

Summations

Question? What is the sum of the first 100 positive integers?

Counting

Question? In how many ways may the first three horses in a 10 horse race finish?

Multiplication Principle: If an event can occur in m ways, and a second event can occur independently in n ways, then the two events can occur in mn ways.

Question? Where is the independence in the horse race example?

Recursive reasoning warmup

A **recursive function** is a function that calls itself during its execution.

Example. Factorials

$$n! = \begin{cases} 0! = 1 \\ n! = n(n-1)! \end{cases}$$

Example. Fibonacci's rabbits satisfy

- ✳ Begin with one pair of immature rabbits,
- ✳ In the second month the rabbits are mature and produce a new pair of rabbits, and
- ✳ Each pair continues to produce a new pair of rabbits each month forever.

Fibonacci

(c. 1170–c. 1250)



Question? How many pairs of rabbits does Fibonacci have after n months?

Reference: Leonardo of Pisa, *Liber abbaci*, 1202.

Example. Call a string of symbols a valid arithmetic expression if

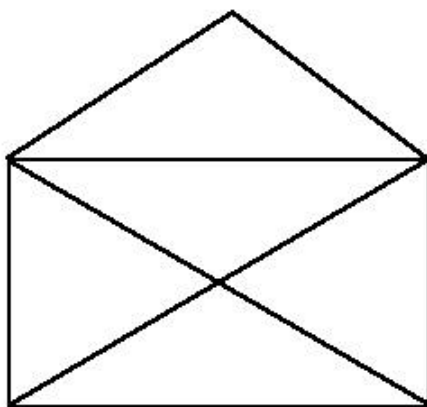
- * It uses only the digits 0, 1, 2, ... , 9, and the operators +, -, /, and *,
- * It begins and ends with a digit, and
- * It does not have two consecutive operators.

Question? How many such valid arithmetic expressions of length n are there?

Addition principle: If the things to be counted are separated into distinct cases, the total number is the sum of the numbers in the various cases.

A slight diversion

Question? Can you draw this without lifting your pencil from the paper? Can you do so covering each line segment once and return to the starting point?



First method of proof

Theorem 2.2.2. (The Principal of Mathematical Induction)

Let $P(n)$ be a statement about the positive integers. If one can prove that

- $P(1)$ is true, and that
- For every positive integer m , whenever $P(1), P(2), \dots, P(m)$ is true, then it follows that $P(m + 1)$ is true,

then $P(n)$ is true for every positive integer n .

Example. Using mathematical induction, prove that

$$1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}.$$

Example. Prove that the sum of the cubes of three successive positive integers is divisible by 9.

Question? In how many ways can n people be seated around a round table?

Definition: A **permutation** is an ordered arrangement of distinct objects.

Proposition 3.2.14. The number of permutations of n things taken r at a time is

$$P(n, r) = \frac{n!}{(n - r)!}.$$

Addition principle: If the things to be counted are separated into distinct cases, the total number is the sum of the numbers in the various cases.

Recall that we added the two cases in the arithmetic expressions.

Question? How many even four digit numbers with distinct digits can be formed from $\{0, 1, 2, 3, 4, 5, 6\}$?

Definition: A **combination** is an unordered arrangement of objects.

Proposition 3.2.18. The number of combinations of n things taken r at a time is

$$C(n, r) = \frac{n!}{r!(n - r)!}$$

A **probability experiment** is an experiment that can be carried out under repeatable conditions with a set of outcomes that are equally likely to occur. The **sample space** of the experiment is the set of all possible outcomes, and a subset of the sample space is called an **event**.

If $n(D)$ denotes the number of elements in a set D , then the **probability** of an event E is

$$p(E) = \frac{n(E)}{n(S)}$$

where S is the sample space of the experiment.

Question? An urn contains four red and three blue balls. An experiment consists of randomly selecting four balls. What is the probability that two red and two blue balls are selected?

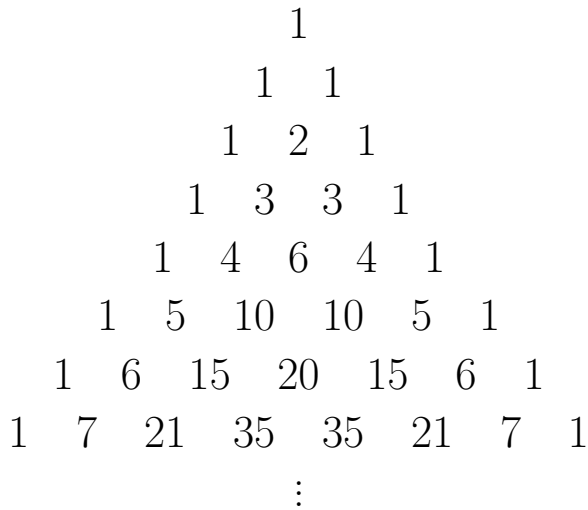
Question? How many full houses are there in poker?

Question? How many straights are there in poker?

Fundamental Theorem of Arithmetic. Every integer $n > 1$ is expressible as $n = p_1 p_2 p_3 \cdots p_r$ where r is a positive integer and each p_i is a prime.

Question? How many positive factors does 1,361,367 have?

Pascal's Triangle:



Blaise Pascal (1623-1662)

Source: Wikimedia Commons

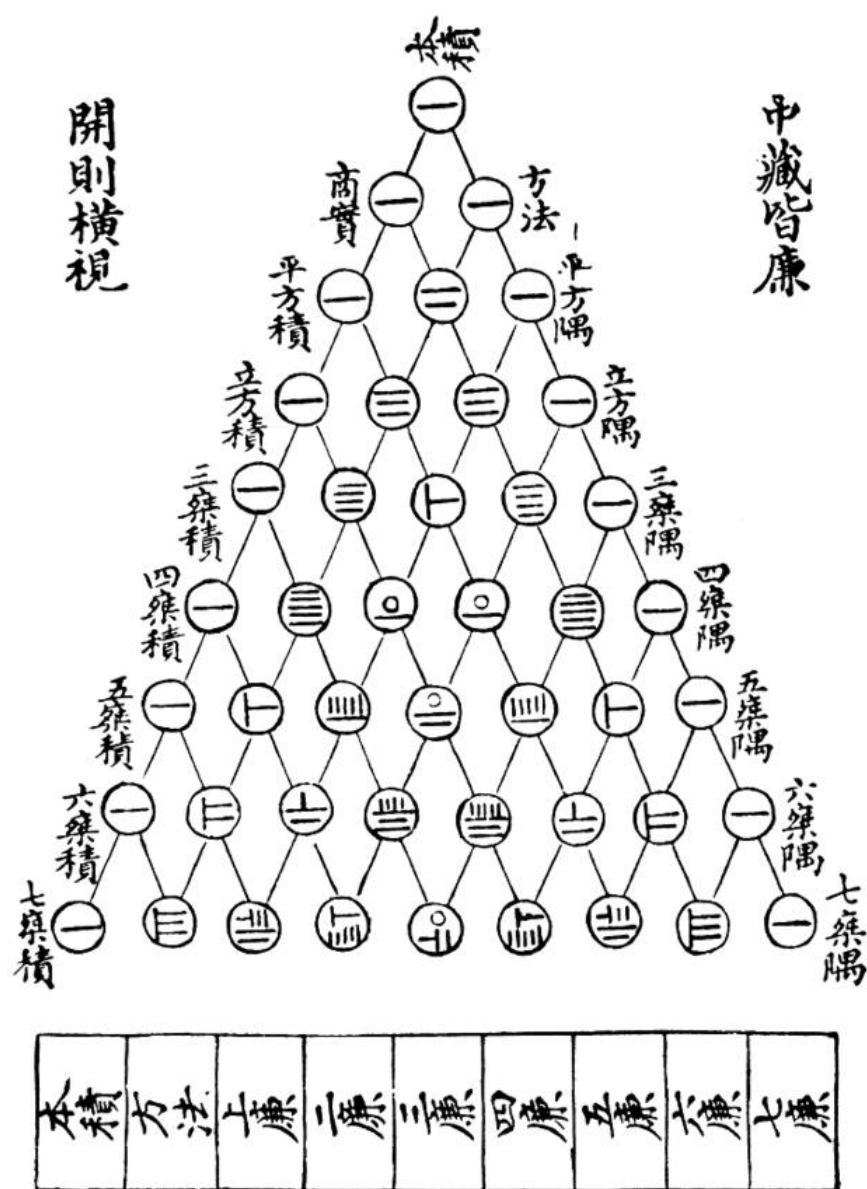
The key identity underlying Pascal's Triangle:

Proposition. $C(n+1, r) = C(n, r) + C(n, r-1)$.

We consider two proofs: One by calculation and a combinatorial proof. A **combinatorial proof** is one accomplished by counting the same set in two different ways to establish an equality.

