Chapter 10

Compound Interest

10.1 Introduction

Financial transactions, both in the world of high finance and in the personal lives of individuals, can be computationally intensive. The principle concept involved in mortgages, credit cards, savings accounts, and retirement annuities is compound interest. The applications discussed here will be of value both in making decisions regarding personal finance as well as in further studies in economics and accounting.

This chapter will be devoted to applications of geometric sequences and series to calculations involving compound interest. It is written to accompany the author's text *Introduction to Mathematical Programming*. Any references to earlier chapters or appendices are to that text.

The reader is presumed to have had at least a brief introduction to sequences and series, but for the most part the chapter is self-contained. Unlike the objective in calculus, where the main goal is to use power series to approximate functions, here we will be using sequences to calculate financial quantities exactly.

This section will be devoted to an introduction to compound interest and geometric sequences. A sum of money P is earning *compound interest* at the rate of i percent per year if at regular intervals interest is credited on interest earned during previous intervals. Therefore, at the end of the first year, the sum P has grown by Pi to

$$A = P + Pi = P(1 + i).$$

Similarly, during the second year the interest earned is $P(1 + i)i$. Thus, by the end of the second year the amount has grown to

$$\begin{aligned} A &= P(1+i) + P(1+i)i \\ &= P(1+i)(1+i) \\ &= P(1+i)^2. \end{aligned}$$

In general, after n years, the original sum P has grown to $P(1+i)^n$. This quantity, the product of an amount of money with a power of 1 plus a rate of interest, is the key building block in calculations involving mortgages, savings, annuities, etc. It also happens that such a quantity is a term in a geometric sequence.

Notational convention: Above we have adopted a convenient convention regarding percentages. In text, "i%" or "i percent" will refer to a percentage such as 5%. However, in a formula, we let i denote the decimal equivalent, i.e., i denotes a decimal such as .05 for 5%. This convention avoids the need to either multiply or divide by 100.

A *sequence* is a set of numbers written in a definite order. Usually the order is indicated by subscripts as below

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

The first set of three dots above indicates a number of terms omitted, the second set indicates that the sequence is infinite, i.e., it continues indefinitely. The number in the position n in the ordering is referred to as the "nth *term*." Thus, in the example below,

$$1, 1, 2, 3, 5, 8, \ldots$$

the fourth term is 3.

A *geometric sequence* is one for which there is a constant r, called the *ratio* of the sequence, such that each term is obtained by multiplying the previous term by r. For instance, the following sequence has ratio 2:

$$3, 6, 12, 24, \ldots.$$

A *series* is an indicated sum of the terms of a sequence. Thus, a series would be written

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots.$$

The dots in such a sum indicate that the series includes the sum of an infinite number of terms, a sum which is, of course, impossible to carry out. In calculus, we consider the convergence of such a series by examining the *sequence of partial sums*, i.e., the sequence whose nth term is the sum of the first n terms of the series.

Here we will not be concerned with infinite series, and will consider only geometric sequences and series. A *geometric series* is obtained by adding the terms of a geometric sequence. Thus, it is a series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

The parameter r is called the *ratio* of the series and we refer to a as the *first term*. From the point of view of the usual calculus sequence, the key facts are that the geometric series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

converges if $|r| < 1$ and its sum is

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \frac{a}{1-r}$$

and it diverges if $|r| \geq 1$.

The only use that we will make of an infinite geometric series will be in situations like the following:

Example 10.1.1. Planners for a convention center have conducted an analysis that reveals that of each \$1 that a visitor attracted to an event spends, \$0.70 falls into the hands of a local worker and is then respent locally. The successive local recipients then continue to respend \$0.70 of each dollar indefinitely. If this is the case, how much additional local spending does this \$1 actually generate?

The initial spending plus the initial local respending generate \$1.70. But 70% of the respent \$0.70 is again respent, and in theory the respending continues indefinitely. The total spent is calculated by summing the geometric series

$$1 + .7 + (.7)^2 + \cdots + (.7)^n + \cdots$$

with ratio $r = .7$ and first term $a = 1$. Then the sum, i.e., the limit of the sequence of partial sums, is

$$\frac{1}{1-.7} = \frac{1}{.3} = 3.33333\ldots$$

and so \$3.33 in additional local spending is generated by each dollar spent by a convention visitor. ■

Exercises

1. If \$2,000 is deposited at 8% compound interest, what will be the amount accumulated after four years?

2. If \$5,000 is deposited at 6% compound interest, what will be the amount accumulated after seven years?

3. For each of the following series, determine the sum if the series converges:

(a) $4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots$

(b) $3 - 1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27} + \cdots$

(c) $2 + 3 + \frac{9}{2} + \frac{27}{4} + \frac{81}{8} + \cdots$

4. Suppose that of the money spent by a family visiting a city for a baseball game, \$0.60 of every \$1 spent is respent repeatedly locally. If the family spends \$40, how much local spending is generated by their visit?

5. Suppose that in the preceeding problem that we assume the money is respent locally only three times. Then how much local spending is generated?

6. A person with a \$35,000 annual salary expects raises of 3% for each of the next several years. What will his salary be after four years if that turns out to be correct?

7. Suppose that a rubber ball rebounds to two-thirds of the height from which it falls and continues to rebound indefinitely. What will be the total distance traveled by the ball if it is dropped from a height of five feet?

10.2 Compound interest and savings

We now consider in detail the applications of geometric sequences and of the sum of finitely many terms of a geometric series. Restating the conclusion reached informally in the introduction:

Proposition 10.2.1. *If P dollars earns interest at a rate of i percent compounded annually for n years, then it will grow to*

$$A = P(1+i)^n.$$

Example 10.2.2. Suppose that \$100 is deposited in a savings account that pays 6% interest per year compounded annually. How much would be in the account after five years?

Using the formula from the previous proposition, we get

$$A = 100(1+.06)^5 = \$133.82.$$

Thus, the amount deposited increases by \$33.82, or roughly one-third of the amount deposited, in five years. ■

The calculation above is easily done on a calculator with a key for exponentiation. On a TI-82, the calculation looks like

```
100(1.06)^5
           133.8225578
```

For this calculation, the MODE was set to Float, causing 10 digits to be shown. To obtain answers rounded to a fixed number of digits after the decimal sign, set the MODE to an integer. An introduction to the TI-82 can be found in Appendix C.

The exponentiation key on a calculator is often indicated by $\boxed{y^x}$ or similar indication of raising a quantity to a power.

Interest is rarely compounded annually as in the example above. The more common situation is to compound the interest more frequently. For instance, if the interest is compounded semi-annually, the interest is credited twice a year at half the annual rate. In the case of a deposit at 6% compounded semiannually, the amount of the deposit would be multiplied by $(1.03)^2$ to get the amount at the end of one year. The next proposition gives the general formula.

Proposition 10.2.3. *If P dollars earns interest at a rate of i percent compounded m times annually for n years, then it will grow to*

$$A = P(1+\frac{i}{m})^{mn}.$$

Example 10.2.4. Now suppose that \$100 is deposited in a savings account that pays 6% interest per year as in the previous example, but that now the interest is compounded monthly. How much would be in the account after five years, and how does the amount differ from that obtained with annual compounding?

Using the formula from the previous proposition, we get

$$A = 100(1 + \frac{.06}{12})^{12 \cdot 5} = 100(1.005)^{60} = \$134.88.$$

Thus, the amount deposited increases by \$34.88, or \$1.06 more than with annual compounding. ■

The two examples indicate that there is an advantage to frequent compounding. Table 10.2.1 shows the effect of different frequencies of compounding.

