ON CLUSTERS AND THE MULTI-ISOPERIMETRIC PROFILE IN RIEMANNIAN MANIFOLDS WITH BOUNDED GEOMETRY

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Abstract. For a complete Riemannian manifold with bounded geometry, we prove a generalized compactness theorem for sequences of clusters (with uniformly bounded perimeter and volume) in a larger space obtained by adding at most countable many limit manifolds at infinity, in the spirit of [FN20]. The arguments presented in the proof of this generalized compactness theorem when applied to minimizing sequences of clusters give a generalized existence theorem for isoperimetric clusters in a larger space obtained by adding, in this case, only finitely many limit manifolds at infinity, as in [Nar14]. To achieve this goal, we show that isoperimetric clusters are bounded and also we prove the continuity of the multi-isoperimetric profile. In fact, we prove a stronger continuity property that is the Hölder continuity of the multi-isoperimetric profile. The multi-isoperimetric profile has been introduced recently in [MN18] in the context of smooth metric measured spaces with a Gaussian-weighted notion of perimeter. This work generalizes to the context of Riemannian isoperimetric clusters some previous results about the classical Riemannian and sub-Riemannian isoperimetric problem, see [GR12, Mor03, MN16, Nar09, Nar14, Nar18], as well as results from clusters theory in the Euclidean setting, see [Mag12, Mor94]. In particular, as a consequence of our generalized existence results, we prove an existence theorem (in the classical sense) for isoperimetric clusters in a quite large class of noncompact Riemannian manifolds (the same considered in [MN16]) which includes, for instance, the space forms.

Key Words: existence of isoperimetric clusters, multi-isoperimetric problem, minimizing clusters.

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Contents

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1. Introduction

In this paper, we study the existence of isoperimetric clusters for prescribed volume vector, i.e. minimizers of the perimeter functional under volume constraint, in a complete Riemannian manifold, assuming some bounded geometry conditions. The difficulty appears when the ambient manifold is noncompact since, in this case, it could happen that for a sequence of clusters with perimeter approaching the infimum, some volume may disappear at infinity and the limit of the sequence could not belong to the ambient manifold. We show that the cluster splits into a finite number of pieces (sub-clusters) that carry a positive fraction of the volume, one of them possibly staying at finite distance and the others concentrating along with divergent directions. Moreover, each of these pieces will converge to an isoperimetric cluster for its volume vector lying in some pointed limit manifold, possibly different from the original. So, isoperimetric clusters exist in this generalized sense as stated in Theorem 2. The range of applications of these results is wide as it generalizes all the well known ideas carried by clusters in Euclidean spaces. We show that isoperimetric clusters are bounded in Proposition 3 by standard arguments used to prove the boundedness of clusters in Euclidean spaces and isoperimetric regions in Riemannian manifolds with bounded geometry. We also study properties of the multi-isoperimetric profile, defined in (8), as its continuity stated in Theorem 5. The theory that we will construct has its own importance and value, although we show how to apply it to prove a classical existence theorem of isoperimetric clusters in Theorem 4. The vague notions invoked in this introductory paragraph will be made clear and rigorous in the sequel.

1.1. **Finite perimeter sets and its basic concepts.** In the remaining part of this paper we always assume that all the Riemannian manifolds (M^n, g) considered are complete and smooth with smooth Riemannian metric g and $n \geq 2$. We denote by $\mathrm{Vol}_g()$ the canonical Riemannian measure induced on M by g, and by $\mathcal{H}^{n-1}_g$ the $(n-1)$-Hausdorff measure associated to the canonical Riemannian length space metric d_g of M. When it is already clear from the context, explicit mention of the metric g will be suppressed in what follows. First of all, we shall denote the metric ball centered at $p \in M$ of radius r by $\mathrm{B}_g(p, r)$ and we define two concepts from Riemannian geometry.

Definition 1. *Let (M^n, g) be a complete Riemannian manifold, the* **injectivity radius** *of M, denoted inj_M, is defined as follow*

$$\mathrm{inj}_M := \inf_{p \in M} \{\mathrm{inj}_{p,M}\},$$

where for every point $p \in M$, $\mathrm{inj}_{p,M}$ is the injectivity radius at p of M, i.e. the largest radius r for which the exponential map $\exp_p : B(0, r) \to \mathrm{B}_g(p, r)$ is a diffeomorphism.

The next definition places the concept of bounded geometry which will be crucial for all the theory developed in this work.

Definition 2. *A complete Riemannian manifold (M^n, g), is said to have* **bounded geometry** *if there exists a constant $k \in \mathbb{R}$, such that $\mathrm{Ric}_g \geq k(n-1)$ (i.e., $\mathrm{Ric}_g \geq k(n-1)g$ in the sense of quadratic forms) and $\mathrm{Vol}_g(\mathrm{B}_M(p, 1)) \geq v_0$ for some positive constant v_0, where $\mathrm{B}_M(p, r)$ is the geodesic ball (or equivalently the metric ball) of M centered at p and of radius r.*

The reason to consider only manifolds with bounded geometry is explained in this introduction. Let us define the basic notions from the geometric measure theory.

Definition 3. *Let (M^n, g) be a Riemannian manifold of dimension n, $U \subseteq M$ an open subset, $\mathfrak{X}_c(U)$ the set of smooth vector fields with compact support on U. Given a function $u \in L^1(M, g)$, define the* **total variation of** *u by*

$$|\mathrm{D}u|_g(M) := \sup \left\{ \int_M u \mathrm{div}_g(X) dv_g : X \in \mathfrak{X}_c(M), ||X||_{\infty,g} \leq 1 \right\},$$

where $||X||_{\infty,g} := \sup\{|X_p|_g : p \in M\}$ *and* $|X_p|_g$ *is the norm of the vector* X_p *in the metric* g *on* T_pM*. We say that a function* $u \in L^1(M,g)$*, has* **bounded variation**, *if* $|\mathrm{D}u|_g(M) < +\infty$ *and we define the set of all functions of bounded variations on* (M^n, g) *by* $\mathrm{BV}(M,g) := \{u \in L^1(M,g) : |\mathrm{D}u|_g(M) < +\infty\}$.

Definition 4. *A function* $u \in L^1_{\mathrm{loc}}(M)$ *has* **locally bounded variation in** (M^n, g)*, if for each open set* $U \subset\subset M$,

$$|\mathrm{D}u|_g(U) := \sup\left\{\int_U u\mathrm{div}_g(X)dv_g : X \in \mathfrak{X}_c(U), \|X\|_{\infty,g} \leq 1\right\} < +\infty,$$

and we define the set of all functions of locally bounded variations on (M^n, g) *by* $\mathrm{BV}_{\mathrm{loc}}(M,g) := \{u \in L^1_{\mathrm{loc}}(M,g) : |\mathrm{D}u|_g(U) < +\infty, U \subset\subset M\}$*. So for any* $u \in \mathrm{BV}(M,g)$*, we can associate a vector Radon measure on* (M^n, g)*, denoted* $\mathrm{D}u$ *with total variation* $|\mathrm{D}u|_g$.