An earlier source is the Yáng Huī (ca. 1238-1298) triangle below. So in China it is known as the Yáng Huī Triangle.

古 法 七 乘 方 圖



Vandermonde's Identity. Let m, n , and r be nonnegative integers with r not exceeding either m or n . Then

$$\begin{aligned} C(m+n, r) &= \sum_{k=0}^r C(m, r-k)C(n, k) \\ &= C(m, r)C(n, 0) + C(m, r-1)C(n, 1) \\ &\quad + \cdots + C(m, 0)C(n, r) \end{aligned}$$

To illustrate this identity, the **21** in row seven of the triangle below is the sum of the products of the entries in rows three and four having the same color:

$$\begin{array}{ccccccc} & & & & 1 & & \\ & & & & 1 & & 1 \\ & & & 1 & 2 & 1 & \\ & & \textcolor{blue}{1} & \textcolor{red}{3} & \textcolor{green}{3} & 1 & \\ & \textcolor{green}{1} & \textcolor{red}{4} & \textcolor{blue}{6} & 4 & 1 & \\ & 1 & 5 & 10 & 10 & 5 & 1 \\ & 1 & 6 & 15 & 20 & 15 & 6 & 1 \\ & 1 & 7 & \textcolor{orange}{21} & 35 & 35 & 21 & 7 & 1 \\ & & & & \vdots & & & & \end{array}$$

$$\textcolor{orange}{C(7, 2)} = \textcolor{blue}{C(4, 2)} \cdot \textcolor{blue}{C(3, 0)} + \textcolor{red}{C(4, 1)} \cdot \textcolor{red}{C(3, 1)} + \textcolor{green}{C(4, 0)} \cdot \textcolor{green}{C(3, 2)}$$

$$\textcolor{orange}{21} = \textcolor{blue}{6} \cdot \textcolor{blue}{1} + \textcolor{red}{4} \cdot \textcolor{red}{3} + \textcolor{green}{1} \cdot \textcolor{green}{3}$$

The Binomial Theorem. The coefficient of the $a^{n-k}b^k$ term in the expansion of $(a+b)^n$ is $C(n, k)$, i.e.

$$\begin{aligned}(a+b)^n &= \sum_{k=0}^n C(n, k)a^{n-k}b^k \\&= a^n + C(n, 1)a^{n-1}b + C(n, 2)a^{n-2}b^2 + C(n, 3)a^{n-3}b^3 + \cdots + b^n \\&= a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 \\&\quad + \cdots + b^n\end{aligned}$$

To expand $(a+b)^n$:

- The first term is a^n
- Each successive term is gotten from the previous one by
 - Decreasing the exponent on a by 1,
 - Multiplying by the old exponent on a ,
 - Increasing the exponent on b by 1, and
 - Dividing by the new exponent on b .

Example. Expand $(2-x)^6$.

Proposition. A set with n elements has **?** subsets.

Question? Suppose that a baseball player has a batting average of .300 and comes to the plate five times in a game. What is the probability that he gets three hits? Or that he gets at least three hits?

Example. Prove that for any integer $n \geq 0$ that

$$11^{n+2} + 12^{2n+1}$$

is divisible by 133.

One-to-one correspondences

A function $f : A \rightarrow B$ is **one-to-one** if $f(x) = f(y)$ implies that $x = y$.

A function $f : A \rightarrow B$ is **onto** if for any element b in B there is an element a in A such that $f(a) = b$.

A function that is both one-to-one and onto is called a **one-to-one correspondence**. Sets A and B with a one-to-one correspondence between them are said to be in one-to-one correspondence, or to have the same number of elements, or to have the same cardinality.

Question? How many four element subsets of

$$\{1, 2, 3, \dots, 20\}$$

are there?

Question? How many integer solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 16$$

with each x_i non-negative?

Proposition. The solutions to the two examples above are the same.

A **binary sequence** of length n is a sequence of n 0's and 1's.

Proposition. There are $C(n, r) = C(n, n - r)$ binary sequences of length n containing exactly r 1's.

Proposition. The number of solutions in non-negative integers to

$$x_1 + x_2 + \cdots + x_n = r$$

is $C(n + r - 1, r) = C(n + r - 1, n - 1)$.

Question? In how many ways may 75 shares of stock be distributed among 10 men and 10 women in such a way that each woman receives at least two shares and each man at least one share?

Now that we have conquered counting solutions involving non-negative integers, how about the case of positive integers?

Question? How many solutions in positive integers are there to the equation $x_1 + x_2 + x_3 + x_4 = 12$?

Now what about putting an upper bound on the positive variables, say 6?

Question? In how many ways may the sum of three dice add up to 14?

Question? What is the probability of getting a sum of 11 in a toss of three fair dice?

In general, a **generating function** is a polynomial such that the coefficient of x^r is the answer to a counting problem involving r .

We have seen one generating function in that we “generated” $C(n, r)$ as the coefficient of x^r in the expansion of $(1 + x)^n$. For instance,

$$(1 + x)^7 = x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1$$

Example. Carry out the multiplication

$$(1 + x + x^2)(1 + x + x^3 + x^5).$$

Question? How many ways are there to make change for 25 cents using pennies, nickels and dimes?

Question? How many terms are there in the expansion of $(a + b + c)^6$?

Back to Pascal's Triangle:

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Recall the identity that governs Pascal's Triangle:

$$C(n+1, r) = C(n, r) + C(n, r-1).$$

Example 3.3 no. 11. Compute exact numerical answers for the first two parts and then answer the third.

1. How many binary sequences of length at most 5 have exactly 3 1's?
2. How many binary sequences of length 6 have exactly 4 1's?
3. Why do the first two questions have the same answer?

Question? (Sec. 3.4 no. 3) What entry in Row 7 of Pascal's Triangle is the same as

$$C(5, 2)C(2, 0) + C(5, 1)C(2, 1) + C(5, 0)C(2, 2)?$$

Hint: Apply the Binomial Theorem to

$$(1 + x)^2(1 + x)^5 = (1 + x)^7.$$

Derangements

A **derangement** of a set of items with a natural order is a permutation of them such that none of the items is in its natural position.

Question? How many derangements of a set of n items are there?

The number D_n of derangements of n objects can be calculated recursively. But determining the recursive formula is tricky, so here it is in full.

First, a list consisting of only one item cannot be re-arranged, so $D_1 = 0$. Second, $D_2 = 1$ since the only way to get two items completely out of order is to transpose them.

Now consider the number D_n of derangements of the set of the first n positive integers: $\{1, 2, 3, \dots, n\}$.

The number 1 can go to any of $n - 1$ places, so we can break the set of derangements into $n - 1$ subsets depending on the location of 1 and calculate the number x of derangements in each of these subsets.

So we now know that $D_n = (n - 1)x$, and we need to calculate x .

Suppose that 1 goes to position k . There now are two possibilities, so we will use the Addition Principle.

The possibilities are that k goes to position 1, or it doesn't.

Case 1: k goes to position 1. Then the other $n - 2$ integers are deranged among themselves and this can be done in D_{n-2} ways.

Case 2: k does not go to position 1. With the position of 1 known, each of the other integers has one restriction on where it can move: Each integer i other than k cannot go to position i and integer k cannot go to position 1. This is equivalent to deranging the $n - 1$ integers other than 1 and can be done in D_{n-1} ways.

Hence, $x = D_{n-1} + D_{n-2}$ and $D_n = (n - 1)(D_{n-1} + D_{n-2})$.

Some sources use the notation $!n$ for the number of derangements. Then here is the formula that we derived:

$$!n = D_n = \begin{cases} 0 & \text{if } n = 1, \\ 1 & \text{if } n = 2, \text{ and} \\ (n - 1)(D_{n-1} + D_{n-2}) & \text{if } n > 2 \end{cases}$$

Question? When the power goes out in a restaurant 10 diners select their coats from the check room at random. What is the probability that none of them select their own coats?

The derangements of $\{1, 2, 3, 4\}$ grouped by the location of 1:

2	1	4	3
3	1	4	2
4	1	2	3
3	4	1	2
2	4	1	3
4	3	1	2
4	3	2	1
2	3	4	1
3	4	2	1

Note that for each position of 1 there is $D_2 = 1$ derangement – colored **red** – where 1 exchanges place with the integer whose position it occupies and $D_3 = 2$ derangements where the integer in whose place 1 is is not in first place.