Table 10.2.1

Interest on \$1,000 at 8% per year for 10 years	
No compounding	\$800.00
Annual	1,158.92
Semi-annual	1,191.12
Quarterly	1,208.04
Monthly	1,219.64
Daily	1,225.35

From the previous proposition, we see that the entries in the table are calculated by

$$1000\left(1 + \frac{.08}{m}\right)^{10m}$$

with increasing values of m corresponding to increasing frequencies of compounding. We can gain insight into what would happen if we were able to compound the interest instantly, called *continuous compounding*, by recalling the definition of e from calculus:

$$e = \lim_{x \to \infty}\left(1 + \frac{1}{x}\right)^{x} = 2.7182818\ldots.$$

The key to relating this definition of e to compounding interest m times a year is to consider the limit of the expression in Proposition 10.2.3 as $m \to \infty$. We first multiply and divide by i in the exponent:

$$\begin{aligned}\lim_{m\to\infty} P\left(1+\frac{i}{m}\right)^{mn} &= \lim_{m\to\infty} P\left(1+\frac{i}{m}\right)^{\frac{m}{i}in}\\ &= P\left[\lim_{m\to\infty}\left(1+\frac{i}{m}\right)^{\frac{m}{i}}\right]^{in}\end{aligned}$$

Now we can make the change of variable $x = \dfrac{m}{i}$ to obtain the definition of e:

$$P\left[\lim_{m\to\infty}\left(1+\frac{i}{m}\right)^{\frac{m}{i}}\right]^{in} = P\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right]^{in} = Pe^{in}$$

We have proven the following result, which is the analog of Proposition 10.2.3 for continuous compounding:

Proposition 10.2.5. *If P dollars earns interest at a rate of i percent compounded continuously for n years, then it will grow to*

$$A = Pe^{in}.$$

In order to compare the effects of different combinations of rates of interest and frequencies of compounding, we need to introduce the effective rate of compounding. The *effective rate* of compounding is the interest rate which would give the same growth at annual compounding. The following example shows that the effective rate is obtained by determining how \$1 would grow if deposited for a year with the given frequency of compounding.

Example 10.2.6. Determine the effective rate of 8% compounded monthly.

According to Proposition 10.2.3, \$1 deposited at 8% compounded monthly for one year would increase to

$$A = (1+\frac{.08}{12})^{12} = 1.08300$$

Therefore, the effective rate of 8% compounded monthly is 8.3%. ■

Example 10.2.7. Compare the effective rates of daily compounding and of continuous compounding for the interest rate of 8%.

We need to compare the calculations obtained by the formulas in Proposition 10.2.3 and Proposition 10.2.5. For daily compounding, we get

$$A = \left(1 + \frac{.08}{365}\right)^{365} = 1.083277572$$

and for continuous compounding, we get

$$A = e^{.08} = 1.083287068.$$

Thus, continuous compounding is more effective, but only very slightly so.

A final way to view the effectiveness of a rate of interest is to determine the length of time required for a sum to double. We illustrate with an example.

Example 10.2.8. Determine how long it would take for a sum of money to double when it is deposited at 8% compounded monthly.

The size of the deposit is not relevant, so we will assume that it is \$1 and determine how long until it grows to \$2. From Proposition 10.2.3, we need to solve the following equation for n :

$$2 \quad = \quad 1\left(1 + \tfrac{.08}{12}\right)^{12n}$$

The key to the solution is to use logarithms:

$$\begin{aligned} 2 &= (1.0066667)^{12n} \\ \ln(2) &= 12n \ln(1.0066667) \\ n &= \tfrac{\ln(2)}{12 \ln(1.0066667)} \\ &= 8.6932. \quad \blacksquare \end{aligned}$$

So a deposit at 8% compounded monthly would double in about eight years and eight months.

We next derive an important summation formula for geometric sequences. It is the key to considering situations where an amount is deposited into an account each year and allowed to accumulate.

Proposition 10.2.9. *The sum of the first n terms of a geometric series is given by*

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}, \quad r \neq 1.$$

Proof. Let S denote the required sum, and consider the two equations

$$\begin{array}{lrcl} (1) & S & = & a + ar + ar^2 + \cdots + ar^{n-1} \\ (2) & rS & = & ar + ar^2 + ar^3 + \cdots + ar^n \end{array}$$

Subtracting equation (2) from equation (1) yields the equation

$$S - rS = a - ar^n$$

since all but one term in the right hand side of each equation drops out. Solving for S yields the result:

$$S = \frac{a(1 - r^n)}{1 - r}$$

since $r \neq 1$. □

A sequence of equal deposits made at regular intervals to an account – for instance a retirement account – is called an *annuity.* If, in addition, the frequency of the payments is equal to the frequency of the compounding, the annuity is called an *ordinary annuity.* The next proposition applies the formula from Proposition 10.2.9 to calculate the amount accumulated in an ordinary annuity.

Proposition 10.2.10. *If R dollars is deposited into an account paying i percent compounded annually at the end of each year for n years, then the total amount V accumulated will be*

$$V = R\left(\frac{(1+i)^n - 1}{i}\right).$$

Proof. The \$R deposited at the end of the first year accumulates for $n-1$ years to the end of the n year period, and thus at a rate of $i\%$ it will contribute $\$R(1+i)^{n-1}$ to the total. Similarly, the $\$R$ deposited at the end of year j will contribute $\$R(1+i)^{n-j}$ and the R deposited at the end of the nth year contributes only $\$R$ to the total. Thus, the total accumulation V can be written as the sum of n terms of a geometric sequence, and by Proposition 10.2.9, we get

$$\begin{aligned} V &= R + R(1+i) + R(1+i)^2 + \cdots + R(1+i)^{n-1} \\ &= \frac{R(1-(1+i)^n)}{1-(1+i)} \\ &= R\frac{(1+i)^n - 1}{i} \end{aligned}$$

since R is the first term and $1+i$ is the ratio of the geometric sequence. □

The quantity developed in the previous proposition is tabled in Table 10.2.3 following the exercises at the end of this section.

Example 10.2.11. How much can be saved in five years by depositing $1,000 into an account at the end of each year paying 7% interest compounded annually?

Using the previous proposition, the answer is

$$A = 1000 \cdot \left(\frac{(1.07)^5 - 1}{.07}\right) = \$5,570.74. \qquad \blacksquare$$

In the next example, we adjust Proposition 10.2.10 to the case where the compounding is not annual.

Example 10.2.12. Suppose that a person planning to retire at age 65 delays beginning his retirement savings until reaching the age of 50. He then deposits $350 at the end of each month into an account paying 8% a year compounded monthly. How much will he accumulate by the time he is 65?

To adjust for the more frequent monthly compounding, we multiply the number of years by 12 and divide the interest rate by 12. This gives

$$\begin{aligned} A &= 350 \cdot \left(\frac{(1+\frac{.08}{12})^{12\cdot 15} - 1}{\frac{.08}{12}}\right) \\ &= 350 \cdot \left(\frac{(1.006667)^{180} - 1}{.006667}\right) \\ &= \$121,113.42 \qquad \blacksquare \end{aligned}$$

On the TI-82, this calculation looks like

```
350*((1.00666667
)^180-1)/.006666
67
          121113.42
```

Two kinds of retirement plans are based on the idea of saving through an annuity.

The simpler is called a *defined contribution* plan in which the amount contributed is defined. The retirement benefit is then whatever income the accumulated sum will generate at the specified time. We will consider such a plan in the exercises at the end of this section.

A *defined benefit* plan contracts to provide a specified benefit at retirement. The calculations for such a plan are more complicated and will be delayed until the next section.

In the next example we consider the problem of determining the regular annuity payment needed to reach a particular savings goal.

Example 10.2.13. The Smiths want to save \$12,000 in six years for the down payment on a house. How much must they save at the end of each quarter if an account paying 6% compounded quarterly is available to them?