Given $E \subset M$ measurable with respect to the canonical Riemannian measure and $U \subset M$ an open subset, the **perimeter of** E **in** U, $\mathrm{Per}_g(E,U) \in [0, +\infty]$, is

$$\mathrm{Per}_g(E,U) := |\mathrm{D}\chi_E|_g(U) = \sup\left\{\int_U \chi_E \mathrm{div}_g(X)dv_g : X \in \mathfrak{X}_c(U), ||X||_{\infty,g} \leq 1\right\}. \tag{1}$$

If $\mathrm{Per}_g(E,U) < +\infty$ for every open set $U \subset\subset M$, we call E a **locally finite perimeter set**. Let us set $\mathrm{Per}_g(E) := \mathrm{Per}_g(E,M)$. Finally, if $\mathrm{Per}_g(E) < +\infty$ we say that E **is a set of finite perimeter**.

In the case that the boundary of the set E is not smooth, the topological boundary ∂E of a set of finite perimeter is not a good candidate to measure the perimeter because its Hausdorff measure exceeds, in general, such value. The correct boundary in this context is the reduced boundary introduced by Ennio De Giorgi whose definition we recall below.

Definition 5. *The* **reduced boundary** *of a set of finite perimeter* E *of* (M^n, g)*, denoted by* $\partial^* E$ *is defined as the collection of points* p *at which the limit*

$$\nu_{E,g}(p) := \lim_{r\to 0} \frac{\mathrm{D}^g\chi_E\left(\mathrm{B}_M(p,r)\right)}{|\mathrm{D}^g\chi_E|\left(\mathrm{B}_M(p,r)\right)}$$

exists and has length equal to one, i.e.

$$\left|\nu_{E,g}(p)\right|_g = 1.$$

The function $\nu_{E,g} : \partial^* E \to \mathbb{S}^{n-1}$ *is called the* **generalized inner normal to** E.

Remark. *By the very definition,* $\partial^* E$ *could depend on the metric* g*, but in fact it can be show that it does not depend on the choice of the metric* g*. To see this is a straightforward argument.*

By standard results of the theory of sets of finite perimeter, we have that $\mathrm{Per}_g\left(E,F\right) = \mathcal{H}^{n-1}_g\left(\partial^* E \cap F\right)$ where $\partial^* E$ is the reduced boundary of E and $F \subset M$ is any Borel set. In particular, if E has smooth boundary, then $\partial^* E = \partial E$, where ∂E is the topological boundary of E. In the sequel, we will not distinguish between the topological boundary and reduced boundary when no confusion can arise.

Definition 6. *Let* (M^n, g) *be a Riemannian manifold. We denote by* $\tilde{\tau}_M$ *the set of finite perimeter subsets of* (M^n, g)*. The function* $\tilde{\mathrm{I}}_M : [0, \mathrm{Vol}_g(M)) \to [0, +\infty)$ *defined by*

$$\tilde{\mathrm{I}}_M(v) := \inf\{\mathrm{Per}_g\left(\Omega\right) : \Omega \in \tilde{\tau}_M, \mathrm{Vol}_g\left(\Omega\right) = v\}$$

is called the **isoperimetric profile function** (*or shortly the* **isoperimetric profile**) *of the Riemannian manifold* (M^n, g)*. If there exist a finite perimeter set* $\Omega \in \tilde{\tau}_M$ *satisfying* $\mathrm{Vol}_g\left(\Omega\right) = v$, $\tilde{\mathrm{I}}_M(\mathrm{Vol}_g\left(\Omega\right)) = \mathrm{Per}_g\left(\Omega\right)$ *such an* Ω *will be called an* **isoperimetric region**, *and we say that* $\tilde{\mathrm{I}}_M(v)$ *is* **achieved**.

For further results on the isoperimetric profile, one can consult [P+00], [Bay04], [NP18], [FN14], etc.

Definition 7. *We say that a sequence of finite perimeter sets E_j* **locally converges to another finite perimeter set** *E, and we denote this by writing $E_j \overset{\text{loc}}{\to} E$, if $\chi_{E_j} \to \chi_E$ in $L^1_{\text{loc}}(M,g)$, i.e. if $\operatorname{Vol}_g(E_j \Delta E \cap U) \to 0$, for all $U \subset\subset M$. Here χ_E means the characteristic function of the set E and the notation $U \subset\subset M$ means that $U \subseteq M$ is open and $\overline{U}$ (the topological closure of U) is compact in M.*

Definition 8. *We say that a sequence of finite perimeter sets E_j* **converge in the sense of finite perimeter sets** *to another finite perimeter set E if $E_j \overset{\text{loc}}{\to} E$ and*

$$\lim_{j\to+\infty} \operatorname{Per}_g(E_j) = \operatorname{Per}_g(E).$$

For a more detailed discussion on locally finite perimeter sets and functions of bounded variation, one can consult [JPPP07] to the Riemannian setting and [Mor09], [Mag12], [Giu84] and [AFP00] to the Euclidean setting.

1.2. **On $C^{m,\alpha}$-convergence of manifolds and bounded geometry.** Now, let us recall the basic definitions from the theory of convergence of manifolds, as exposed in [Pet16]. This will help us to state the main result in a precise way.

Definition 9. *For any $m \in \mathbb{N}$, $\alpha \in [0,1]$, a sequence of pointed smooth complete Riemannian manifolds is said to* **converge in the pointed $C^{m,\alpha}$, respectively C^m, topology to a smooth manifold** *M (denoted by $(M_i, g_i, p_i) \to (M, g, p)$), if for every $R > 0$ we can find a domain Ω_R with $B(p,R) \subseteq \Omega_R \subseteq M$, a natural number $\nu_R \in \mathbb{N}$, and C^{m+1} embeddings $F_{i,R} : \Omega_R \to M_i$, for large $i \geq \nu_R$ such that $F_{i,R}(p) = p_i, B(p_i,R) \subseteq F_{i,R}(\Omega_R)$ and $F^*_{i,R}(g_i) \to g$ on Ω_R in the $C^{m,\alpha}$, respectively C^m topology.*

Definition 10. *We say that a sequence of multipointed Riemannian manifolds $(M_i, g_i, p_{1i}, \cdots, p_{ji}, \cdots)$* **converges to the multipointed Riemannian manifold** $(M, g, p_1, \cdots, p_j, \cdots)$ **in the multipointed C^0-topology**, *if for every j we have*

$$(M_i, g_i, p_{ji}) \to (M, g, p_j)$$

in the pointed C^0-topology.