Here are the derangements of $\{1, 2, 3, 4, 5\}$ where 1 goes to position 2:

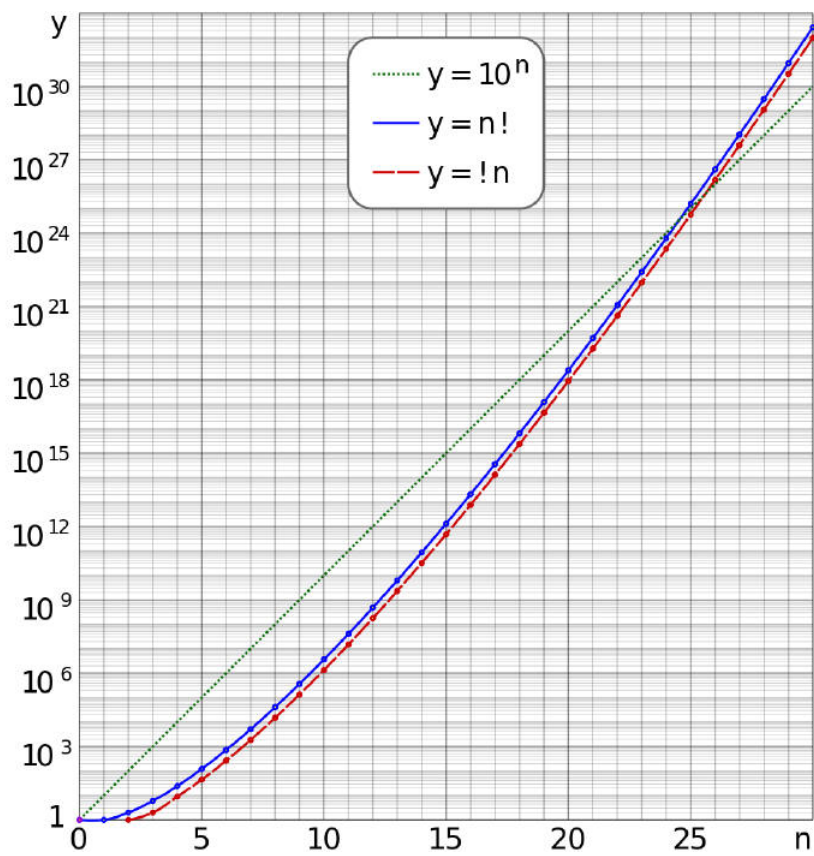
There are $D_3 = 2$ where 1 and 2 change places:

2	1	4	5	3
2	1	5	3	4

There are $D_4 = 9$ where 1 and 2 do not change places but 1 is in second place:

4	1	2	5	3
4	1	5	3	2
4	1	5	2	3
3	1	2	5	4
3	1	4	5	2
3	1	5	2	4
5	1	2	3	4
5	1	4	2	3
5	1	4	3	2

There is an interesting relationship between factorials and derangements. Note that the plot below uses $!n$ for the number of derangements.



Source: Wikimedia Commons

It turns out that $n!$ has a recursion very similar to D_n :

$$n! = \begin{cases} 1 & \text{if } n = 1, \\ 2 & \text{if } n = 2, \text{ and} \\ (n-1)((n-1)! + (n-2)!) & \text{if } n > 2 \end{cases}$$

The Excel spreadsheet below shows that the ratio of the number of derangements to the number of permutations approaches $\frac{1}{e}$.

	A	B	C	D	E	F
1						
2			Derangements vs. Factorials			
3						
4		n	D_n	n!	D_n/n!	
5		1	0	1	0.0000000000000000	
6		2	1	2	0.5000000000000000	
7		3	2	6	0.3333333333333330	
8		4	9	24	0.3750000000000000	
9		5	44	120	0.3666666666666670	
10		6	265	720	0.3680555555555560	
11		7	1,854	5,040	0.3678571428571430	
12		8	14,833	40,320	0.3678819444444440	
13		9	133,496	362,880	0.3678791887125220	
14		10	1,334,961	3,628,800	0.3678794642857140	
15		11	14,684,570	39,916,800	0.3678794392336060	
16		12	176,214,841	479,001,600	0.3678794413212820	
17		13	2,290,792,932	6,227,020,800	0.3678794411606910	
18		14	32,071,101,049	87,178,291,200	0.3678794411721620	
19		15	481,066,515,734	1,307,674,368,000	0.3678794411713970	
20		16	7,697,064,251,745	20,922,789,888,000	0.3678794411714450	
21		17	130,850,092,279,664	355,687,428,096,000	0.3678794411714420	
22		18	2,355,301,661,033,950	6,402,373,705,728,000	0.3678794411714420	
23		19	44,750,731,559,645,100	121,645,100,408,832,000	0.3678794411714420	
24		20	895,014,631,192,902,000	2,432,902,008,176,640,000	0.3678794411714420	
25		21	18,795,307,255,050,900,000	51,090,942,171,709,400,000	0.3678794411714420	
26		22	413,496,759,611,121,000,000	1,124,000,727,777,610,000,000	0.3678794411714420	
27		23	9,510,425,471,055,780,000,000	25,852,016,738,885,000,000,000	0.3678794411714420	
28		24	228,250,211,305,339,000,000,000	620,448,401,733,239,000,000,000	0.3678794411714420	
29		25	5,706,255,282,633,470,000,000,000	15,511,210,043,331,000,000,000,000	0.3678794411714420	
30						
31						
32					v	
33						
34					1/e	
35						

Closed forms for recursions

A **closed form** expression for a recursive sequence is one that allows the evaluation of a term without referring to prior terms.

The next proposition suggests how to obtain a closed form for some recursions.

Proposition. If s_n and t_n are both solutions of a recurrence $u_n = au_{n-1} + bu_{n-2}$ and c and d are any constants, then $cs_n + dt_n$ is also a solution.

The expression $cs_n + dt_n$ is for constants c and d is called a **linear combination** of s_n and t_n .

Example. We use the proposition above and the assumption that $F_n = t^n$ to determine the closed form expression for the Fibonacci recursion

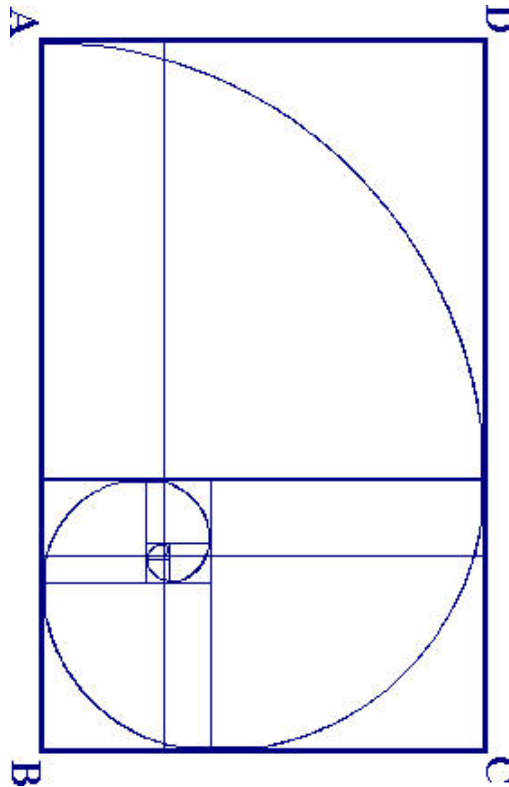
$$F_0 = 1, F_1 = 1, F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.$$

Fibonacci numbers and Phi

The nautilus, the logarithmic spiral, and additive squares



Source:Chris 73 / Wikimedia Commons

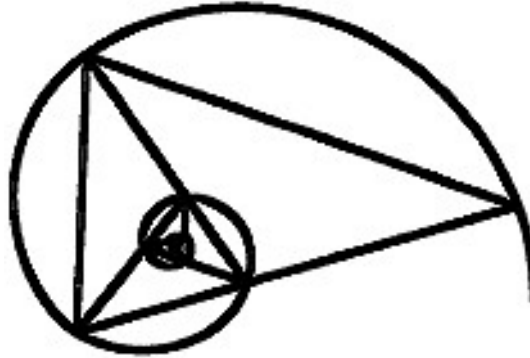
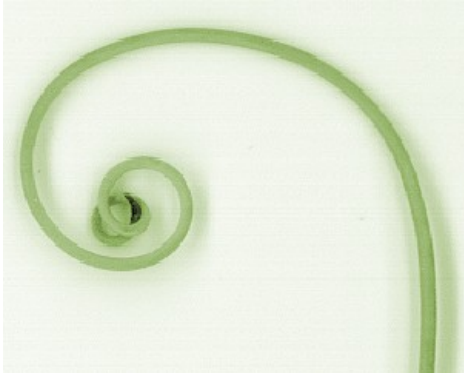


Source:dgleahy.com/dgl/p14.html

By using the closed form expression for F_n , one can show that the quotient

$$\frac{F_{n+1}}{F_n} \rightarrow \phi = \frac{1 + \sqrt{5}}{2} = 1.61803... \text{ as } n \rightarrow \infty$$