The calculation for this problem is similar to that of the previous problem except that in this case we must divide by the quantity calculated in Proposition 10.2.10. Letting R be the amount that must be saved each quarter, we adjust for the quarterly compounding by dividing the rate of interest by 4 and multiplying the number of years by 4. This yields

$$\begin{aligned} R\left(\frac{(1+\frac{.06}{4})^{4\cdot 6}-1}{\frac{.06}{4}}\right) &= 12,000 \\ R\left(\frac{1.015^{24}-1}{.015}\right) &= 12,000 \\ 28.63352R &= 12,000 \\ R &= \$419.09 \quad \blacksquare \end{aligned}$$

The preceding examples based on individuals and families using savings plans would seem to indicate that saving systematically is a desireable thing to do. However, many economists attribute difficulties with the United States economy to the low savings rates of Americans. The following newspaper article indicates the advantages of an early savings program for individuals, let alone for the national economy. Some exercises at the end of the section will be based on the estimates in this article.

SUNDAY, MARCH 12, 1995 **TRIBUNE-REVIEW**

Teaching teens about IRAs

By Vivian Marino
AP BUSINESS WRITER

NEW YORK (AP) — Have you had that serious talk with your teenage kids yet? You know, those three little letters — I-R-A.

Granted, telling teens to invest in an individual retirement account may be like suggesting they stay home and study during spring break. But parents and even grandparents can get the ball rolling by opening an account for them, perhaps as a graduation or birthday gift.

Parents can direct a child's wages into an IRA and make a gift to the child of an equal amount of cash.

That's how Jill Boese got started. The 23-year-old Hoboken, N.J. research assistant says her parents had $1,000 put into her first IRA when she was just 17. She contributes to it on her own each tax season, watching the tax-deferred income grow every year. (She also qualifies for a tax deduction.)

"I just happen to be a planner by nature, I guess," Boese said. "I'd like to retire comfortably and do it slowly over a lifetime."

Anyone with reported earned income can contribute up to $2,000 a year — even infants working as professional child models would qualify, provided a tax return is filed for them.

Money from weekly allowances or neighborhood chores doesn't count as earned income. The contributions for those who do qualify, though, can be as little as $100, or whatever a financial institution will accept as a minimum for opening an account.

While young IRA contributors will benefit from the account's tax-deferred status, or could qualify for a deduction that would reduce or negate their federal income taxes, the biggest draw lies in the compounding power.

The earlier you start, the bigger the war chest when it comes time to retire, said Norman Fosback. He is editor of a group of investment advisory services published by the Institute for Econometric Research in Fort Lauderdale, Fla.

"In my view, the biggest mistake that people commit in their investment program for retirement is that they don't start early enough," he said.

Fosback provides these scenarios to prove his point:

- An individual who starts an IRA at age 26 and puts away $2,000 each year for 40 years in an account with a 10 percent return, probably something containing stocks or stock funds, would end up with $973,704 by the time he or she turns 65. Total contributions: $80,000.

- A person starting an IRA at 19 who puts away $2,000 annually for seven years in an account with a 10 percent return would end up with $944,641 by age 65. Total contributions: $14,000.

- Someone starting at 14 who contributes $2,000 annually for five years in that same account would have $1.18 million by 65. Total contributions: $10,000.

Fosback also notes that a one-time $2,000 contribution at infancy would yield around $1 million by retirement age. He says even smaller annual contributions can grow substantially over time.

Some financial institutions support the creation of head-start IRA's for all children, regardless of their earned-income status, arguing that setting up these accounts early on would get individuals into the habit of saving more. However, it appears unlikely Congress will address that issue this year.

The main goal of the nearly two dozen IRA- and IRA-related bills introduced in the current Congress is to get more adults to save by loosening some of the current restrictions on IRA's

Julie Heim is a researcher for the monthly industry newsletter *IRA Reporter.* She says some smaller financial firms actually discourage working teens from opening
IRA's because it sometimes causes extra paperwork down the road.

"Their complaint is that they (the young clients) want to withdraw the money in a few years ... despite the huge penalties they face," for most withdrawals before age 59.5, she said "They see the money there and they want it now."

Reprinted from the **TRIBUNE-REVIEW** with permission from the Associated Press.

Previous examples assumed that contributions to a savings program were constant over an extended period of time. Usually this is not actually the case since one hopes that the ability to save increases as one advances in a career. The next example considers an approach to that case.

Example 10.2.14. Michelle, after having moved about for a few years following college graduation, has landed an excellent job she can envision holding for a lengthy period. Having read an article not unlike the preceeding one, she decides to embark on a retirement savings plan. She opens an account paying 7% annually, and determines that she can afford to save \$1,000 in the first year. Further, her expectation is that she can increase her savings by 3% each year. If she makes her deposits at the end of each year, how large a sum can she accumulate in 30 years?

We consider the amounts deposited each year and the number of years for which they earn interest to try to determine the pattern so that we can make the calculation:

- The \$1,000 deposited in the first year earns interest for 29 years, and thus accumulates to $\$1,000(1.07)^{29}$.
- Increasing by 3%, \$1,000(1.03) is deposited at the end of the second year. It accumulates interest over 28 years to $\$1,000(1.03)(1.07)^{28}$.
- At the end of the 30^{th} year, $\$1,000(1.03)^{29}$ is deposited, and it earns no interest.
- In general, $\$1,000(1.03)^{i-1}$ is deposited at the end of year i, and accumulates interest for $30-i$ years, yielding $\$1,000(1.03)^{i-1}(1.07)^{30-i}$.

When these terms are added up, we obtain:

$$\begin{aligned} A &= 1000(1.07)^{29} + 1000(1.03)(1.07)^{28} + 1000(1.03)^2(1.07)^{27} \\ &\qquad + \cdots + 1000(1.03)^j(1.07)^{29-j} + \cdots + 1000(1.03)^{29} \\ &= \sum_{j=0}^{29} 1000(1.03)^j(1.07)^{29-j} \end{aligned}$$

On the TI-82 with MODE set to 2 decimal places, this calculation looks like

```
sum seq(1000(1.0
3)^J*(1.07)^(29-
J),J,0,29,1)
        129624.81
```

■

In the calculation above, the sum operation is item 5 on the menu accessed by $\boxed{\text{2nd}}$ LIST MATH. The sum command is then applied to a sequence which is the last item in the menu found by $\boxed{\text{2nd}}$ LIST OPS. The last three entries in the seq(command – 0,29,1 – are the first , last, and step values of the index J. The TI-82 will only permit the summation of up to 99 terms.

Exercises

1. If \$10,000 is invested at 6% compounded quarterly for seven years, how much will it grow to? What would it grow to if the interest were compounded daily?

2. If \$1,000 is invested at 5% compounded monthly for 40 years, how much will it grow to? What would it grow to if the interest were compounded daily?

3. Suppose that the population of a city grows continuously from an initial population of 45,000 at 3% per year. What will the population of the city be after five years?

4. Check the estimate in the article by Vivian Marino that a single deposit of \$2,000 at infancy (use age 0) at 10% annual interest would yield around \$1 million by retirement age (use age 65).

5. How long would be required for a sum to double at 9% compounded quarterly?

6. How long would be required for a sum to triple at 10% compounded monthly?

7. Determine if the Fosbeck estimate for an IRA with \$2,000 deposited annually for seven years beginning at the end of a person's 19th year is high or low.

8. Suppose that the person in Example 10.2.11 had started to save 10 years earlier under the same terms.

(a) How much less per month would need to have saved to achieve the same goal at age 65?

(b) If he started to save \$350 ten years earlier, and continued to age 65, how much more would he have saved?

(c) How much must a person beginning at age 30 need to deposit each month into an account paying 6% interest compounded monthly to save \$1,000,000 by the age of 65?

9. Michelle in Example 10.2.14 increased her savings amount by 3% each year. What constant savings amount over the same time period at the same 7% rate would yield the same result?