It is easy to see that this type of convergence implies pointed Gromov-Hausdorff convergence. We now define the notion of norm of sets in a manifold which is the **$C^{m,\alpha}$-norm at scale** r, for the precise definition of $C^{m,\alpha}$ and $\|\cdot\|_\alpha$ we refer to [Pet16, 11.3.1]. This notion can be taken as a possible definition of bounded geometry.

Definition 11 ([Pet16])**.** *A subset A of a Riemannian n-manifold M has* **bounded $C^{m,\alpha}$ norm on the scale of** *r, $||A||_{C^{m,\alpha},r} \leq Q$, if every point p of M lies in an open set U with a chart ψ from the Euclidean r-ball into U such that*

- *(i): $|\mathrm{D}\psi| \leq e^Q$ on $B(0,r)$ and $|\mathrm{D}\psi^{-1}| \leq e^Q$ on U,*
- *(ii): $r^{|j|+\alpha}||\mathrm{D}^j g||_\alpha \leq Q$ for all multi indices j with $0 \leq |j| \leq m$, where g is the matrix of functions of metric coefficients in the ψ coordinates regarded as a matrix on $B(0,r)$.*

We define the set $\mathcal{M}^{m,\alpha}(n,Q,r)$ as the set of pointed manifolds (M,g,p) which satisfies $||M||_{C^{m,\alpha},r} \leq Q$.

In the sequel, unless otherwise specified, we will make use of the technical assumption on $(M, g, p) \in \mathcal{M}^{m,\alpha}(n, Q, r)$ that $n \geq 2$, $r, Q > 0$, $m \geq 1$, $\alpha \in]0, 1]$. Roughly speaking, $r > 0$ is a positive lower bound on the injectivity radius of M, i.e. $\mathrm{inj}_M > C(n, Q, \alpha, r)$.

In general, a lower bound on Ric_M and on the volume of unit balls, i.e. the bounded geometry requirements (Definition 2), does not ensure that the pointed limit metric spaces at infinity are still manifolds, this motivates the following definition.

Definition 12. *We say that a smooth Riemannian manifold (M^n, g) has $C^{m,\alpha}$***-bounded geometry** *if it is of bounded geometry and if for every diverging sequence of points (p_j), there exist a subsequence (p_{j_l}) and a pointed smooth manifold $(M_\infty, g_\infty, p_\infty)$ with g_∞ of class $C^{m,\alpha}$ such that the sequence of pointed manifolds $(M, g, p_{j_l}) \to (M_\infty, g_\infty, p_\infty)$, in $C^{m,\alpha}$-topology.*

We observe here that Definition 12 is weaker than Definition 11. In fact, using [Pet16, Theorem 11.3.6], one can show that if a manifold M has bounded $C^{m,\alpha}$ norm on the scale of r for $\alpha > 0$ in the sense of Definition 11 then M has $C^{m,\alpha}$-bounded geometry in the sense of Definition 12, while in general the converse is not true.

In the absence of the extra condition of Definition 12, just assuming bounded geometry in the sense of Definition 2, *a priori* the resulting limit space is merely a length space (Y, d_Y, y). Nevertheless, as pointed out by L. Ambrosio in [Amb18, Section 7], the right axiomatization for these limit spaces may be the RCD-metric spaces as suggested by Gromov in [Gro91] and by the works of Cheeger-Colding ([CC97], [CC00a], [CC00b]), for a more detailed introduction to this topic we refer to reader to [AGS14]. The RCD-metric spaces are being broadly studied, strong tools and properties are derived from the Riemannian setting generating beautiful generalizations of the theory for this class of metric spaces, see [AGMR15], [ABS19], [BPS19], [DPG18], and the references therein. About the isoperimetric problem in this setting, one can consult [ABFP21] and [APP21]. Regarding the smooth structure of limit spaces (Y, d_Y, y), the reader is referred to Cheeger-Anderson [CA92], Anderson [And92], or [Pet16, Chapter 11] for a more expository discussion.

1.3. **Clusters.** The definition of clusters that we adopt is motivated by the definition given by Francesco Maggi in the Euclidean setting in [Mag12]. For other references in cluster theory in the Euclidean setting, one can consult [Mor09] and [Alm76].

Definition 13. *Let (M^n, g) be a Riemannian manifold of dimension n. An* **N-cluster** *$\mathcal{E}$ of (M^n, g) is a finite family of sets of finite perimeter $\mathcal{E} := \{\mathcal{E}(h)\}_{h=1}^N$, $N \in \mathbb{N}$, $N \geq 1$, with*

$$0 < \mathrm{Vol}_g(\mathcal{E}(h)) < +\infty, \qquad 1 \leq h \leq N,$$
$$\mathrm{Vol}_g(\mathcal{E}(h) \cap \mathcal{E}(k)) = 0, \qquad 1 \leq h < k \leq N.$$

The sets $\mathcal{E}(h)$ are called the **chambers** *of $\mathcal{E}$. When the number N of the chambers of $\mathcal{E}$ is clear from the context, we shall use the term "cluster" in place of "N-cluster". The* **exterior chamber** *of $\mathcal{E}$ is defined as*

$$\mathcal{E}(0) = M^n \setminus \bigcup_{h=1}^N \mathcal{E}(h).$$

In, particular, $\{\mathcal{E}(h)\}_{h=0}^N$ is a partition of M^n (up to a set of null volume). The **volume vector** *$\mathbf{v}_g(\mathcal{E})$ is defined as*

$$\mathbf{v}_g(\mathcal{E}) = (\mathrm{Vol}_g(\mathcal{E}(1)), \ldots, \mathrm{Vol}_g(\mathcal{E}(N))) \in \mathbb{R}^N.$$

We let $\mathbb{R}^N_+$ be the set of those $\mathbf{v} \in \mathbb{R}^N$ such that $\mathbf{v}(h) > 0$ (the h-th component of a vector $\mathbf{v}$) for every $h = 1, \ldots, N$. Notice that if $\mathcal{E}$ is an N-cluster, then $\mathbf{v}_g(\mathcal{E}) \in \mathbb{R}^N_+ = [0, \mathrm{Vol}_g(M)[^N$ as $\mathbf{v}_g(\mathcal{E})(h) = \mathrm{Vol}_g(\mathcal{E}(h)) > 0$ for every $h = 1, \ldots, N$.

Remark. *It is important to notice that the chambers of a cluster are not assumed to be indecomposable, it is known that indecomposability is the commonly accepted notion of connectedness in the framework of sets of finite perimeter.*

Definition 14. *The* **interfaces** *of the N-cluster $\mathcal{E}$ in (M^n, g) are the $\mathcal{H}_g^{n-1}$-rectifiable sets*

$$\mathcal{E}(h,k) = \partial^*\mathcal{E}(h) \cap \partial^*\mathcal{E}(k), \quad 0 \leq h, k \leq N, h \neq k.$$

We define the **relative perimeter of $\mathcal{E}$ in** $F \subset M^n$ *as*

$$\operatorname{Per}_g(\mathcal{E}, F) = \sum_{0 \leq h < k \leq N} \mathcal{H}_g^{n-1}(F \cap \mathcal{E}(h,k))), \tag{2}$$

where F is any Borel set in (M^n, g). The **perimeter of $\mathcal{E}$** *is denoted* $\operatorname{Per}(\mathcal{E}) \doteq \operatorname{Per}_g(\mathcal{E}, M)$.