Pea tendrils, additive triangles, and the logarithmic spiral

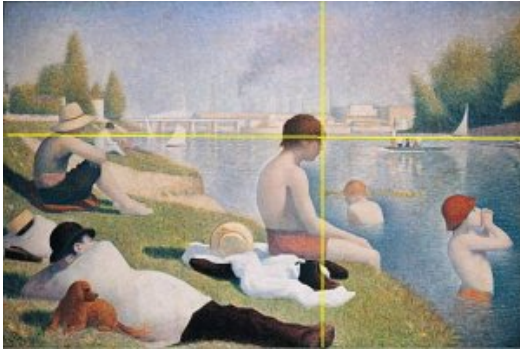


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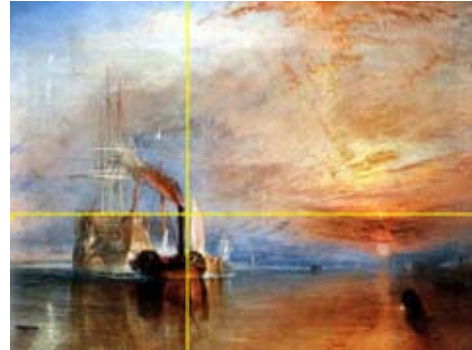
[easyweb.easynet.co.uk/ iany/patterns/spirals.htm](http://easyweb.easynet.co.uk/iany/patterns/spirals.htm)

The steady increase in curvature of the tendril as it tightens around a support creates a curve which is roughly equiangular or logarithmic. Such a curve can be generated mathematically by dividing isosceles triangles (bisecting a base angle each time) or rectangles (the remainder to be similar to the original rectangle) of any shape – there is an infinite family of logarithmic curves of different pitch.

The Golden Rectangle in Art and Architecture



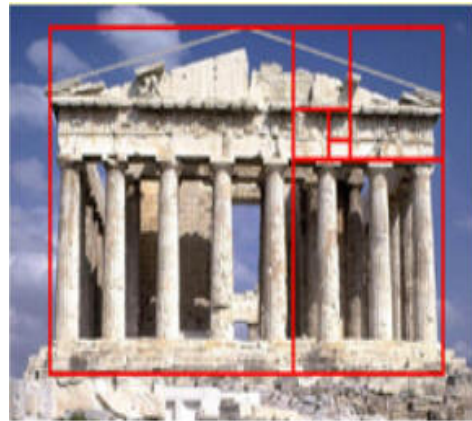
Seurat



Turner



Parthenon



Parthenon

To pique your interest, here is a relationship between Fibonacci numbers and binomial coefficients:

$$\begin{aligned}\text{For } n \geq 0, f_n &= C(n, 0) + C(n-1, 1) + C(n-2, 2) + \cdots \\ &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} C(n-i, i)\end{aligned}$$

As an illustration, the binomial coefficients summing to $f_6 = 13$ are **red** and those summing to $f_7 = 21$ are **blue** in the triangle below. Note that the sum for f_7 stops at $C(4, 3)$ since $\lfloor \frac{7}{2} \rfloor = 3$.

$$\begin{array}{ccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & & & 1 & & 1 & & & \\ & & & & & & 1 & & 2 & & 1 & & \\ & & & & & 1 & & 3 & & 3 & & \mathbf{1} & \\ & & & 1 & & 4 & & \mathbf{6} & & \mathbf{4} & & 1 & \\ & & 1 & & \mathbf{5} & & \mathbf{10} & & 10 & & 5 & & 1 \\ & \mathbf{1} & & \mathbf{6} & & 15 & & 20 & & 15 & & 6 & & 1 \\ 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\ & & & & & & & \vdots & & & & & & \end{array}$$

Getting ready for fun with Fibonacci

To prepare for things to come regarding Fibonacci numbers, here is an alternate way to define them:

Let f_n count the number of ways for an n by 1 board to be covered by a combination of squares and dominoes. A square covers one square of the board, and a domino covers two squares. We start to count with the empty board which can be covered in one way, i.e. $f_0 = 1$. The one board can be covered by a square in one way so that $f_1 = 1$. For $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$.

The five coverings of a 4-board are illustrated below:



This interpretation of the Fibonacci numbers is identical to the one involving rabbits except that we start to count with a zero subscript.

Collectively, squares and dominoes are referred to as “tiles”. So the middle three examples above each involve three tiles while the first uses four and the last only two.

For any n there is obviously a single tiling involving only squares, and for an odd n any tiling must involve at least one square.

Combinatorial fun with Fibonacci

Proposition. For $n \geq 0$,

$$\begin{aligned} F_{2n-1} &= C(n, 1)F_0 + C(n, 2)F_1 + \cdots + C(n, k)F_{k-1} + \cdots + C(n, n)F_{n-1} \\ &= \sum_{k=1}^n \binom{n}{k} F_{k-1} \end{aligned}$$

Call a tiling of a board **breakable** at square k if the squares k and $k+1$ are not covered by a domino.

Proposition. For $n \geq 1$, $F_n^2 + F_{n+1}^2 = F_{2n+2}$.

Proposition. For $n \geq 0$, $F_0 + F_2 + F_4 + \cdots + F_{2n} = F_{2n+1}$

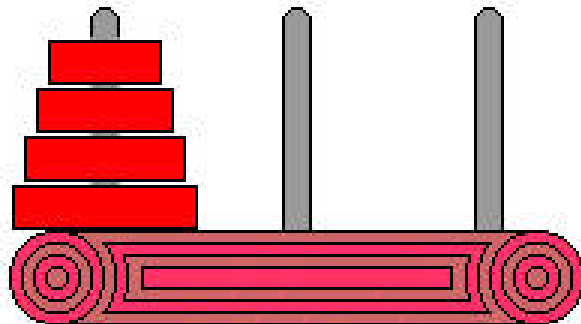
And now here are two identities for which apparently no combinatorial proof is known:

Proposition. $\sum_{i=0}^{2n} \binom{2n}{i} F_{2i-1} = 5^n F_{2n-1}$.

Proposition. $\sum_{i=0}^{2n+1} \binom{2n+1}{i} F_{i-1}^2 = 5^n F_{2n}$.

The Towers of Hanoi

In the great temple of Brahma in Benares, India, on a brass plate under the dome that marks the center of the world, there are 64 disks of pure gold that the priests carry one at a time between three diamond needles according to Brahma's immutable law: No disk may be placed on a smaller disk. In the beginning of the world all 64 disks formed the Tower of Brahma on one needle. Now, however, the process of transfer of the tower from one needle to another is in mid course. When the last disk is finally in place, once again forming the Tower of Brahma but on a different needle, then will come the end of the world and all will turn to dust.



Question? Suppose n disks are to be moved. How many moves are necessary?

A bit of logic

To prove a proposition like “If P , then Q ”, abbreviated “ $P \Rightarrow Q$ ” there are two options: prove the proposition directly, or prove the logically equivalent **contrapositive** of the proposition $\neg Q \Rightarrow \neg P$.

In the second method, called **proof by contradiction**, one assumes that Q is false and shows that that assumption leads to a contradiction of P .

Proposition. The square root of 2 is irrational.

The Pigeonhole Principle: If $k + 1$ or more pigeons are placed into k pigeonholes, then there must be one pigeonhole containing two or more pigeons.

Finally, we will need a bit of number theory:

Proposition. If two numbers have the same remainder when divided by c , then their difference is divisible by c .

Problem. A group of 30 students wrote a dictation. John Bull made 13 errors, and each of the rest made fewer than 13 errors. Prove that at least 3 students made the same number of errors.

Problem. Prove that, given any 12 distinct natural numbers, we can choose two of them such that their difference is divisible by 11.

Problem. Given a set S of 10 distinct numbers between 1 and 100, inclusive, there exist two distinct, disjoint subsets A and B of S whose elements sum to the same number.

Modular arithmetic

Modular arithmetic was developed by Carl Friedrich Gauss and published in his *Disquisitiones Arithmeticae* in 1801 when he was 21. It offers a convenient approach to questions of divisibility.



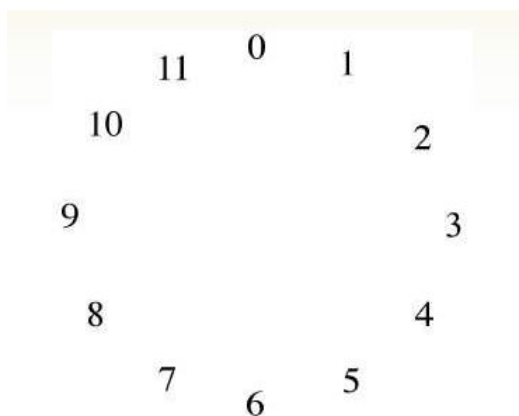
Carl Friedrich
Gauss (1777–1855)

Definition: Let m be a fixed integer. For integers a and b we say that a is **congruent** to b **modulo** m and write

$$a \equiv b \pmod{m}$$

whenever $m \mid (a - b)$. **If** $m \nmid (a - b)$, **we write** $a \not\equiv b \pmod{m}$.