10. How much of Michelle's total accumulation in Example 10.2.14 is interest?

11. Suppose that Michelle in Example 10.2.14 waits 10 years and initially saves \$1,500 at 7% and increases the amount by 4% each year. How much could she accumulate in 20 years?

12. The proud father of a new born daughter, Mike vows to immediately start a savings program to send his daughter to his alma mater. His research shows that the current tuition there is \$15,000 and is expected to increase at 4% per year. Assuming that his daughter will enter college at age 18 and his broker feels that 8% compounded quarterly is reasonable for planning purposes, how much must Mike save each quarter? (Hint: Consider the four years of college separately.)

13. Suppose that a grandparent deposits \$1,000 for a newborn grandchild at 6% compounded monthly. How much would be available to the child at age 65?

14. As a lifetime 2.68 hitter who has never come near a Golden Glove award or an All Star ballot, Shoeless Joe Stoneglove recognizes that he is probably overpaid at \$2.1 million a season. He decides that his career can last at best another five years, and resolves to tighten his belt and start to save. If he deposits \$100,000 each quarter into an ordinary annuity account at 9%, how much can he accumulate in five years?

15. If inflation is increasing consumer prices at 3.5% per year, how long until prices double?

16. This problem is Example 7.6.5 with an additional assumption that allows interest to be earned on unspent capital.

Consider the capital budgeting problem where five projects are being considered for completion over the next three years. The expected returns for each project and the yearly expenditures (in thousands of dollars) are tabled below.

Table 10.2.2

Project	Expenditures for Year 1	Year 2	Year 3	Returns
1	5	1	8	20
2	4	7	10	40
3	3	9	2	20
4	7	4	1	15
5	8	6	10	30
Maximum available funds	25	25	25	

Formulate an integer program to choose the projects that will maximize the return assuming that capital not spent in one year earns interest at 5% for a year and can then be used the next year.

17. In 1626 Peter Minuit, governor of the Dutch West India Company, bought Manhattan for $24 in trinkets. Suppose the Native Americans converted their trinkets to cash and invested it at annual interest. By the year 2000, what would be the advantage to 3% versus 2%?

18. The Olon Corporation has a defined contribution pension plan. Bob, an employee of the firm, retired this year. He has been employed by the firm for 10 years and the company has contributed $10,000 annually into Bob's pension fund since he was hired. If Bob takes a lump sum payment upon retirement and the interest rate is 12%, how much will Bob receive?

19. Another Olon employee, Eva, has been with the company for 16 years at retirement. For the first 5 years Olon paid $9,000 a year into her defined contribution retirement plan, and then $10,000 a year for the last 11 years. If the interest rate was 12%, how much accumulation does Eva have?

20. Edward, at age 28, has finally landed a steady, though low paying, job. He figures that he can manage to save $5 from each of his 26 equally spaced bi-weekly paychecks. He hopes to move on, or up, after no more than 5 years. If he follows this plan for five years and leaves the resulting amount on deposit at daily interest until he is 60, how much will he have, and how much will be interest earned? Assume that the savings plan available to Edward pays 6%.

Table 10.2.3

Amount of an Annuity with Regular Payment $1

No. Yrs.	6%	6.5%	7%	7.5%	8%	8.5%	9%	9.5%
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0600	2.0650	2.0700	2.0750	2.0800	2.0850	2.0900	2.0950
3	3.1836	3.1992	3.2149	3.2306	3.2464	3.2622	3.2781	3.2940
4	4.3746	4.4072	4.4399	4.4729	4.5061	4.5395	4.5731	4.6070
5	5.6371	5.6936	5.7507	5.8084	5.8666	5.9254	5.9847	6.0446
6	6.9753	7.0637	7.1533	7.2440	7.3359	7.4290	7.5233	7.6189
7	8.3938	8.5229	8.6540	8.7873	8.9228	9.06050	9.20043	9.34265
8	9.8975	10.0769	10.2598	10.4464	10.6366	10.8306	11.0285	11.2302
9	11.4913	11.7319	11.9780	12.2298	12.4876	12.7512	13.0210	13.2971
10	13.1808	13.4944	13.8164	14.1471	14.4866	14.8351	15.1929	15.5603
11	14.9716	15.3716	15.7836	16.2081	16.6455	17.0961	17.5603	18.0385
12	16.8699	17.3707	17.8885	18.4237	18.9771	19.5492	20.1407	20.7522
13	18.8821	19.4998	20.1406	20.8055	21.4953	22.2109	22.9534	23.7236
14	21.0151	21.7673	22.5505	23.3659	24.2149	25.0989	26.0192	26.9774
15	23.2760	24.1822	25.1290	26.1184	27.1521	28.2323	29.3609	30.5402
16	25.6725	26.7540	27.8881	29.0772	30.3243	31.6320	33.0034	34.4416
17	28.2129	29.4930	30.8402	32.2580	33.7502	35.3207	36.9737	38.7135
18	30.9057	32.4101	33.9990	35.6774	37.4502	39.3230	41.3013	43.3913
19	33.7600	35.5167	37.3790	39.3532	41.4463	43.6654	46.0185	48.5135
20	36.7856	38.8253	40.9955	43.3047	45.7620	48.3770	51.1601	54.1222
21	39.9927	42.3490	44.8652	47.5525	50.4229	53.4891	56.7645	60.2638
22	43.3923	46.1016	49.0057	52.1190	55.4568	59.0356	62.8733	66.9889
23	46.9958	50.0982	53.4361	57.0279	60.8933	65.0537	69.5319	74.3529
24	50.8156	54.3546	58.1767	62.3050	66.7648	71.5832	76.7898	82.4164
25	54.8645	58.8877	63.2490	67.9779	73.1059	78.6678	84.7009	91.2459
26	59.1564	63.7154	68.6765	74.0762	79.9544	86.3546	93.3240	100.9143
27	63.7058	68.8569	74.4838	80.6319	87.3508	94.6947	102.7231	111.5012
28	68.5281	74.3326	80.6977	87.6793	95.3388	103.7437	112.9682	123.0938
29	73.6398	80.1642	87.3465	95.2553	103.9659	113.5620	124.1353	135.7877
30	79.0582	86.3749	94.4608	103.3994	113.2832	124.2147	136.3075	149.6875
31	84.80168	92.98923	102.0730	112.1544	123.3459	135.7730	149.5752	164.9078
32	90.88978	100.0335	110.2182	121.5659	134.2135	148.3137	164.0370	181.5741
33	97.34316	107.5357	118.9334	131.6834	145.9506	161.9203	179.8003	199.8236
34	104.1838	115.5255	128.2588	142.5596	158.6267	176.6836	196.9823	219.8068
35	111.4348	124.0347	138.2369	154.2516	172.3168	192.7017	215.7108	241.6885
36	119.1209	133.0969	148.9135	166.8205	187.1021	210.0813	236.1247	265.6488
37	127.2681	142.7482	160.3374	180.3320	203.0703	228.9382	258.3759	291.8855
38	135.9042	153.0269	172.5610	194.8569	220.3159	249.3980	282.6298	320.6146
39	145.0585	163.9736	185.6403	210.4712	238.9412	271.5968	309.0665	352.0731
40	154.7620	175.6319	199.6351	227.2565	259.0565	295.6825	337.8824	386.5200
41	165.0477	188.0480	214.6096	245.3008	280.7810	321.8156	369.2919	424.2394
42	175.9505	201.2711	230.6322	264.6983	304.2435	350.1699	403.5281	465.5421
43	187.5076	215.3537	247.7765	285.5507	329.5830	380.9343	440.8457	510.7686
44	199.7580	230.3517	266.1209	307.9670	356.9496	414.3137	481.5218	560.2917
45	212.7435	246.3246	285.7493	332.0645	386.5056	450.5304	525.8587	614.5194
46	226.5081	263.3357	306.7518	357.9694	418.4261	489.8255	574.1860	673.8987
47	241.0986	281.4525	329.2244	385.8171	452.9002	532.4606	626.8628	738.9191
48	256.5645	300.7469	353.2701	415.7533	490.1322	578.7198	684.2804	810.1164
49	272.9584	321.2955	378.9990	447.9348	530.3427	628.9110	746.8656	888.0775
50	290.3359	343.1797	406.5289	482.5299	573.7702	683.3684	815.0836	973.4448

10.3 Present value

In most of the examples in the previous section, we have calculated the value of some investment in the future. In this section, we consider another calculation, called present value, which we will see is useful in comparing investments. We will also determine how to calculate the income that a sum will generate over a fixed period of years. The *present value* of an investment is the amount that could be deposited today to yield the future value of the investment.