Definition 15. *The* **flat distance or flat norm in $F \subset M^n$ of two N-clusters $\mathcal{E}$ and $\mathcal{E}'$ of** (M^n, g) *is defined as*

$$d_{\mathcal{F},g}^F(\mathcal{E}, \mathcal{E}') := \sum_{h=1}^N \operatorname{Vol}_g\left(F \cap (\mathcal{E}(h) \Delta \mathcal{E}'(h))\right).$$

We set $d_{\mathcal{F},g}(\mathcal{E}, \mathcal{E}') = d_{\mathcal{F},g}^{M^n}(\mathcal{E}, \mathcal{E}')$. With this notation at hand, we say that a sequence of N-clusters $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ in (M^n, g) **locally converges to** $\mathcal{E}$, *and write $\mathcal{E}_k \overset{\text{loc}}{\to} \mathcal{E}$, if for every compact set $K \subset M^n$ we have $d_{\mathcal{F},g}^K(\mathcal{E}, \mathcal{E}_k) \to 0$ as $k \to +\infty$. If $d_{\mathcal{F},g}(\mathcal{E}, \mathcal{E}_k) \to 0$ as $k \to +\infty$, we say that $\mathcal{E}_k$* **converges to** $\mathcal{E}$ *and we denote $\mathcal{E}_k \to \mathcal{E}$.*

In order to simplify our formula for the relative perimeter, we will prove the following result which interestingly makes easy to prove the lower semicontinuity for sequences of clusters. In (2), we have the problem of working with the interfaces of the clusters which can be a tough task, since the intersection of the reduced boundary is not the same set of the reduced boundary of the intersection. To avoid this kind of problem, the formula provided by the following proposition turns out to be one of the key ideas to work with clusters in the way that it permits us to work with the perimeter of each chamber *separately* in a sum, as shown in the following proposition which is the Riemannian counterpart of Proposition 29.4 of [Mag12].

Proposition 1. *If $\mathcal{E}$ is an N-cluster in (M^n, g), then for every $F \subset M^n$ we have*

$$\operatorname{Per}_g(\mathcal{E}; F) = \frac{1}{2} \sum_{h=0}^N \operatorname{Per}_g(\mathcal{E}(h); F). \tag{3}$$

In particular, if A is open in M^n and $\mathcal{E}_k \overset{\text{loc}}{\to} \mathcal{E}$, then

$$\operatorname{Per}_g(\mathcal{E}; A) \leq \liminf_{k\to+\infty} \operatorname{Per}_g(\mathcal{E}_k; A). \tag{4}$$

For completeness, let us state the classical compactness criterion for a sequence of clusters which are contained in some fixed ball. The assumptions of the sequence of clusters be subset of a compact manifold (the ball they are contained in) is crucial for this compactness result, since we can use a simple tool that is the compactness criterion for **BV** functions which prove the criterion almost automatically. Without this assumptions, the problem of showing the existence of a "limit cluster" turns out to be quite hard and in general for a complete Riemannian manifold this result is no longer true, because, as it is well known, some part with positive volume of the sequence of clusters could disappear at infinity.

Proposition 2 (Compactness criterion for clusters)**.** *If $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ is a sequence of N-clusters in (M^n, g),*

$$\sup_{k\in\mathbb{N}} \operatorname{Per}_g(\mathcal{E}_k) < +\infty,$$
$$\inf_{k\in\mathbb{N}} \min_{1\le h\le N} \operatorname{Vol}_g(\mathcal{E}_k(h)) > 0$$

and

$$\mathcal{E}_k(h) \subset \mathrm{B}_g(p, R), \quad \forall k \in \mathbb{N}, h = 1, \ldots, N,$$

$R > 0$, for some $p \in M$, then there exist an N-cluster $\mathcal{E}$ in (M^n, g) with $\mathcal{E}(h) \subset \mathrm{B}_g(p, R)$ such that up to a subsequence $\mathcal{E}_k \to \mathcal{E}$ as $k \longrightarrow +\infty$.

Proof. The proof goes along the same lines of the Euclidean proof, see Proposition 29.5 in [Mag12]. □

Remark (Density properties at interfaces)**.** *If $\mathcal{E}$ is a N-cluster in (M^n, g), $p \in \mathcal{E}(h, k)$, $0 \le h < k \le N, j \ne h, k$, then we get*

$$\nu_{\mathcal{E}(h)}(p) = -\nu_{\mathcal{E}(k)}(p), \tag{5}$$
$$\theta_n(\mathcal{E}(j))(p) = 0, \tag{6}$$
$$\theta_{n-1}(\partial^*\mathcal{E}(j))(p) = 0. \tag{7}$$

We set the following notations that were used in the previous remark:

$$\theta_n(E)(p) = \lim_{r\to 0} \frac{\operatorname{Vol}_g(E \cap \mathrm{B}_g(p, r))}{\omega_n r^n}$$

and

$$\theta_{n-1}(E)(p) = \lim_{r\to 0} \frac{\operatorname{Per}_g(E, \mathrm{B}_g(p, r))}{\omega_{n-1} r^{n-1}}$$

for any finite perimeter set $E \subset M$ and $x \in M$, where ω_n is the volume of the unit ball on the Euclidean space of dimension n.

Remark. *If $\mathcal{E}$ is an N-cluster and $\Lambda \subset \{0, \ldots, N\}$, then*

$$\mathcal{H}_g^{n-1}\left(\partial^*\left(\bigcup_{h\in\Lambda} \mathcal{E}(h)\right) \setminus \bigcup_{h\in\Lambda, k\notin\Lambda} \mathcal{E}(h, k)\right) = 0.$$

1.4. **Isoperimetric clusters.** The main goal of this paper is to prove the existence of isoperimetric clusters in a generalized sense. An **isoperimetric cluster for volume $\mathbf{v} \in \mathbb{R}_+^N$** is an N-cluster $\mathcal{E}$ that solves the minimizing problem below which is also known as **multi-isoperimetric problem**, i.e. such that $\mathbf{v}_g(\mathcal{E}) = \mathbf{v}$ and

$$\operatorname{Per}_g(\mathcal{E}) = \inf\left\{\operatorname{Per}(\mathcal{E}') : \mathcal{E}' \text{ is an N-cluster with } \mathbf{v}_g(\mathcal{E}') = \mathbf{v}\right\}.$$

Similarly to the **isoperimetric problem** context, i.e. $N = 1$, we can define the **multi-isoperimetric profile function**, or **multi-isoperimetric profile**, as a function I_M from $[0, \operatorname{Vol}_g(M))^N$ to $[0, +\infty)$ given by

$$\mathrm{I}_M(\mathbf{v}) = \inf\left\{\operatorname{Per}(\mathcal{E}) : \mathcal{E} \text{ is an N-cluster in } (M^n, g) \text{ with } \mathbf{v}_g(\mathcal{E}) = \mathbf{v}\right\}. \tag{8}$$

This generalization of the isoperimetric profile for clusters has been explored in [MN18] with a Gaussian-weighted notion of perimeter which is induced by the Gaussian probability measure. These notions are all used in this work to solve and investigate the Gaussian Multi-Bubble Conjecture.