Example. Clocks and modular arithmetic.



Proposition. Let m be a fixed integer. Let a , b , and c be integers. Then

- (i) $a \equiv a \pmod{m}$.
- (ii) If $a \equiv b \pmod{m}$, then $b \equiv a \pmod{m}$.
- (iii) If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $a \equiv c \pmod{m}$.

The properties enumerated in the Proposition are called, **reflexive**, **symmetric**, and **transitive**, respectively.

Proposition. If $a \equiv a' \pmod{m}$ and $b \equiv b' \pmod{m}$, then

- (i) $a + b \equiv a' + b' \pmod{m}$
- (ii) $a - b \equiv a' - b' \pmod{m}$
- (iii) $a \cdot b \equiv a' \cdot b' \pmod{m}$

Question? So addition, subtraction, and multiplication behave nicely! How about division? The following example illustrates the difficulty with division.

Example. Note that $28 \equiv 10 \pmod{6}$. Also, 2 divides both 28 and 10 so we might expect to divide by 2 and have 14 be equivalent to 5 modulo 6.

But $14 \not\equiv 5 \pmod{6}$.

What happened? Why can't we “cancel” the 2 in this case?

The answer requires a couple of definitions...

The **greatest common divisor** of two integers a and b is the largest integer d such that $d|a$ and $d|b$. The greatest common divisor of two integers is denoted by $\gcd(a, b)$.

Integers a and b are said to be **relatively prime** if $\gcd(a, b) = 1$.

This allows an answer to when we can “cancel” in modular arithmetic...

Proposition. If $ac \equiv bc \pmod{m}$ and $\gcd(c, m) = 1$, then $a \equiv b \pmod{m}$.

So we should not have expected 14 to be equivalent to 5 modulo 6 since $\gcd(2, 6) = 2$.

Example. $20 \cdot 2 \equiv 9 \cdot 2 \pmod{11}$ and therefore $20 \equiv 9 \pmod{11}$ since $\gcd(11, 2) = 1$.

So we can cancel a constant in a modular equivalence only when the constant and the modulus are relatively prime.

Proposition. $a \equiv b \pmod{m}$ if and only if a and b have the same remainder when divided by m .

Proposition. An integer n is divisible by 9 if and only if the sum of its digits is divisible by 9.

Example. Determine all solutions to $8x + 12y = b$ where x and y are positive integers and $75 < b < 80$.

Example. Determine the remainder when 2^{37} is divided by 7.

Introduction to Graph Theory

A **graph** is a set of vertices and a set of edges such that each edge is associated with an (unordered) pair of vertices.

A graph is **simple** if it contains at most one edge between any pair of vertices and no loops, i.e., no edges that start and end on the same vertex.

A **path** is a sequence of vertices such that consecutive vertices are adjacent via an edge and no edge is used twice.

A graph is **connected** if there is a path joining each pair of distinct vertices.

A path is **Eulerian** if it uses every edge in the graph exactly once.

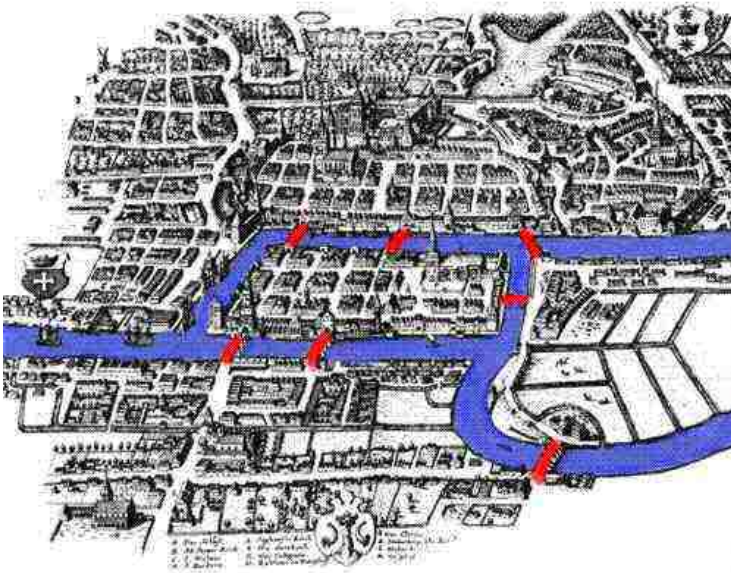
A **circuit** is a path that returns to its starting point.

The **degree** of a vertex is the number of edges incident on the vertex.

The Königsberg Bridge Problem

The river Pregel flows through Königsberg, and in Euler's day there were seven bridges connecting the North and South shores and the two islands in the river. A popular puzzle in the town was whether or not it was possible to walk in such a way as to cross each bridge exactly once.

The Bridges of Königsberg



Leonhard
Euler (1707–1783)



Euler's Theorem. A graph has an Euler path if and only if ...

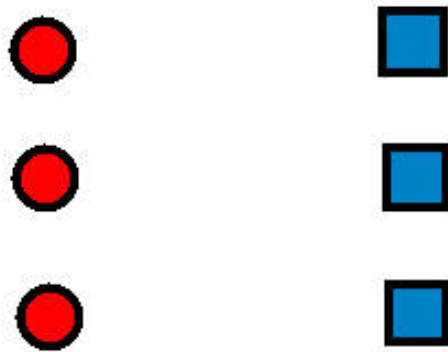
A graph is **planar** if it can be drawn in the plane without its edges crossing.

A graph is **complete** if every pair of distinct vertices is connected by an edge. The complete graph with n vertices is denoted by K_n .

Example. K_4 is planar while K_5 is not.

Utilities Puzzle

Connect all of the circles with all of the squares without arcs crossing.



Unsolved problem. The question of the the minimum number of edge crossings required to draw K_n is unsolved for most cases where $n \geq 13$.

A graph is **bipartite** if it includes two disjoint sets of vertices such that all edges connect a vertex in one of the sets with a vertex in the other.

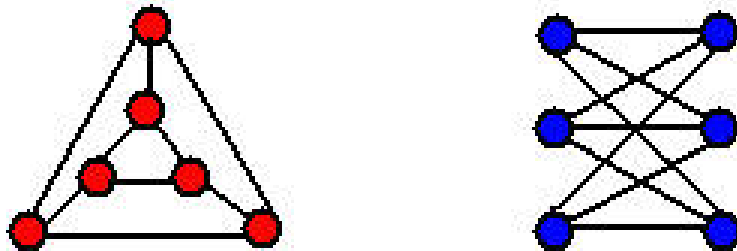
$K_{n,m}$ denotes the bipartite graph that includes all possible edges joining a set of n vertices with a set of m vertices.

A graph H is a **subgraph** of a graph G if every vertex and every edge of H is also a vertex or an edge of G .

Kuratowski's Theorem. A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a subgraph.

Two graphs are **isomorphic** if (ignoring their vertex labels) one can be re-drawn to look like the other.

Thus, unlike a one-to-one function that establishes an isomorphism for sets, the one-to-one correspondence between the vertices of isomorphic graphs must be accompanied by a one-to-one correspondence between their edges in such a way that the incidence relationships are preserved.



Question? So who cares if a graph is planar, or at least, close to planar?

In addition to the number of edges, e , and the number of vertices, v , for a planar graph there is also the number f of faces.

When a planar graph is drawn without edge intersections, the plane is divided into contiguous regions called **faces**. There is also an unbounded face consisting of the region lying entirely outside the graph.

Theorem. (Euler's Formula) In a connected, planar graph

$$e - v + 2 = f,$$

i.e., the number of edges minus the number of vertices plus two equals the number of faces.

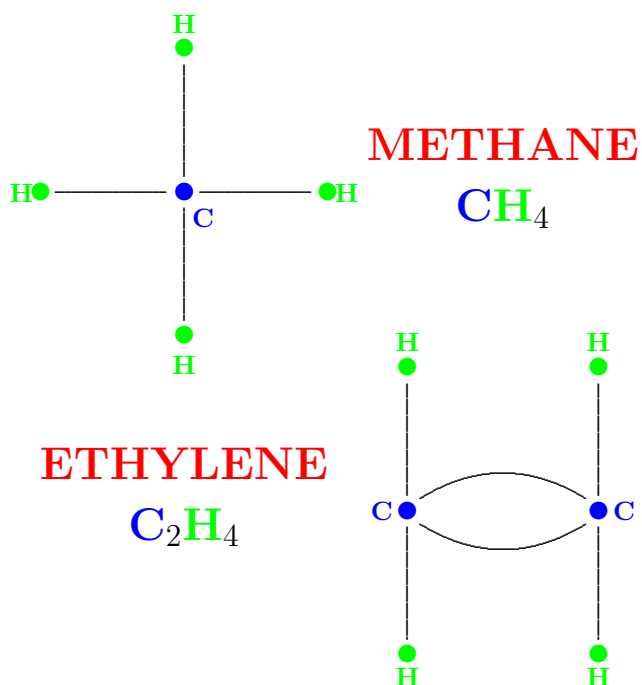
Proof. Use induction on the number of edges . . .