Example 10.3.1. Rich Uncle Albert wishes to provide his nephew Clarence, now just six years old, with $50,000 when he turns 21. The most secure and convenient place for Uncle Albert to invest the money to provide for Clarence pays 6% annual interest. How much must Albert deposit?

Letting P be the amount of Albert's deposit, we see that

$$50,000 = (1.06)^{15} P$$

and therefore

$$P = \frac{50,000}{(1.06)^{15}} = \$20,863.25 \qquad \blacksquare$$

We say that $20,863.25 is the present value of $50,000 in 15 years at 6%. The quantity

$$(1.06)^{-15} = .417265$$

is called the present value of a $1 at 6% in 15 years.

One use of present value is to compare different sums of money at different times by calculating the present values of each.

Example 10.3.2. Jerry wants to retire from his tailor shop and move to Sun City. He has two offers for the shop from youngsters starting out in the business. But either will need to work for several years before they can pay Jerry. Carmen offers $44,000 after six years, Ellen offers $3,000 now and $35,000 after four years. If the prevailing rate of interest is 6%, which is the better offer?

We calculate the present value of each offer. For Carmen,

$$PV = \frac{\$44,000}{(1.06)^6} = \$31,018.26$$

and for Ellen,

$$\$3,000 + \frac{\$35,000}{(1.06)^4} = \$30,723.28.$$

so the offer by Carmen has a slightly larger present value. ■

The *present value of an annuity* is the sum that could be deposited at the current time that would yield the same amount as the sequence of regular annuity payments.

Proposition 10.3.3. *The present value of an annuity with regular payment R at i% for n years is*

$$PV = \left(\frac{1-(1+i)^{-n}}{i}\right) R.$$

Proof. We show that the amount PV will yield the same amount when deposited for n years at $i\%$ as the regular annual payment of R into an $i\%$ annuity account. Setting the two quantities equal yields the following equation:

$$PV(1+i)^n = \frac{(1+i)^n - 1}{i} R$$

When solved for PV this yields

$$PV = \frac{1-(1+i)^{-n}}{i} R.$$

which completes the proof. □

The quantity $\dfrac{1-(1+i)^{-n}}{i}$ is called the *present value* of the annuity. We now consider an example and at the same time indicate how to adjust for more frequent compounding.

Example 10.3.4. A man has an investment in municipal bonds that will pay \$250 in interest every quarter for five years. If he deposits the interest into an account paying 8% compounded quarterly, what is the present value of his planned investment of the interest?

To adjust for the quarterly compounding, we adjust the expression for the present value of an annuity by dividing the interest rate by 4, and multiplying the number of years by 4. Using Proposition 10.3.3, we then multiply the \$250 regular payment by the adjusted present value of the annuity:

$$\begin{aligned} PV &= \left(\frac{1-(1+\frac{.08}{4})^{-4\cdot 5}}{\frac{.08}{4}}\right)250 \\ &= \left(\frac{1-(1+.02)^{-20}}{.02}\right)250 \\ &= 16.3514\cdot 250 \\ &= \$4,087.86. \end{aligned}$$

So depositing \$4,087.86 at 8% compounded quarterly would produce the same result as investing the \$250 quarterly dividends.

In the previous section we discussed how to calculate the value of an annuity, and above we considered the present value of an annuity. Since a common and important purpose of annuities is to accumulate savings for retirement, the question of how much income a sum will generate over time is a key one. Stated more carefully, the question is:

If \$$S$ is deposited at i% annual interest, how much can be withdrawn at the end of each year if the income must last for n years?

Let R represent the amount withdrawn at the end of each year, beginning after the \$$S$ has been on deposit for one year. We consider the amount S_j left in the account at the end of year j to see if there is a pattern:

$$S_1 = S(1+i)-R$$

$$\begin{aligned} S_2 &= (S(1+i)-R)(1+i)-R \\ &= S(1+i)^2-((1+i)+1)R \end{aligned}$$

$$\begin{aligned} S_3 &= ((S(1+i)-R)(1+i)-R)(1+i)-R \\ &= S(1+i)^3-((1+i)^2+(1+i)+1)R \end{aligned}$$

We recognize the coefficient of R in the final expression of S_3 as the sum of the first three terms of a geometric series. The general formula after j

years is

$$S_j = S(1+i)^j - ((1+i)^{j-1} + \cdots + (1+i)^2 + (1+i) + 1)R$$

and we recognize the sum as the quantity calculated in the proof of Proposition 10.2.10. Thus,

$$S_j = S(1+i)^j - \left(\frac{(1+i)^j - 1}{i}\right) R.$$

Now to calculate the withdrawl R that is possible if the income is to last for n years, we set S_n equal to zero and solve for R :

$$S(1+i)^n - \left(\frac{(1+i)^n - 1}{i}\right) R = 0$$

Dividing by the coefficient of R yields

$$\begin{aligned} R &= \left(\frac{i}{(1+i)^n - 1}\right)(1+i)^n S \\ &= \left(\frac{i}{1-(1+i)^{-n}}\right) S \end{aligned}$$

We recognize the coefficient of S here as the reciprocal of the present value of an annuity. We have proven the following

Proposition 10.3.5. *The sum \$$S$ deposited at i% annual interest will generate \$$R$ of income at the end of each year for n years where*

$$R = \left(\frac{i}{1-(1+i)^{-n}}\right) S.$$

Example 10.3.6. Suppose that the person in Example 10.2.12 expects to need income for 15 years when he retires at 65. If he can deposit his savings of \$121,113.42 at 8% interest, what will his annual income be?

From the previous Proposition, his annual income R can be calculated from

$$\begin{aligned} R &= \left(\frac{.08}{1-(1.08)^{-15}}\right) \cdot 121,113.42 \\ &= (.1168) \cdot (121,113.42) \\ &= \$14,149.63 \qquad \blacksquare \end{aligned}$$

We now have the tools in hand to do the critical problem for a person evaluating his or her retirement planning.

Example 10.3.7. At age 35, Dana has saved only \$20,000 toward retirement, and wants to have a retirement income of \$60,000 a year beginning at age 65 and lasting for 20 years. After consulting a number of experts, she has decided to use 8% interest compounded monthly to accomodate her monthly deposits, and then 8% annual interest to draw out a year's income at a time during her retirement. How much must she deposit monthly during the remaining years to achieve her goal?