We notice that we prove the Hölder continuity of the multi-isoperimetric profile in Theorem 5. If either M is compact or exists a minimizing sequence contained in a compact subset of M, classical compactness arguments of geometric measure theory, i.e. Proposition 2 , combined with the direct method of the calculus of variations provide existence of isoperimetric clusters in any dimension n. These arguments are derivations from the theory of clusters in Euclidean spaces, for this setting we refer the reader to [CM17], [Mor94], [HM20] and [Mag12].

In the case that $N = 1$, we return to the classical isoperimetric problem which was extensively studied. The existence of isoperimetric regions in noncompact Riemannian manifolds is not a easy task. However, we can find papers in this direction which give pretty good answers in some specific types of manifolds, we refer the reader to [ABFP21], [MJ00, Section 4.6], [Rit01], [RR04], [CR08], [MN16], [Nar09], [MnFN19] and [Nar14]. In the sub-Riemannian setting we refer to [GR12], where Ritoré and Galli proved the existence of isoperimetric regions to the case of noncompact sub-Riemannian manifolds with cocompact isometry group. For more details on regularity theory see either [Mor03], [Mor09] or [Nar18]. Accordingly to these references, we could extract that we need some condition on the geometry of the manifold to prove existence of isoperimetric regions which we will call **bounded geometry**, defined in Definition 2. This condition has been studied by several mathematicians and [NP18] provided a counter example of a manifold which does not satisfy one of the bounded geometry conditions and hence does not contain isoperimetric regions for some volumes.

1.5. **Main theorems.** With the notions of multipointed C^0-convergence and basic clusters concepts, we can enunciate the generalized compactness theorem which assumes that the sequence of clusters has uniformly bounded perimeter and components of the volume vectors and then ensures the existence of a limit cluster in the multipointed C^0-topology at a union (possibly infinite) of multipointed limit manifolds.

Theorem 1 (Generalized Compactness for Sequences of Clusters)**.** *Suppose that (M^n, g) has C^0-bounded goemetry. Let $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ be a sequence of N-clusters in (M^n, g) with $\operatorname{Per}_g(\mathcal{E}_k) \leq P$ and $\mathbf{v}_g(\mathcal{E}_k)(h) \leq \mathbf{v}(h)$, for $h \in \{1, ..., N\}$. Then, up to a subsequence, there exists $J \in \mathbb{N} \cup \{+\infty\}$ such that, for all $j \in \{1, ...J\}$, there exist a sequence of points $(p^h_{jk})_{k\in\mathbb{N}} \subset M$, a manifold $(M^j_\infty(h), g)$, $(p^h_{j\infty})_{k\in\mathbb{N}} \subset M^j_\infty(h)$ and a formal disjoint union of finite perimeter set $\sqcup_{j=1}^J \mathcal{E}^j_\infty(h) = \mathcal{E}_\infty(h) \subset M_\infty(h) = \sqcup_{j=1}^J M^j_\infty(h)$, such that*

$$(\mathcal{E}_k(h), g, p^h_{jk}) \text{ converges to } (\mathcal{E}_\infty(h), g, p^h_{j\infty})$$

in the multipointed C^0-topology. Moreover, if we define the N-cluster $\mathcal{E}_\infty = \{\mathcal{E}_\infty(h)\}_{h=1}^N$ in the formal disjoint union of manifolds $M \sqcup \left(\sqcup_{h=1}^N M_\infty(h)\right)$, then $\mathbf{v}_g(\mathcal{E}_\infty) = \lim_{k\to+\infty} \mathbf{v}_g(\mathcal{E}_k)$ and $\operatorname{Per}_g(\mathcal{E}_\infty) = \lim_{k\to+\infty} \operatorname{Per}_g(\mathcal{E}_k)$.

As discussed in the introductory section, the main theorem of this work is the generalized existence of isoperimetric clusters, i.e. a cluster $\mathcal{E}$ that possibly lives at a formal disjoint union of limit multipointed manifolds which satisfies $\mathrm{I}_M(\mathbf{v}) = \operatorname{Per}_g(\mathcal{E})$, it is stated rigorously below following the notation of the past theorem.

Theorem 2 (Generalized Existence of Isoperimetric Clusters)**.** *Suppose that (M^n, g) has C^0-bounded goemetry. Let $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ be a minimizing sequence of N-clusters for $\mathbf{v} \in \mathbb{R}^N_+$. Then, up to a subsequence, for each $h \in \{1, \cdots, N\}$, there exists $J \in \mathbb{N}$, a formal disjoint union of manifolds*

$(M_\infty(h), g), M_\infty(h) = \sqcup_{j=1}^{J} M_\infty^j(h)$, J sequences of points $(p_{jk}^h)_{k\in\mathbb{N}} \subset M$ converging to J limits $p_{j\infty}^h \in M_\infty^j(h)$ and a formal disjoint union of N-cluster $\mathcal{E}_\infty$ in $(M_\infty, g), M_\infty = \sqcup_{h=1}^{N} M_\infty(h)$, such that

$$(\mathcal{E}_k(h), g, p_{jk}^h) \text{ converges to } (\mathcal{E}_\infty(h), g, p_{j\infty}^h),$$

in the multipointed C^0-topology. Moreover, $\mathbf{v}_g(\mathcal{E}_\infty) = \mathbf{v}$ and $\mathrm{Per}_g(\mathcal{E}_\infty) = \mathrm{I}_{M_\infty}(\mathbf{v}) = \mathrm{I}_M(\mathbf{v})$.

Remark. *The notation I_{M_∞} refers to the multi-isoperimetric profile of a formal disjoint union of manifolds and it is defined in the same way presented in (8).*

Remark. *In both Theorems 1 and 2, the sequence of points p_{jk}^h satisfies the following $d_g(p_{j_1k}^h, p_{j_2k}^h) \to +\infty$ as $k \to +\infty$ and $j_1 \neq j_2$. Such property follows directly by the construction of the* formal disjoint union of clusters at the limit *in Lemma 1.*

Theorem 2 is studied by Vincenzo Scattaglia in the work [Sca], where the author states a similar result for isoperimetric clusters with densities in the Euclidean space. In fact, the author consider densities in the definition of volume and perimeter, both densities functions are assumed to be in L^1_{loc}, lower semicontinuous, and bounded away from 0, in the main theorem which makes it considerably general.