Theorem 6.1.7. In any graph G , the sum of the degrees of the vertices is twice the number of edges.

Corollary 6.1.8. In any graph G , the number of nodes with odd degree is even.

Question? Can there exist a hydrocarbon with five carbon atoms and three hydrogen atoms?

Two hydrocarbons



What is the degree of each hydrogen atom?

What is the degree of each carbon atom?

Question? How many edges are there in K_n ? How about $K_{n,m}$?

Proposition. $C(n + m, 2) = C(n, 2) + C(m, 2) + nm$.

Question? (The Party Problem.) Six guests are invited to a party, and each pair of guests are either friends or strangers. Is it true that among the guests there are always three who are all friends or three who are all strangers?

Example. Provide an example of a graph for a party with five guests without a set of three who are all friends or three who are all strangers.

Ramsey's Theorem. For any pair of positive integers (r, s) , there exists a least positive integer $R(r, s)$ such that for any complete graph on $R(r, s)$ vertices, whose edges are coloured red or blue, there exists either a complete subgraph on r vertices which is entirely blue, or a complete subgraph on s vertices which is entirely red.

Unsolved problems. The Ramsey number $R(r, s)$ is known for small values of r and s , and there are estimates for others, for example $40 \leq R(10, 3) \leq 42$, or $205 \leq R(7, 7) \leq 540$. And $43 \leq R(5, 5) \leq 49$. But the search for exact values is full of open problems.

Just how hard are these problems? Here is a quote from Paul Erdős:

“Imagine an alien force, vastly more powerful than us landing on Earth and demanding the value of $R(5, 5)$ or they will destroy our planet. In that case, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they asked for $R(6, 6)$, we should attempt to destroy the aliens.”

So here is a problem likely not so hard, just “... difficult and is not likely to be solvable by an amateur.”

Unsolved problem. (The $3n + 1$ Problem)

In this problem, a sequence is generated starting with an initial natural number n . The rule for generating the next natural number in the sequence is; if n is even, the next natural number in the sequence is $n/2$, or if n is odd, the next natural number in the sequence is $3n + 1$. For example, if the initial value of n is 17, the sequence generated is $\{17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, \dots\}$. If the natural number 4 is encountered in the sequence, the sequence starts repeating ($\{4, 2, 1\}$ is generated over and over). There are three possibilities;

1. The natural number 4 is encountered in the sequence,
2. The sequence starts repeating, but for a different cycle than 4, 2, 1, or
3. The sequence doesn't repeat.

The $3n + 1$ conjecture states that only the first possibility can occur.

Reference: <http://home.graysoncable.com/dkcox/> which is copyrighted by Darrell Cox.

Instant Insanity

Four blocks have their faces colored **red**, white, **green** and **blue**. The problem is to stack the blocks so that all four colors show on each side of the stack.

Question? How many ways are there to stack the blocks?

Question? The solution is determined by . . .

Trees

A **tree** is a graph with a unique path joining any pair of vertices.

Intuitively, a tree is a graph with just enough edges to be connected.

Proposition. A tree with n vertices has ??? edges.

Proposition. A tree is planar.

One use of trees is guidance in making decisions. In the following example, we will see a use of a “decision tree” in the solution to a counting problem:

Counting “words”

Question? Using the 26 letters of the alphabet and considering the letters AEIOUY to be vowels and the others consonants, how many five letter “words”, i.e., five letter lists, are there subject to the following constraints?

- ★ No vowels are adjacent,
- ★ There are never three adjacent consonants, and
- ★ Adjacent consonants are always different.

In a graph, a **component** is a maximal connected subgraph. Two vertices are in the same connected component if and only if there exists a path between them

Two useful facts:

Lemma. Removing an edge that belongs to a circuit of a connected graph does not disconnect the graph.

Lemma. (**Maximally circuit free**) Adding an edge to a tree creates a unique circuit.

Proposition. (**Characterization Theorem**) The following are equivalent for a graph T having n vertices, no loops, and no parallel edges:

- (a.) T is a tree.
- (b.) T is connected and contains no cycles.
- (c.) T is connected and has $n - 1$ edges.
- (d.) T contains no cycles and has $n - 1$ edges.

A **weighted graph** is a graph with lengths assigned to the edges, e.g., distances, times, costs, etc.

A **spanning tree** for a graph G is a subgraph which is a tree and includes all vertices of G .

If the edges of G are weighted, a **minimal spanning tree** is a spanning tree of minimal total edge weight.

Proposition. Any edge of the least weight belongs to a minimal spanning tree.

Proof. Suppose not; i.e., suppose that edge ij has the minimal weight but belongs to no minimal spanning tree. Let T be a minimal spanning tree. Then adding edge ij to T produces a graph containing a cycle involving ij . Removing any edge from the cycle other than ij produces a spanning tree with total weight no larger than that of T , and hence a minimal spanning tree. Contradiction. $\square$

Prim's algorithm

We assume that G is a weighted, connected graph without loops or parallel edges and having n vertices. Let T_k be the tree formed by the algorithm at step k .

INIT: Choose the shortest edge from G and the vertices v_0 and v_1 on which it is incident to form T_0 .

ADD EDGE: Select the edge of minimal weight which is incident on a vertex in T_k and a vertex not in T_k . Add this edge and vertex to T_k to form T_{k+1} .

END?: Repeat **ADD EDGE** until all vertices of G are in the tree.

Theorem. Prim's algorithm is correct.

Proof. We must show that the algorithm actually produces a minimal spanning tree. Letting T_i be the tree formed by the algorithm at step i , we proceed by induction.

Induction hypothesis: For each i , T_i is contained in some minimal spanning tree.

$i = 1$: This is clear from the proposition above since the shortest edge belongs to some minimal spanning tree.

Now assume $T_{i-1} \subset T$, T a minimal spanning tree. Let pq be the edge and q the node to be added to form T_i .

If $T_i \not\subset T$, we must show that T_i is contained in some minimal spanning tree. Consider the graph $H = T \cup \{pq\}$. H is not a tree since it contains the extra edge pq . Further, since pq joins a node not in T_{i-1} with one of T_{i-1} , H contains a cycle that involves another edge rs joining a node of T_{i-1} with a node not in T_{i-1} .

Removing rs from H produces a spanning tree T' containing T_i . By Prim's algorithm, the weight of rs is at least as great as that of pq . Hence, the weight of T' is no greater than that of T , so T' is a minimal spanning tree containing T_i . $\square$

Theorem. The number of edges examined in executing Prim's algorithm is at worst a cubic polynomial of the number of edges.

The proof will require the following result from the first assignment:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

An application

A **symmetric function** of several variables is one in which an exchange of variables does not alter the value of the function.

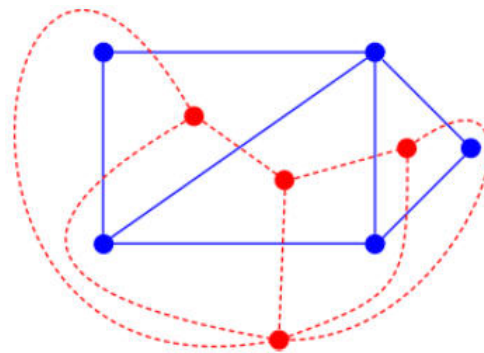
Interesting property. A minimal spanning tree minimizes an increasing symmetric function of the edge lengths.

Question? A message is to be passed to all members of an underground organization. Each member knows certain other members and has a means of contacting them. Associated with each contact i to j is a probability p_{ij} that the message will be compromised.

Assuming that these probabilities are independent, how should a message be distributed to minimize the overall chance of compromise?

The Dual of a Graph and The Four Color Theorem

The **dual** G' of a planar graph G is the graph obtained by placing a vertex of G' in each face of G and connecting the vertices of G' if the corresponding faces of G share an edge.



Source: http://en.wikipedia.org/wiki/File:Duals_graphs.svg

Viewing G as a map with each face a country, coloring the map so that countries sharing a border have different colors can be reduced to coloring the vertices of the dual graph so that the vertices on opposite ends of each edge have different colors.

