end{aligned}$$

The proof of Theorem 5(c) is now complete. □

6 Hypergraphs: Proof of Theorem 6

We let $m = ckn^r/\log n$ for some constant $c > 0$ and consider the following edge colouring algorithm: The degree of a set X of vertices in a hypergraph $H = (V, \{E_1, E_2, \ldots, E_M\})$ is the number of edges E_i which contain it. The hypergraph $H(k,r)$ has vertex set $[r-1+k]$ and edges $\{[r-1] \cup \{i\} : i = r, \ldots, r-1+k\}$. Its *centre* is $[r-1]$. Initially $G_1 = K_m^{(r)}$ and we obtain G_{i+1} from G_i in the following manner.

(a) Choose a random $(r-1)$-set of vertices X of degree at least k.

(b) Choose a random copy H_i of $H(k,r)$ which has centre X and uses only edges of G_i.

(c) Give the edges of H_i a new colour in $K_m^{(r)}$ and then delete the edges of H_i from G_i to produce G_{i+1}.

This process continues until there are no uncoloured copies of $H(k,r)$. Then we give each of the uncoloured edges of K_m a new distinct colour. Then let $t = \lfloor \frac{\varepsilon_1 m^r}{kr!} \rfloor$ and let δ_i denote the minimum degree of an $(r-1)$-set of vertices in G_i and let $m_0 = (1-5\varepsilon_1)m$. Let $\mathcal{D}_i$ be the event: $\{\delta_i \geq m_0\}$.

Lemma 4.

$$\Pr(\overline{\mathcal{D}_t}) = o(1).$$

Proof Let X_A denote the number of times that the $(r-1)$-set A is chosen as X in Step (a) of the procedure. We count up to time t or the occurrence of $\overline{\mathcal{D}_i}$, $i \leq t$. Then X_A is dominated by $Bin(t, \binom{m}{r-1}^{-1})$. Using Chernoff bounds we see that

$$\Pr(X_A \geq 2\varepsilon_1 m/k) \leq \exp\{-\varepsilon_1 m/(13k)\}.$$

Next let Y_A be the number of times that $|X \cap A| = r-2$ and $A \setminus X$ is a vertex of H_i. Now Y_A is dominated by $Bin(t, p_1)$ where $p_1 = \frac{(r-1)(m-r)}{\binom{m}{r-1}} \frac{k}{m-r-1} \leq \frac{kr}{\binom{m}{r-1}}$ and then

$$\Pr(Y_A \geq 2\varepsilon_1 m) \leq \exp\{-\varepsilon_1 m/(13k)\}.$$

Finally note that the degree of A is $m - r - kX_A - Y_A$. □

Then for a fixed $K = K_n^{(r)}$ we let $\mathcal{A}_i$ be the event: {Exactly one edge of K gets coloured in round i}. Let $Y_i = 1_{\mathcal{A}_i}$ and let $\mathcal{E}_1 = \mathcal{E}_1(K)$ be the event: $\left\{\sum_{i=1}^t Y_i < 2\varepsilon_1 n^r\right\}$. Arguing in a similar manner to Lemma 2 we can bound $\sum_i Y_i$ by $Bin\left(t, \frac{kn}{(1-5\varepsilon_1)m} \frac{\binom{n}{r-1}}{\binom{m}{r-1}}\right)$. Thus

Lemma 5.

$$\Pr(\overline{\mathcal{E}_1} \cap \mathcal{D}_t) \leq e^{-\varepsilon_1 n^r/4}.$$

□

Finally, we argue as in Lemma 3 and obtain

$$\begin{aligned} \Pr\left(\bigcap_{i=1}^t \mathcal{B}_i \cap \mathcal{E}_1 \cap \mathcal{D}_t\right) &\leq \left(1 - \frac{\binom{n}{r-1}}{\binom{m}{r-1}} \frac{k^2 n^2}{3m^2}\right)^t \\ &\leq \exp\{-\varepsilon_1 \frac{kn^{r+1}}{3r!m}\}. \end{aligned}$$

The proof of Part (a) can now be completed in a similar manner to the proof of Theorem 1.

For Part (b) we assume that we are given a k-bounded colouring of the edges of $K_m^{(r)}$ where $m > 2kn^r$. We choose a random subset S by including each vertex with probability $p = \frac{2n}{m}$. Let the colour classes be $C_1, C_2, \ldots, C_M$ and let $k_i = |C_i|$. Define the event $\mathcal{A}_i$: {S contains two edges from C_i}. Then $\Pr(\mathcal{A}_i) \leq k_i^2 p^{r+1}/2$ and the proof can be completed as in the proof of Theorem 3.

7 Open problems

We leave it as an open problem to resolve the logarithmic gaps between the upper and lower bounds in the above theorems.

References

[1] M. J. Albert, A. M. Frieze and B. Reed, *Multicoloured Hamilton Cycles.* Electronic Journal of Combinatorics 2 (1995) R10.

[2] B. Alspach, M. Gerson, G. Hahn and P. Hell, *On sub-Ramsey numbers*, Ars Combinatoria 22 (1986) 199-206.

[3] C. Cooper and A. M. Frieze, *Multi-coloured Hamilton cycles in randomly coloured random graphs*, Combinatorics, Probability and Computing 11 (2002) 129–134.

[4] P. Erdős, J. Nešetril and V. Rödl, *Some problems related to partitions of edges of a graph* in Graphs and other Combinatorial topics, Teubner, Leipzig (1983) 54-63.

[5] T. I. Fenner and A. M. Frieze, *On the existence of polychromatic sets of edges in graphs and digraphs*, Progress in Graph Theory, Edited by J.A. Bondy and U.S.R. Murty, Academic Press (1984) 219-232.

[6] P. Fraisse, G. Hahn and D. Sotteau, *Star sub-Ramsey numbers*, Annals of Discrete Mathematics 34 (1987) 153-163.

[7] A. M. Frieze and B. A. Reed, *Polychromatic Hamilton cycles*, Discrete Mathematics 118, (1993) 69-74.

[8] G. Hahn and C. Thomassen, *Path and cycle sub-Ramsey numbers and an edge-colouring conjecture*, Discrete Mathematics 62 (1986) 29-33.

[9] G. Hahn, *Some star anti-Ramsey numbers*, Congressus Num. 19 (1977) 303-310.

[10] G. Hahn, *More star anti-Ramsey numbers*, Discrete Mathematics 43 (1981) 131-139.

[11] P. Hell and J. J. Montellano, *Polychromatic cliques*, to appear.