We also show in the next result that isoperimetric clusters are always bounded which is the analogous of Theorem 3 of [Nar14] which was generalized to the set of metric spaces. Indeed, Antonelli, Pasqualetto, and Pozzetta in [APP21, Theorem 3.28] adapted the proof of [Nar14, Theorem 3] to the context of isoperimetric sets ($N = 1$) in RCD-metric space.

Theorem 3 (Boundedness of Isoperimetric Clusters)**.** *Let (M^n, g) be a Riemannian manifold with bounded geometry, then isoperimetric clusters are bounded.*

Since the results that we have been working with are supposed to generalize the classical existence and compactness results, it is natural to provide a proof of the classical result statement applying the previous results of this paper. [AFP21, Theorem 1.3] provides the analogue of the following theorem to the context of isoperimetric sets and just assuming Gromov-Hausdorff asymptoticity to the models which is a weaker condition than the one required in the next theorem and in [MN16] that is the C^0-locally asymptoticity.

Theorem 4 (Classical Existence of Isoperimetric Clusters)**.** *Let (M^n, g) be C^0-locally asymptotically the n-dimensional space form $\mathbb{M}_k^n$ of curvature k, $\mathrm{Ric}_g \geq k(n-1)$ (i.e. $\mathrm{Ric}_g \geq k(n-1)g$ in the sense of quadratic forms). Then, for every $\mathbf{v} \in \mathbb{R}_+^N$, there exist an isoperimetric cluster in M, i.e. an N-cluster $\mathcal{E}$ with*

$$\mathrm{I}_M(\mathbf{v}) = \mathrm{Per}_g(\mathcal{E}).$$

Now, we state the Hölder continuity of the multi-isoperimetric profile, the proof of the Hölder continuity of the classical isoperimetric profile was given by Nardulli and Flores in [MnFN19, Theorem 2], moreover, it was improved in [ABFP21, Corollary 3.5] to the Lipschitz continuity of the classical isoperimetric profile for Riemannian manifolds with Ricci curvature bounded by below and an Euclidean volume growth which surely includes the manifolds with bounded geometry.

Theorem 5 (Local Hölder continuity of the multi-isoperimetric profile)**.** *Let (M^n, g) be a manifold with bounded geometry. Then for any $\mathbf{v} \in]0, \mathrm{Vol}_g(M)[^N$ there exist constants $C := C(n, k) > 0$ and $R_\mathbf{v} := R_\mathbf{v}(n, k, N, v_0, \mathbf{v})$ such that for every $\mathbf{v}_1, \mathbf{v}_2 \in \mathrm{B}_{\mathbb{R}^N}(\mathbf{v}, R_\mathbf{v})$, we have that*

$$|\mathrm{I}_M(\mathbf{v}_1)-\mathrm{I}_M(\mathbf{v}_2)|\leq C(n,k,N,v_0)\left(\frac{|\mathbf{v}_1-\mathbf{v}_2|}{v_0}\right)^{\frac{n-1}{n}},$$

where $0<v_0=\inf\{\mathrm{Vol}_g\left(\mathrm{B}_M(p,1)\right):p\in M\}$ *is as in Definition 2 and*

$$R_{\mathbf{v}}=\frac{1}{C(n,k)}\min\left\{v_0,\sum_{h=1}^{N}\left(\frac{|\mathbf{v}(h)|}{\mathrm{I}_M(\mathbf{v})+C(n,k)}\right)^n\right\}.$$

1.6. Plan of the article.

(1) Section 1 constitutes the introduction of the basic concepts, the contextualization of the problem, a basic notion of manifold's convergence theory and the statements of the main results.
(2) In Section 2, we prove the main theorem (Theorem 2), the generalized compactness theorem (Theorem 1) and also the Hölder continuity of the multi-isoperimetric profile (Theorem 5). Moreover, we verify that isoperimetric clusters are bounded (Theorem 3).
(3) In Section 3, we show how the generalized existence theorem (Theorem 2) is used to prove classical existence theorems for isoperimetric clusters (Theorem 4).

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2. Proof of the main theorem

2.1. **The structure lemma and the generalized compactness theorem.** We will state the Lemma that provides a kind of structure for sequence of clusters with uniformly bounded perimeter and components of the volume vector. It will be useful since it splits each chamber of the cluster in various pieces (which can be infinite) with properties of volume and perimeter being well preserved. Moreover, some convergence properties in the $C^{m,\alpha}$ or Gromov-Hausdorff convergence sense will work very well too.

Lemma 1 (Structure Lemma for Sequences of Clusters)**.** *Assume that* (M^n,g) *has bounded geometry. Let* $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ *be a sequence of* N*-clusters in* (M^n,g) *with* $\mathrm{Per}_g(\mathcal{E}_k)\leq P$ *and* $\mathbf{v}_g\left(\mathcal{E}_k\right)(h)\leq\mathbf{v}(h),\forall h\in\{1,...,N\}$*, where* $P\in\mathbb{R}_+$ *and* $\mathbf{v}\in\mathbb{R}^N_+$*. Then there exists* $J\in\mathbb{N}\cup\{+\infty\}$ *such that, for all* $j\in\{1,...,J\}$ *and* $h\in\{1,...,N\}$*, there exist a sequence of points* $(p^h_{jk})_{k\in\mathbb{N}}$*, a sequence of radii* $R^h_{jk}\to+\infty$ *as* k *goes to* $+\infty$ *and a volume vector* $\mathbf{v}_j\in\mathbb{R}^N$*, such that, if we set* $\mathcal{E}^0_k(h)=\mathcal{E}_k(h)\cap\mathrm{B}_g\big(p^h_{0k},R^h_{0k}\big)$*,*

$$\mathcal{E}^j_k(h)=\mathcal{E}_k(h)\cap\mathrm{B}_g\left(p^h_{jk},R^h_{jk}\right)\setminus\mathring{\bigcup}_{i=0}^{j-1}\mathcal{E}^i_k(h),$$

for $j\geq1$*, and the* N*-cluster* $\mathcal{E}'_k=\{\cup_{j=0}^{J}\mathcal{E}^j_k(h)\}_{h=1}^N$*, the following properties hold*

$(i):$ $\mathbf{v}_j(h)=\lim_{k\to+\infty}\mathrm{Vol}_g\left(\mathcal{E}^j_k(h)\right)$*, for each* $h\in\{1,...,N\}$*,*

$(ii):$ $0 < \mathbf{v}_g\left(\mathcal{E}'_k\right)(h) = \sum_{j=0}^{J} \mathbf{v}_j(h) \leq \mathbf{v}(h)$. *Moreover, if* $\mathbf{v}_g\left(\mathcal{E}_k\right) \to \mathbf{v}$, *then* $\mathbf{v}_g\left(\mathcal{E}'_k\right) \to \sum_{j=0}^{J} \mathbf{v}_j = \mathbf{v}$,

$(iii):$ $\lim_{k\to+\infty} \operatorname{Per}_g(\mathcal{E}'_k) \leq \lim_{k\to+\infty} \operatorname{Per}_g(\mathcal{E}_k) \leq P$.