We first calculate the sum S that she will need on hand to generate the retirement income. Using Proposition 8.3.5, we need to solve the following equation

$$\begin{aligned} 60,000 &= \left(\frac{.08}{1-(1.08)^{-20}}\right)\cdot S \\ S &= \left(\frac{1-(1.08)^{-20}}{.08}\right)\cdot 60,000 \\ &= (9.8181)\cdot(60,000) \\ &= \$589,086.00 \end{aligned}$$

Next we need to determine how much of this sum her \$20,000 will account for if left to grow at 8% compounded monthly. Let's call this amount CP, and use Proposition 10.2.3:

$$CP = \left(1+\frac{.08}{12}\right)^{12\cdot 30} = (20,000)\cdot(10.9357) = \$218,714.59.$$

So the amount that she needs to make up is \$370,371.41 = 589,086.00 - 218,714.59. Using Proposition 10.2.10, we need to solve:

$$\begin{aligned} 370,371.41 &= \left(\frac{(1+\frac{.08}{12})^{12\cdot 30}-1}{\frac{.08}{12}}\right)\cdot R \\ R &= \left(\frac{\frac{.08}{12}}{(1+\frac{.08}{12})^{12\cdot 30}-1}\right)\cdot 370,371.41 \\ &= (.0006798)\cdot(370,371.41) \\ &= \$248.51. \end{aligned}$$

So the additional accumulation that Dana needs to reach her goal can be achieved by a contribution of \$248.51 per month for thirty years. ■

Exercises

1. Determine the present value of \$8,000 at 6% annual compounding for 10 years.

2. Determine the present value of \$5,000 at 8% compounded quarterly for 5 years.

3. What is the present value of an ordinary annuity with 9% quarterly payments of \$1,000 over 30 years?

4. What is the present value of an ordinary annuity with 8% monthly payments of \$500 over 20 years?

5. What is the present value of Mike's savings plan for his daughter's education in Exercise 12 at the end of the previous section?

6. What sum would be needed to generate \$15,000 in yearly income for 25 years if the sum can be deposited at 6%?

7. What yearly income would a lottery winning of \$1,000,000 generate for 45 years if it were deposited at 7%?

8. Suppose that \$275,000 is to be used to generate income for a 15 year period. Two investments are available – one at 7% and one at 6%. How much greater would the annual income be with the 7% investment?

9. If \$1,000,000 were deposited at 7% annual interest, for how long would it generate an annual income of \$75,000?

10. LVW, Inc. has a defined benefit pension plan. The firm wishes to contribute an amount to its pension fund so that each employee will receive an annual income of \$20,000 upon retirement. The company's consulting actuary estimates that the average employee will survive fifteen years after his or her retirement and will be employed 25 years prior to retirement. The firm expects to be able to obtain a return of 8% per year on contributions into the fund. How much should the company contribute per year, per employee, to the pension fund?

11. A fund raiser for a charity sold 3,500 chances at \$5 each for a drawing with \$1,000 a year for 20 years going to the winner. If he can invest the necessary funds at 7% annual interest, how much money did he raise with this drawing?

12. Suppose that Eva of Exercise 19 of the previous section decides to take her retirement in equal payments for the next 15 years. How much will her annual retirement income be if the amount remaining on deposit is assumed to earn interest at 11%?

10.4 Mortgages

In one of the examples in the previous section, the Smiths were saving to get the down payment for a house. Once they have their down payment, they will probably borrow to pay the balance. The process of paying off a loan is called *amortizing* it. A *mortgage* is an agreement to pay off a debt secured by a piece of property over a fixed period of time. Usually, the terms of a mortgage require a sequence of equal payments, and that is the case that we will consider here.

Table 10.4.2 at the end of the section is designed to calculate the monthly payment needed to amortize a loan when the interest rate and the amount of the loan in thousands of dollars is known. Because the table shows only two decimal places, payments calculated in this way are not exact.

The most common lengths of time for amortizing a home loan are 20, 25, and 30 years. However, in order to be able to conveniently observe the pattern of payments over the entire period of the mortgage, we will consider a shorter period of time in our first example.

Example 10.4.1. Determine the monthly payment required and the total interest paid on a loan of \$30,000 over 10 years at an interest rate of 8.5%.

From Table 10.4.2 at the end of the section, we obtain \$12.40 from the 8.5% column and the 10 year row as the payment per month for each \$1,000 borrowed. Therefore, the monthly payment is approximately

$$30 \cdot 12.40 = \$372.00.$$ [1]

To determine the total interest paid, we note that we are paying \$372 a month for 10 years, and subtract the loan amount:

$$10 \cdot 12 \cdot 372 - 30,000 = \$14,640.$$

So under these terms, the total interest is almost half the amount borrowed.
■

The payments for the first year of the mortgage shown in Table 10.4.1 reflect the fact that over the course of the mortgage the amount of principal repaid gradually increases and the amount of interest paid gradually declines.

[1] We shortly show that the actual payment is \$371.96.

An additional aspect, one that we will not discuss in any detail here, is the tax deduction for interest. A family with a 28% marginal tax rate could have saved $692.62 if eligible to itemize.

Table 10.4.1

For the calendar year 1996

Monthly payment: $371.96

Month	Beginning principal outstanding	Interest	Principal Repayment
January	$30,000.00	$212.50	$159.46
February	$29,840.54	$211.37	$160.59
March	$29,679.95	$210.23	$161.73
April	$29,518.22	$209.09	$162.87
May	$29,355.35	$207.93	$164.03
June	$29,191.32	$206.77	$165.19
July	$29,026.13	$205.60	$166.36
August	$28,859.77	$204.42	$167.54
September	$28,692.23	$203.24	$168.72
October	$28,523.51	$202.04	$169.92
November	$28,353.59	$200.84	$171.12
December	$28,182.47	$199.63	$172.33

Interest paid during 1996	=	$2,473.66
Principal repaid during 1996	=	$1,989.86
Principal outstanding at year end	=	$28,010.14

We can calculate the allocation of the first couple of month's payments to illustrate the process that leads to Table 10.4.1.

- Interest is paid on the entire $30,000 in the first month at one-twelfth the annual rate, so the interest is

$$\left(\frac{1}{12}\right) \cdot .085 \cdot 30,000 = \$212.50$$

  and thus the repaid principal is $159.46 =371.96 − 212.50.

- In the second month, interest is paid on the remaining balance of

$$\$29,840.54 = 30,000 - 159.46.$$

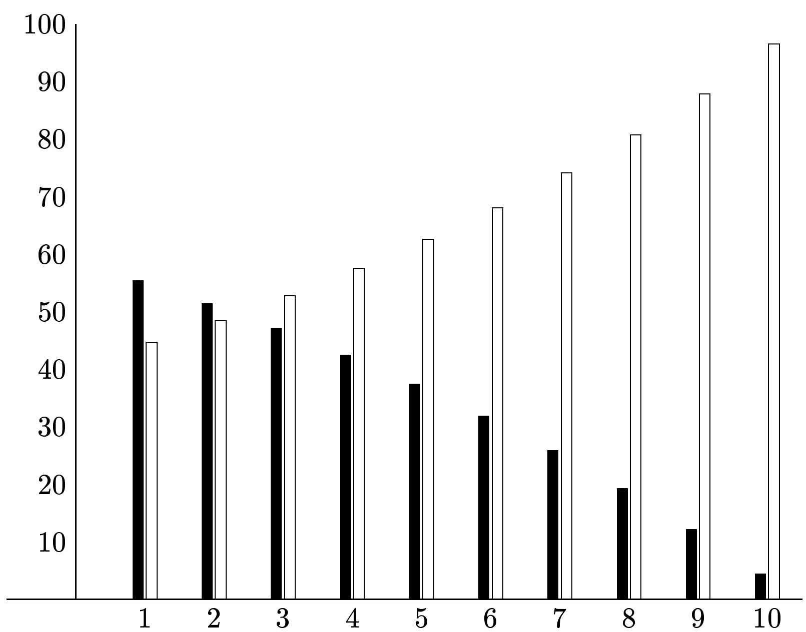
Thus, the interest is

$$\left(\frac{1}{12}\right) \cdot .085 \cdot 29,840.54 = \$211.37$$

and the repaid principal is $\$160.59 = 371.96 - 211.37$.

So over time we see a pattern of slowly decreasing interest and increasing principal. The following histogram displays the change in proportion of interest and principal repaid on a yearly basis over the course of the mortgage.