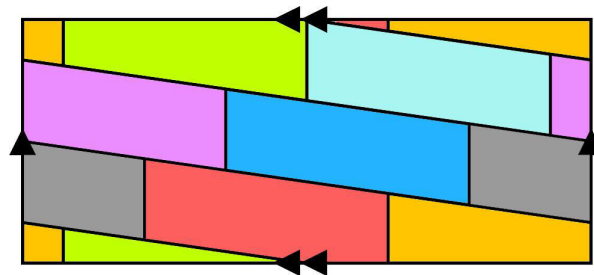
For many years, it was known that any planar graph could be colored with five colors, and that there were graphs that could not be colored with three colors. No one could find a graph that required five colors or a proof that all maps could be colored with four colors.

The problem was resolved by Appel and Haken in 1976 who reduced the problem to showing that 1955

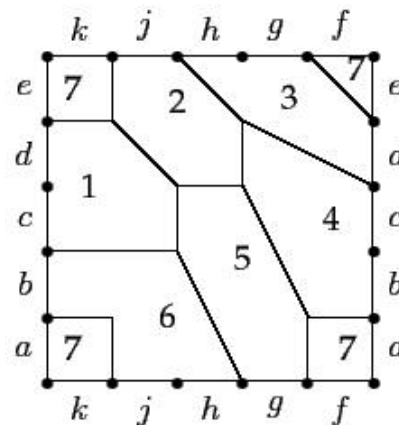
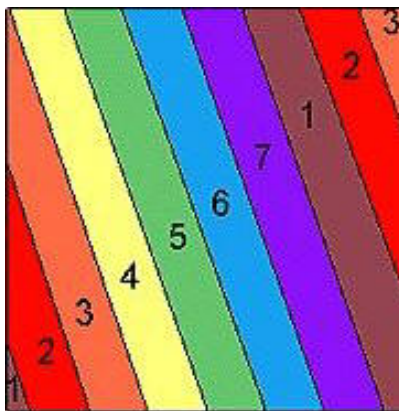
cases needed to be colored and used a computer for a case-by-case analysis of those cases.

Theorem. (Four Color Theorem) Any planar graph can be colored by four colors.

Maps on surfaces that are not flat can require more colors. For instance, by joining the single arrows together and the double arrows together in the figure below, one obtains a torus with seven mutually touching regions; therefore seven colors are necessary.



Source: http://en.wikipedia.org/wiki/Four_color_theorem



www.meru.org/Posters/hextorus.html www.parabola.unsw.edu.au/vol35_no2/node2.html

A bit of perfection

An integer n is **perfect** if it is the sum of its positive divisors other than itself.

Examples. 6, 28, and 496 are perfect.

Proposition. (Euclid) If $N = 2^{k-1}(2^k - 1)$ and $2^k - 1$ is prime, then N is perfect.

$2^{k-1}(2^k - 1)$ being perfect can be shown to be equivalent to $2^k - 1$ being prime. Such primes are called **Mersenne primes** and much work has been devoted to finding them. It can also be shown that if $2^k - 1$ is prime, then k is prime.

Marin Mersenne
(1588 – 1648)



<http://commons.wikimedia.org>

On January 25, 2013, the 48th Mersenne prime $2^{57,885,161} - 1$ was discovered by Curtis Cooper and has 17,425,170 digits. The associated perfect number has 34,850,339 digits.

You can be a part of the search for Mersenne primes:

<http://www.mersenne.org/>

Three unsolved problems:

Unsolved problem. Are there infinitely many Mersenne primes?

Unsolved problem. Is every Mersenne number $2^p - 1$ square free?

Source: <http://primes.utm.edu/mersenne/index.html>

Unsolved problem. Are there any odd perfect numbers?

Many properties that an odd perfect number must satisfy have been determined, among them its largest prime factor is greater than 10^8 , and it has at least 101 prime factors and at least 9 distinct prime factors.

A little about prime numbers and their distribution

Theorem. (Euclid) There are infinitely many primes.

Theorem. There are arbitrarily large gaps between successive primes.

Proof. Can be done in one line!

- ★ The primes 29 and 31 are called **twin primes**. Another pair is 59 and 61. Two others, each having 4,932 digits, are $697,053,813 \cdot 2^{16,352} + 1$ and $697,053,813 \cdot 2^{16,352} - 1$. How many such pairs are there?
- ★ **(Dirichlet, 1837)** Any arithmetic progression whose terms do not all share a common factor contains an infinite number of primes.
- ★ **The Goldbach Conjecture.** Every even integer greater than 2 can be written as the sum of two primes.
- ★ For every positive integer m there is an even number $2n$ such that there are more than m pairs of consecutive primes with difference $2n$.

“God may not play dice with the universe,
but something strange is going on with the
prime numbers.”

Paul Erdős (1913-1996)

Attitudes toward infinity at the time of Cantor

Expressions like this basic limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

were understood by saying that if n grew arbitrarily then $\frac{1}{n}$ would get arbitrarily close to zero.

But the notion of an infinite *set* was not accepted or understood. Descartes said “The infinite is recognizable, but not comprehensible.”

Gauss wrote that “In mathematics infinite magnitude may never be used as something final; infinity is only a *façon de parler*, meaning a limit to which certain ratios may approach as closely as desired when others are permitted to increase indefinitely.”

Henri Poincare called Cantor’s work a “grave disease” affecting mathematics. Cantor was called a “scientific charlatan” and a “corruptor of youth”.

Cantor realized that he was breaking sharply with his predecessors and said in 1883: “I place myself in a certain opposition to widespread views on the mathematical infinite and to oftdefended opinions of the essence of number.”

But Cantor had his supporters. David Hilbert defended him saying “No one shall expel us from the Paradise that Cantor has created.”

Infinite sets

We say that two sets have the same size, or the same **cardinality**, if the elements of the two sets can be placed into one-to-one correspondence.

A set is **finite** if it can be placed in one-to-one correspondence with the set of the first n integers for some n ; otherwise, a set is **infinite**.

Cantor developed a special notation to express the **cardinal numbers**, i.e. the sizes, of various sets:

- * $\aleph_0$ denotes the cardinality of the integers. Sets having this cardinality are said to be **countable**.
- * $\aleph_1$ denotes the cardinality of the next larger infinite set. Sets having this or larger cardinal number are said to be **uncountable**.
- * $\mathfrak{c}$ denotes the cardinality of the set of all real numbers.

Cantor's Theorem(s):

Theorem 1. The set of integers has the same size as the set of even integers.

This result caused Cantor to define a set as **infinite** if could be placed in one-to-one correspondence with one of its proper subsets.

Theorem 2. The set of rational numbers has the same size as the set of natural numbers.

Proof. Establish a one-to-one correspondence between the representations of the rational numbers and the natural numbers as below:

n	Rational representations						
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\dots$
3	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$		
6	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$			
10	$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\dots$			
15	$\frac{5}{1}$	$\frac{5}{2}$					
21	$\frac{6}{1}$						
$\vdots$	$\vdots$						

Question? Do you recognize the numbers in the n column?

Question? Can you associate them with a geometric shape?

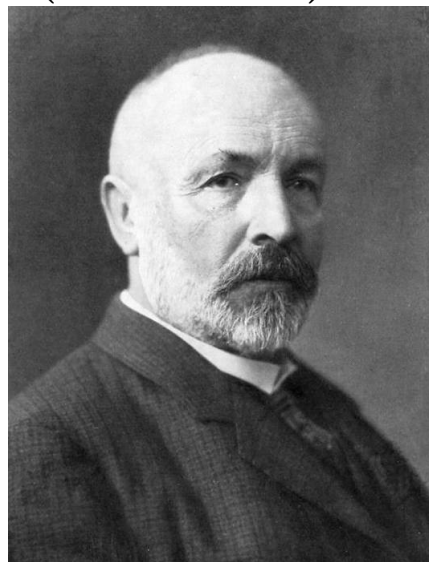
Theorem 3. The set of all possible computer programs has the same size as the set of natural numbers.

Theorem 4. There are more real numbers in the interval $(0, 1)$ than there are natural numbers.

Proof. We argue by contradiction. Suppose not. Then one could subscript **all** the reals in the interval $(0, 1)$ with natural numbers and list them together with their decimal expansions. For example,

$$\begin{aligned} a_0 &= 0. 1 2 1 3 4 5 4 . . . \\ a_1 &= 0. 0 7 7 2 2 1 6 . . . \\ a_2 &= 0. 0 0 0 1 1 1 5 . . . \\ a_3 &= 0. 9 9 9 7 1 1 3 . . . \\ a_4 &= 0. 3 3 3 3 3 3 9 . . . \\ a_5 &= 0. 2 5 0 0 0 0 8 . . . \\ a_6 &= 0. 1 1 1 2 2 2 4 . . . \\ &\vdots \end{aligned}$$

Georg Cantor
(1845 – 1918)



Source:

<http://i12bent.tumblr.com/post/36221807>

Now construct a number x according to the following rules:

If digit n of a_n is even, make digit n of x 1. If digit n of a_n is odd, make digit n of x 2.