Proof. By the Concentration-Compactness Lemma (Lemma 2.1 of [Nar14], the bounded geometry hypothesis, Definition 2, guarantees that property (iii) of the Lemma 2.1 holds), there exist $\mathbf{v}_0(h) \in (0, \mathbf{v}(h))$ and a sequence of points $(p^h_{0k})_{k\in\mathbb{N}}$ such that for any sequence $\epsilon^{0,k} \to 0$ there exists a sequence of radii $(R^{0,k}_h)_{k\in\mathbb{N}}$ which satisfies

$$\left|\operatorname{Vol}_g\left(\mathcal{E}_k(h) \cap \mathrm{B}_g\left(p^h_{0k}, R'\right)\right) - \mathbf{v}_0(h)\right| < \epsilon^{0,k},$$

whenever $R' \geq R^{0,k}_h$ and k sufficiently large. By the mean value property for integrals, we find a radii $R^h_{0k} \in [R^{0,k}_h, R^{0,k}_h + k]$ such that

$$\operatorname{Per}_g\left(\mathcal{E}_k(h) \cap \mathrm{B}_g\left(p^h_{0k}, R^h_{0k}\right)\right) = \frac{1}{R^{0,k}_h + k - R^{0,k}_h} \int_{R^{0,k}_h}^{R^{0,k}_h + k} \operatorname{Per}_g\left(\mathcal{E}_k(h) \cap \mathrm{B}_g\left(p^h_{0k}, r\right)\right) \mathrm{d}r. \tag{9}$$

Without loss of generality, we can assume that $\mathcal{H}^{n-1}_g\left(\partial^*\mathcal{E}_k(h) \cap \partial\mathrm{B}_g\left(p^h_{0k}, R^h_{0k}\right)\right) = 0$. Therefore, under this assumption, by the operations with the perimeter measure, (9) and the coarea formula, we obtain the following

$$\begin{aligned}
&\sum_{h=1}^{N} \operatorname{Per}_g\left(\mathcal{E}_k(h) \cap \mathrm{B}_g\left(p^h_{0k}, R^h_{0k}\right)\right) \\
&= \sum_{h=1}^{N} \frac{1}{k} \int_{R^{0,k}_h}^{R^{0,k}_h + k} \operatorname{Per}_g\left(\mathcal{E}_k(h), \mathrm{B}_g\left(p^h_{0k}, r\right)^{(1)}\right) + \operatorname{Per}_g\left(\mathrm{B}_g\left(p^h_{0k}, r\right), \mathcal{E}_k(h)^{(1)}\right) \mathrm{d}r \\
&\leq \sum_{h=1}^{N} \operatorname{Per}_g\left(\mathcal{E}_k(h)\right) + \frac{1}{k} \sum_{h=1}^{N} \mathbf{v}(h) \\
&\leq P + \frac{1}{k} \sum_{h=1}^{N} \mathbf{v}(h).
\end{aligned} \tag{10}$$

We will fix the notation $\mathcal{E}^0_k(h) \doteq \mathcal{E}_k(h) \cap \mathrm{B}_g\left(p^h_{0k}, R^h_{0k}\right)$. In order to repeat this process and therefore obtain a better approximation for the volume of the chamber $\mathcal{E}_k(h)$, we apply the Concentration-Compactness Lemma for the set

$$\mathcal{E}_k(h) \setminus \mathcal{E}^0_k(h).$$

Thus, we get the existence of $\mathbf{v}_1(h) \in (0, \mathbf{v}_1(h))$ and a sequence of points (p^h_{1k}) such that for all sequence $\epsilon^{1,k} \to 0$ there exists a sequence of radii $(R^{1,k}_h)_{k\in\mathbb{N}}$ with the following property

$$\left|\operatorname{Vol}_g\left(\mathcal{E}_k(h) \cap \mathrm{B}_g\left(p^h_{1k}, R'\right) \setminus \mathcal{E}^0_k(h)\right) - \mathbf{v}_1(h)\right| < \epsilon^{1,k},$$

whenever $R' \geq R_h^{1,k}$ and k sufficiently large and then we also set $\mathcal{E}_k^1(h) = \mathcal{E}_k(h) \cap \mathrm{B}_g\left(p_{1k}^h, R_{1k}^h\right) \setminus \mathcal{E}_k^0(h)$ with R_{1k}^h such that (10) holds for $\mathcal{E}_k^1(h)$. Now, we are in position to apply the Concetration-Compactness Lemma inductively, using in the j-th step, $j \geq 2$, the set

$$\mathcal{E}_k(h) \setminus \bigcup_{i=0}^{j-1} \mathcal{E}_k^i(h),$$

hence obtaining $\mathbf{v}_j(h) \in (0, \mathbf{v}_{j-1}(h))$ and (p_{jk}^h) such that for all sequence $\epsilon^{j,k} \to 0$ there exists a sequence of radii $(R_h^{j,k})_{k\in\mathbb{N}}$ with the following property

$$\left|\mathrm{Vol}_g\left(\mathcal{E}_k(h) \cap \mathrm{B}_g\left(p_{jk}^h, R'\right) \setminus \bigcup_{i=0}^{j-1} \mathcal{E}_k^i(h)\right) - \mathbf{v}_j(h)\right| < \epsilon^{j,k}, \tag{11}$$

whenever $R' \geq R_h^{j,k}$ and k sufficiently large, we inductively denote by

$$\mathcal{E}_k^j(h) = \mathcal{E}_k(h) \cap \mathrm{B}_g\left(p_{jk}^h, R_{jk}^h\right) \setminus \bigcup_{i=0}^{j-1} \mathcal{E}_k^i(h),$$

where R_{jk}^h is taken to satisfy (10) holds for $\mathcal{E}_k^j(h)$. So, we shall iterate the algorithm until we reach the desired $J \in \mathbb{N} \cup \{+\infty\}$. Finally, we certainly have that

$$\mathbf{v}_j(h) = \lim_{k\to+\infty} \mathrm{Vol}_g\left(\mathcal{E}_k^j(h)\right) \leq \mathbf{v}(h).$$

We can easily see by the construction that the first assertion of item (ii) is already proved. Therefore we suppose that $\mathbf{v}_g(\mathcal{E}_k) \to \mathbf{v}$ and $\sum_{j=0}^J \mathbf{v}_j(h) < \mathbf{v}(h)$, it is a direct consequence of Lemma 2.5 of [Nar14] applied to the set $\mathcal{E}_k(h) \setminus \mathcal{E}_k^j(h)$ that exists $p' \in M$ such that

$$\mathbf{v}_{j+1}(h) \geq \mathrm{Vol}_g\left(\mathrm{B}_g\left(p', R_h^{j,k}\right) \cap \left(\mathcal{E}_k(h) \setminus \mathcal{E}_k^j(h)\right)\right) \geq c_3(n,k,v_0)\frac{\mathrm{Vol}_g\left(\mathcal{E}_k(h) \setminus \mathcal{E}_k^j(h)\right)^n}{\mathrm{Per}_g(\mathcal{E}_k(h) \setminus \mathcal{E}_k^j(h))^n}.$$