Percentage Principal and Interest by Year
$30,000 at 8.5% for 10 years.



Solid bar represents interest paid.
Open bar represents principal repaid.
Figure 10.4.1

Now we use the ideas of compound interest to establish how to compute the monthly payment for a mortgage.

Proposition 10.4.2. *The amount of the monthly payment R on a mortgage of \$$P_0$ at the annual rate of i% for n years is*

$$R = \frac{\frac{i}{12}P_0}{1-(1+\frac{i}{12})^{-12n}}.$$

Proof. Let R be the monthly payment, P_j be the principal remaining to be be paid after month j, and I_j the amount of interest paid during month j. Thus, P_0 is the full amount borrowed. For notational convenience, we let $k = \dfrac{1}{12}i$. Then, as in our sample monthly calculations for the example above, the principal is reduced by $R - I_j$ each month, so

$$P_j = P_{j-1} - (R - I_j) = P_{j-1} + I_j - R.$$

Now, since $I_j = kP_{j-1}$, we have

$$\begin{aligned} P_j &= P_{j-1} + kP_{j-1} - R \\ &= (1+k)P_{j-1} - R \end{aligned}$$

Looking at the beginning of the amortization we will see a pattern:

$$P_1 = (1+k)P_0 - R$$

$$\begin{aligned} P_2 &= (1+k)P_1 - R \\ &= (1+k)[(1+k)P_0 - R] - R \\ &= (1+k)^2P_0 - [(1+k)+1]R \end{aligned}$$

$$\begin{aligned} P_3 &= (1+k)P_2 - R \\ &= (1+k)[(1+k)^2P_0 - [(1+k)+1]R] - R \\ &= (1+k)^3P_0 - [(1+k)^2 + (1+k) + 1]R \end{aligned}$$

We recognize the coefficient of R as the sum of the first three terms in a geometric sequence in the final expression for P_3. The general formula for month j is

$$P_j = (1+k)^jP_0 - [1 + (1+k) + (1+k)^2 + \cdots + (1+k)^{j-1}]R$$

and we recognize the sum as the quantity calculated in the proof of Proposition 10.2.10. Thus,

$$P_j = (1+k)^j P_0 - R\left(\frac{(1+k)^j - 1}{k}\right).$$

Now, to calculate R, we consider the case of the last month, we'll call it month N, when the mortgage is paid off, i.e., when $P_N = 0.$:

$$0 = (1+k)^N P_0 - R\left(\frac{(1+k)^N - 1}{k}\right).$$

By solving this equation for R, we get

$$R = \frac{(1+k)^N P_0}{\frac{(1+k)^N - 1}{k}} = \frac{kP_0}{1-(1+k)^{-N}}.$$

Now putting $k = \dfrac{i}{12}$ in the expression for R in the equation above yields the formula for R. □

We now use the Proposition to recalculate the monthly payment gotten earlier from Table 10.4.2. Recall that the amount of the mortgage was \$30,000 and that the interest rate was 8.5% and the duration was 10 years. Then

$$\begin{aligned} R &= \frac{\left(\frac{.085}{12}\right)30,000}{1-(1+\frac{.085}{12})^{-120}} \\ &= \frac{(.07083)(30,000)}{1-(1.07083)^{-120}} \\ &= \$371.96 \end{aligned}$$

Example 10.4.3. A bank offers the Smiths a \$50,000 mortgage on their new home at 8% for either 20 or 30 years. If they can afford the larger monthly payments for the 20 year mortgage, how much interest can they save by taking the 20 year mortgage?

Let M and N denote the monthly payments on the 20 and 30 year mortgages, respectively. Then from Table 10.4.2, we have

$$M = 8.37 \cdot 50 = \$418.5 \text{ and } N = 7.34 \cdot 50 = \$367.00.$$

Following the previous example, the total interest associated with the two mortgages would be

$$\begin{aligned} \text{Interest over 20 years} &= 20 \cdot 12 \cdot 418.50 - 50,000 = \$50,440 \\ \text{Interest over 30 years} &= 30 \cdot 12 \cdot 367.00 - 50,000 = \$82,120 \end{aligned}$$

Thus, being able to manage an additional \$51.50 a month could save the Smiths \$31,680 in interest and also allow them to pay off their loan ten years sooner.

The above examples concerning mortgages show that each periodic payment includes a portion of the principal. In another type of repayment, only the interest is paid until some future time at which point the entire principal is paid. In order to be certain that the full amount of the principal is on hand at the agreed future time, periodic payments may be deposited into an account for the purpose, called a *sinking fund.*

Example 10.4.4. The Morrison's, in cleaning out their attic in preparation for a move to their retirement home, discover a Boudin painting of a beach scene left behind by an earlier owner of their home. Wanting to avoid publicity, they reach a private agreement to sell the Boudin to the Frick for \$800,000. They decide to defer receiving payment for four years rather than receive it immediately after the proceeds from the sale of their home. The museum agrees to pay them interest at 6% for the four years, and then pay them for the Boudin.

How much interest must the Frick pay the Morrisons each year, and how much must it deposit each year into a sinking fund paying 9% in order to have the \$800,000 on hand in four years?

The annual interest payment is 6% of \$800,000, or $.06 \cdot 800,000 = \$48,000$.

The payment into the sinking fund is a payment to an annuity that must accumulate to \$800,000 in four years. Thus, from Table 10.2.2, the payment P is obtained by solving

$$800,000 = 4.5731P$$

which yields $P = \$174,936.04$. ■

Table 10.4.2

Monthly Payments Required to Repay a $1,000 Loan

No. Yrs	**6%**	$6\frac{1}{2}$%	**7%**	$7\frac{1}{4}$%	$7\frac{1}{2}$%	$7\frac{3}{4}$%	**8%**	$8\frac{1}{4}$%	$8\frac{1}{2}$%	$8\frac{3}{4}$%	**9%**
5	19.34	19.57	19.81	19.92	20.04	20.16	20.28	20.40	20.52	20.64	20.76
6	16.58	16.81	17.05	17.17	17.30	17.42	17.54	17.66	17.78	17.91	18.03
7	14.61	14.85	15.10	15.22	15.34	15.47	15.59	15.72	15.84	15.97	16.09
8	13.15	13.39	13.64	13.76	13.89	14.01	14.14	14.27	14.40	14.53	14.65
9	12.01	12.26	12.51	12.64	12.77	12.89	13.02	13.15	13.28	13.42	13.54
10	11.11	11.36	11.62	11.75	11.88	12.01	12.14	12.27	12.40	12.54	12.67
11	10.37	10.63	10.89	11.02	11.15	11.29	11.42	11.56	11.69	11.83	11.97
12	9.76	10.02	10.29	10.42	10.56	10.69	10.83	10.97	11.11	11.25	11.39
13	9.25	9.52	9.79	9.92	10.06	10.20	10.34	10.48	10.62	10.75	10.90
14	8.82	9.09	9.36	9.50	9.64	9.78	9.92	10.06	10.20	10.35	10.49
15	8.44	8.72	8.99	9.13	9.28	9.42	9.56	9.71	9.85	10.00	10.15
16	8.12	8.40	8.68	8.82	8.96	9.11	9.25	9.40	9.55	9.70	9.85
17	7.84	8.12	8.40	8.55	8.69	8.84	8.99	9.14	9.29	9.44	9.59
18	7.59	7.87	8.16	8.31	8.45	8.60	8.75	8.91	9.06	9.21	9.37
19	7.37	7.65	7.95	8.10	8.25	8.40	8.55	8.70	8.86	9.02	9.17
20	7.17	7.46	7.76	7.91	8.06	8.21	8.37	8.53	8.68	8.84	9.00
25	6.45	6.76	7.07	7.23	7.39	7.56	7.72	7.89	8.06	8.23	8.40
30	6.00	6.33	6.66	6.83	7.00	7.17	7.34	7.52	7.69	7.87	8.05

To use the table, multiply the loan amount in $1,000 units by the table entry for the number of years of the loan and the percentage rate of the loan.