For example, for the above start we would have $x = 0.2212211\dots$ Now x is a real number in the interval $(0, 1)$ which differs in at least one decimal place from every number on the list, and is therefore not accounted for in the one-to-one correspondence, a contradiction. $\square$

cantor-german-
mathematician-and-
philosopher

Another method of proof

The Well Ordering Principle is another of Cantor's contributions.

The Well Ordering Principle. Every non-empty set of non-negative integers has a least element.

Fundamental Theorem of Arithmetic. Every integer $n > 1$ is expressible as $n = p_1 p_2 p_3 \cdots p_r$ where r is a positive integer and each p_i is a prime.

Proof. We use contradiction and well-ordering. Assume that not all integers have prime factorizations, and let n be the smallest positive integer without a prime factorization.

n cannot be prime because then it would be its own factorization. So n is composite.

Thus $n = ab$ with a and b both strictly between 1 and n . Since n is the smallest integer without a prime factorization and a and b are smaller, then they each have prime factorizations $a = p_1 p_2 p_3 \cdots p_r$ and $b = q_1 q_2 q_3 \cdots q_s$. But this means that $n = ab = p_1 p_2 p_3 \cdots p_r q_1 q_2 q_3 \cdots q_s$ gives a prime factorization of n , which is a contradiction.

Lehman's Lemma. There are no positive integer solutions to the following equation

$$8a^4 + 4b^4 + 2c^4 = d^4.$$

Peano's Axioms for Arithmetic

The following axioms were developed by Guiseppe Peano (1858-1932) as the culmination of his effort to formalize arithmetic. They are included here only because they are referenced in the next major theorem.

1. 1 is a natural number.
2. For every natural number x there exists another natural number x' called the successor of x .
3. $1 \neq x'$ for every natural number x (x' being the successor of x).
4. If $x' = y'$ then $x = y$.
5. If Q is a property such that:
 - (a) If 1 has the property Q , and
 - (b) if x has property Q then x' has property Q ,then property Q holds for all natural numbers.

Kurt Gödel
(1906 – 1978)



Source:

https://en.wikipedia.org/wiki/Kurt_Gödel

Gödel's Incompleteness Theorem.

In any consistent formal mathematical system that is expressive enough to state Peano's Axioms for arithmetic, there is a statement that is true but not provable.

Two undecidable statements:

The Continuum Hypothesis states that there is no subset of the real line, classically called the continuum, that has more elements than the integers but fewer than the whole line.

The **Axiom of Choice** states that given any collection of non-empty sets, however large, one element can be picked from each of the sets.

Problems potpourri

Proposition.

$$1 \cdot C(n, 1) + 2 \cdot C(n, 2) + 3 \cdot C(n, 3) + \cdots + n \cdot C(n, n) = n2^{n-1}.$$

Problem. A winning configuration in the game of Mini-tetris is a complete tiling of a $2 \times n$ board by 2×2 squares and 2×1 dominos. A domino can be placed so that it lies on a side 2 units long or stands on a 1 unit end.

1. Determine a recursion to count the number T_n of winning configurations of a $2 \times n$ board, $n \geq 1$.

2. Show that $T_n = \frac{2^{n+1} + (-1)^n}{3}$ for $n \geq 1$.

Proposition. If $n + 1$ integers are selected from the set $\{1, 2, 3, \dots, 2n\}$, then one of the integers divides another.

Proposition. Every positive integer can be written as a sum of distinct terms in the Fibonacci sequence.

Proposition. All persons have the same eye color.

Proposition. If all vertices of a simple graph have degree less than or equal to n , then the graph can be

colored with $n+1$ colors so that no pair of adjacent vertices have the same color.

Proposition. Some two vertices in a graph with n vertices must have the same degree.

Problem. Prove the power rule of differentiation:

$$\frac{d}{dx}(x^n).$$

Proposition. For $n \geq 0$, prove that

$$\sum_{i \geq 0} \sum_{j \geq 0} \binom{n-i}{j} \binom{n-j}{i} = f_{2n+1}.$$

Proposition. $\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}.$

Proposition. For $n \geq 2$, 6 divides $2n^3 - 3n^2 + n$.

Proposition. A positive integer n is divisible by 11 if and only if the alternating sum of its digits, e.g. for $n = 8,237$, $-8 + 2 - 3 + 7$, is divisible by 11.

Problem. If $17! = 355,687,ad8,096,000$ what are a and b .

Problem. Compute $5^{21} \pmod{31}$.

Problem. Show that if a country has only three cent and five cent stamps then it can arrange postage for any amount over eight cents.

Proposition. $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^k} = 1 - \frac{1}{2^k}$.

Problem. Show that if five points are located in the interior of a 1×1 square then two of them must be within $\frac{\sqrt{2}}{2}$ units of each other.

Student submitted problem. If opposite corners are removed from a standard checker board can the remaining board be tiled by dominos that cover two adjacent squares?

Proposition. If a sequence of positive integers is defined by $a_0 = 1$, $a_1 = 2$, $a_2 = 3$ and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, show that for all $n \geq 2$, $a_n \leq 2^n$.

Poker again. Answer each of the following:

- ❖ How many poker hands have exactly three cards higher than 8?
- ❖ How many poker hands contain cards from exactly two ranks?

Remarks on Calculus and Discrete Mathematics

Disclaimers. Given the time constraints, the definitions, etc., are a bit casual. Being a discrete course, limits are, of course, avoided.

The **derivative** of a function f is the rate of change of the function, or geometrically, the slope of the tangent line to the graph of the function. If the derivative of a function f exists for every point in an interval it defines a function on the interval whose value at x is denoted by $f'(x)$.

The process of obtaining a derivative is called **differentiation**.

A substantial part of early calculus involves calculating the derivatives of functions built up from combinations of other functions. In other words, differentiation is recursive.

One such rule is the product rule:

$$\frac{d}{dx} (u(x)v(x)) = u(x) \frac{d}{dx} (v(x)) + \frac{d}{dx} (u(x)) v(x)$$

There are of course rules for the differentiation of specific functions:

❖ For n other than 0, $\frac{d}{dx} (x^n) = nx^{n-1}$

❖ $\frac{d}{dx} (\sin(x)) = \cos(x)$

$$\blacklozenge \frac{d}{dx} (\cos(x)) = -\sin(x)$$

Another part of basic calculus is the calculation of areas. This historically is the first step in calculus going back to Archimedes. It is based on Riemann sums, which involves summing an increasingly large number of rectangles.

Some discrete math connections:

- * The power rule can be proven by mathematical induction.
- * Differentiation is a recursive operation.
- * The typical examples of Riemann sums rely on formulas that most calculus students do not really understand, but you realize that they are just the results of routine mathematical induction exercises and that they have their origins in Pascal's Triangle:

$$\blacklozenge 1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

$$\blacklozenge 1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\blacklozenge 1^3 + 2^3 + \dots + n^3 = \left(\frac{n(n+1)}{2} \right)^2$$

- * The harmonic series diverges: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{2^n} + \dots \geq 1 + \frac{n}{2}$ for all n .

Calculus Points to Ponder

- ★ Why is the continuity hypothesis appropriate on a closed interval whereas the differentiability hypothesis applies on an open interval?
- ★ How is the continuity hypothesis used in proving the Fundamental Theorem of calculus?
- ★ The Intermediate Value Theorem and the existence of absolute maxima and minima are two important consequences of continuity to understand.
- ★ Pay close attention to how the inverse of various functions and implicit differentiation can be used to derive differentiation formulae for additional functions, e.g., the natural log and the exponential function, and trig functions and their inverses.
- ★ Keep things straight - don't assume that the converse of a proposition is automatically true. It helps to have counterexamples handy, e.g.,
 - ◆ a continuous function that is not differentiable.
 - ◆ a point where a function has a zero derivative but no local maximum or minimum.
 - ◆ an important continuous function with no elementary antiderivative.

- ◆ a divergent infinite series whose n^{th} term approaches 0.
- ★ Appreciate the beauty of the connections with science, e.g.,
 - ◆ Fermat's Principle implies Snell's Law,
 - ◆ The percentage increase in the flux through an artery is four times the increase in the radius.
- ★ Pay attention to how the Mean Value Theorem can be applied in the development of the methods of locating maxima and minima. You can ignore it in many courses, but there are MANY applications of the Mean Value Theorem later in analysis, so learn it now!
- ★ Don't ignore Riemann Sums and the definition of the definite integral just because the Fundamental Theorem of Calculus seems so much easier. It may happen that the functions that turn out to be important to you don't show up in convenient closed form with a nice neat antiderivative.
- ★ Watch for the analogy between the solution of the closed form of the Fibonacci numbers and the solution of an initial value problem involving a second order linear differential equation with constant coefficients.

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