Exercises

1. A $12,000 loan on a new car is to be amortized over 5 years at an annual interest rate of 8.75%. How much will the monthly payments be?

2. A borrower has been offered home mortgages of 20 or 25 years on a $60,000 loan at 9% interest. The borrower is considering whether he can commit the extra money to increase the down payment to reduce the amount borrowed so that the monthly payments on the 20 year mortgage would be the same as for the 25 year mortgage. How much additional down payment will he need?

3. Suppose that the Smiths of Example 10.4.3 find a second bank which will give them a 20 year mortgage at 7.75% but will charge $500 more than the other bank to initiate the loan. Will the difference in rate make it worth the extra \$500 to borrow from the second bank?

4. A company purchases a new computer system for \$8,000 and will pay it off in four annual payments at 10% annual interest. Prepare an amortization schedule for this purchase showing the allocation of each payment to principal and interest.

5. Should Farmer Brown borrow? Recall Farmer Brown's problem from Exmple 1.3.1:

Farmer Brown is planning his planting for the coming year. He expects to raise potatoes and wheat. He has 100 acres of land available, and will be able to devote 160 days of labor to his crops. He expects an acre of wheat to require four days of labor while an acre of potatoes requires only one day. He has \$1,100 that he can use for the start-up costs of planting and cultivating. It costs \$10 an acre to plant and cultivate potatoes, while the corresponding costs for an acre of wheat are \$20. He expects a revenue of \$40 per acre for potatoes and \$120 an acre for wheat.

As he is planning, he receives an offer of a loan for up to \$1,500 at 5% for one year. This possibility adds two new issues to his planning: should he borrow, and if so, how should he use the money? He finds that he could rent land from his neighbor Farmer Jones for \$20 an acre and that he could hire an additional farm hand for \$25 a day. If he reduces his revenue by the amount needed to repay the loan, how many acres of each should he plant in order to achieve the maximum possible revenue, how much should he borrow, and how should he use the borrowed money?

6. Voracious Consumer has run up a \$7,500 balance on his National Credit Card (NCC) which charges 18% annual interest. Voracious resolves to reform. He decides to stop using his NCC and pay off the balance in two years. What will Voracious need to pay each month, and what will his total interest charge be?

7. How much would Voracious of the previous exercise save in interest if he could pay off his NCC balance in 1 year?

10.5 Summary and objectives

In this chapter we have considered basic questions surrounding compound interest. The basis of all the calculations was found in geometric sequences. The key formula is the sum of the first n terms:

Proposition 10.5.1. *The sum of the first n terms of a geometric series is* *given by*

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}, \quad r \neq 1.$$

With the first term a chosen to be the amount P on deposit, then the amount A that that deposit would grow to in n years at i percent compounded m times yearly is a term in a geometric sequence:

$$A = P(1 + \frac{i}{m})^{mn}.$$

If instead of a single deposit, an annual deposit of R is made at i percent annual interest, then the amount V that accumulates in n years is given by the sum of a geometric sequence:

$$V = R\left(\frac{(1+i)^n - 1}{i}\right).$$

Once one has accumulated a sum S, the next question to ask is how much income R can be generated for a period of n years if the accumulation can be deposited at i percent interest. The answer again is derived using a geometric sequence and is

$$R = \left(\frac{i}{1-(1+i)^{-n}}\right) S.$$

The other basic application we considered was the calculation of the amount needed to pay off a mortgage:

Proposition 10.5.2. *The amount of the monthly payment R on a mortgage of \$$P_0$ at the annual rate of i% for n years is*

$$R = \frac{\frac{i}{12}P_0}{1-(1+\frac{i}{12})^{-12n}}.$$

Objectives

After studying this chapter, the student should be able to do the following calculations. Illustrative examples and exercises are suggested for each calculation.

1. Use an infinite geometric series to determine the impact of additional spending on an economy. Example 10.1.1. Exercise 4 of Section 10.1.

2. Determine the amount resulting from a single deposit at compound interest. Example 10.2.4. Exercises 2 and 5 of Section 10.2.

3. Determine how long it will take for a deposit at compound interest to double. Example 10.2.8. Exercise 5 of Section 10.2.

4. Determine the amount resulting from annual deposits for a period of years at annual compound interest. Example 10.2.11. Exercise 7 of Section 10.2.

5. Determine the amount resulting from annual deposits for a period of years at interest compounded several times a year. Example 10.2.13. Exercise 14 of Section 10.2.

6. Determine the present value of a one time deposit left to gain interest for a fixed time. Example 10.3.1. Exercise 2 of Section 10.3.

7. Determine the present value of an annuity. Example 10.3.4. Exercise 4 of Section 10.3.

8. Determine the annual income generated from a deposit at interest for a period of years. Example 10.3.6. Exercise 7 of Section 10.3.

9. Determine the monthly payment needed to pay off a mortgage. Example 10.4.1. Exercise 1 of Section 10.4.

Appendix A

Selected Answers to Compound Interest Exercises

Section 10.1

1. $2000(1.08)^4 = 2,720.98$.

3(a). The sum is $\dfrac{4}{1-\frac{1}{2}} = 8$.

3(c). The ratio is $\dfrac{3}{2}$ which is greater than 1 so the series diverges.

5. $40 + 40 \cdot \left(\dfrac{3}{5}\right) + 40 \cdot \left(\dfrac{3}{5}\right)^2 + 40 \cdot \left(\dfrac{3}{5}\right)^3 = 87.04$.

7. $5 + 2 \cdot \dfrac{2}{3} \cdot 5 \cdot (1 + \dfrac{2}{3} + \cdots + \left(\dfrac{2}{3}\right)^n + \cdots) = 25$.

Section 10.2

1. $(1 + \frac{.06}{4})^{4 \cdot 7} \cdot 10,000 = \$15,172.22$. Daily compounding, \$15,219.11.

3. $52,167\frac{1}{3}$.

5. 7.787957428 years.

7. It seems a little high. The 7 year accumulation is calculated from the amount of an annuity table is \$18,974.34. Then compounding continuously for 39 years at 10% yields $18,974.34e^{.1 \cdot 39} = 937,378.87$

9. Using the entry for 30 years and 7% from the amount of an annuity table, \$1372.26.

11. \$83,928.07.

12. Assuming 19 years until the Sophomore year, tuition will be $(1.04)^{19} \cdot 15,000 = \$31,602.73764$. The quarterly deposit required at an annual rate of 8% is \$180.373.

13. $48,924.70.
15. 20.14879168 years.
17. $1,478,707.88.
19. $405,434.11.
20. $13,797.56 will be interest.

Section 10.3

1. $4,467.16.
3. $41,366.79.
5. $7,016.36 is the present value for the Sophomore year.
7. $75,009.14.
9. 40.025 years.
11. $6,905.98.
13. $56,381.79.

Section 10.4

1. $247.68.
3. The amount of interest on the 20 year mortgage at 7.75% will be $48,520, so the Smiths can afford the extra $500 loan initiation fee.
5. From Proposition 10.4.2 it will cost $1.027289707 to repay each dollar borrowed. Letting `b` be the amount borrowed, solving the following model shows that by borrowing $100 and putting it all into capital, i.e., `bc = 100`, Brown can plant 80 acres of potatoes and 20 acres of wheat and raise his revenue by $97.21.

```
max 40 x1 + 120 x2 - 1.027289707 b
st
x1 + x2 - ra < 100
x1 + 4 x2 - hh < 160
10 x1 + 20 x2 - bc < 1100
20 ra + 25 hh + bc - b = 0
b < 1500
end
```

7. $735.12.

Index