Passing through the limit as k and j goes to $+\infty$, we obtain that

$$\lim_{j\to+\infty} \mathbf{v}_j(h) \geq c_3(n,k,v_0)\frac{\left(\mathbf{v}(h) - \sum_{j=1}^J \mathbf{v}_j(h)\right)^n}{\mathrm{Per}_g(\mathcal{E}_k(h) \setminus \mathcal{E}_k^j(h))^n} > 0.$$

which is a contradiction with the fact that $\sum_{j=0}^J \mathbf{v}_j(h)$ is a convergent series and thus ensuring the validity of (ii). Since (10) is true for all $j \in \{0, 1, ..., J\}$, Proposition 1 finishes the proof of (iii) and of this Lemma. □

Let us prove the generalized compactness for sequences of clusters (Theorem 1).

Proof of Theorem 1. By Gromov's Compactness Theorem and a diagonalization process applied to the sets $\mathcal{E}_k^j(h)$ of Lemma 1, we ensure the existence of the formal disjoint union of manifolds $(M_\infty(h), g)$, the sequence of points $(p_{j\infty}^h)$ and the formal disjoint union of finite perimeter set $\mathcal{E}_\infty(h)$. T. Colding in [Col97] shows that the volume vectors will converge as desired, in view of the convergence of the perimeter, the C^0-bounded geometry assumption do all the work since all the notions in the definition of perimeter (1) are well transported to the limit manifolds by C^0-convergence of metrics. □

2.2. **Hölder continuity of the multi-isoperimetric profile.** The Hölder continuity of the isoperimetric profile will be used in the next step of our framing of the proof of Theorem 2.

Proof of Theorem 5. The proof goes in the same steps taken in Theorem 2 of [MnFN19] with minor modifications. Let us briefly record it here, given $\epsilon > 0$, we can find $\mathcal{E}$ an N-cluster such that $\mathbf{v}_g(\mathcal{E}) = \mathbf{v}$ and $\mathrm{Per}_g(\mathcal{E}) \leq \mathrm{I}_M(\mathbf{v}) + \epsilon$. Let us define Λ_1 as the set of those $h \in \{1, ..., N\}$ which $\mathbf{v}(h) \leq \mathbf{v}'(h)$. We take p_h^1 and $r_{\mathbf{v}'(h)}$, for $h \in \Lambda_1$, in order to have $\mathcal{E}_1(h) = \mathcal{E}(h) \cup \mathrm{B}_M(p_h^1, r_{\mathbf{v}'(h)})$ with $\mathrm{Vol}_g(\mathcal{E}_1(h)) = \mathbf{v}'(h)$. From the spherical Bishop-Gromov's Theorem, we obtain that

$$\mathrm{Per}_g\left(\mathrm{B}_M(p_h^1, r_{\mathbf{v}'(h)})\right) \leq C_1(n,k) r_{\mathbf{v}'(h)}^{n-1} \leq C_2(n,k)\left(\frac{\mathbf{v}'(h) - \mathbf{v}(h)}{v_0}\right)^{\frac{n-1}{n}}, \tag{12}$$

for $h \in \Lambda_1$ and $\mathbf{v}'(h) - \mathbf{v}(h) \leq v_0$. Let us define Λ_2 as the complementary set of Λ_1, i.e. $\Lambda_2 = \{1, ..., N\} \setminus \Lambda_1$. Then, for $h \in \Lambda_2$, we apply Lemma 2.5 of [Nar14] for $\mathcal{E}(h)$ which furnishes, for any $\mathbf{v}'(h) \in]\mathbf{v}(h) - l, \mathbf{v}(h)[$, the inequality

$$\begin{aligned} &\mathrm{Vol}_g\left(\mathcal{E}(h) \cap \mathrm{B}_g\left(p_h^2, \left(\frac{\mathbf{v}(h) - \mathbf{v}'(h)}{cv_0}\right)^n\right)\right) \\ &\geq \min\left\{\mathbf{v}(h) - \mathbf{v}'(h), c\left(\frac{\mathbf{v}(h)}{\mathrm{I}_M(\mathbf{v}) + \epsilon}\right)^n\right\} \\ &= \mathbf{v}(h) - \mathbf{v}'(h), \end{aligned}$$

where $l = c\min\{v_0, \left(\frac{\mathbf{v}(h)}{\mathrm{I}_M(\mathbf{v})+\epsilon}\right)^n\}$. Note that, since the last inequality holds for $\mathbf{v}'(h) \in]\mathbf{v}(h) - l, \mathbf{v}(h)[$, we need that

$$\mathbf{v}(h) - \mathbf{v}'(h) \leq l = c\min\{v_0, \left(\frac{\mathbf{v}(h)}{\mathrm{I}_M(\mathbf{v}) + \epsilon}\right)^n\}, \tag{13}$$

if we look at the condition to establish (12), take ϵ small enough and possibly adjusting the constant $C_2(n,k)$, we see that $|\mathbf{v}' - \mathbf{v}| \leq R_{\mathbf{v}}$ is sufficient to ensure the validity of (13) and (12) in each correspondent case. We thus choose $r_{\mathbf{v}'(h)}$, for $h \in \Lambda_2$, such that the finite perimeter set $\mathcal{E}_2(h) = \mathcal{E}(h) \setminus \mathrm{B}_g(p_h^2, r_{\mathbf{v}'(h)})$ satisfies $\mathrm{Vol}_g(\mathcal{E}_2(h)) = \mathbf{v}'(h)$ and again by Bishop-Gromov's Theorem we have that

$$\mathrm{Per}_g\left(\mathrm{B}_M(p_h^2, r_{\mathbf{v}'(h)})\right) \leq C_2(n,k)\left(\frac{\mathbf{v}(h) - \mathbf{v}'(h)}{v_0}\right)^{\frac{n-1}{n}}. \tag{14}$$

Finally, we define the finite perimeter sets $\mathcal{E}'(h) = \mathcal{E}_1(h)$, for $h \in \Lambda_1$, $\mathcal{E}'(h) = \mathcal{E}_2(h)$, for $h \in \Lambda_2$ and thus the cluster $\mathcal{E}' = \{\mathcal{E}'(h)\}_{h=1}^N$ satisfies $\mathbf{v}_g(\mathcal{E}') = \mathbf{v}'$. We put Proposition 1, (12) and (14) into account to obtain

$$\begin{aligned}
\mathrm{I}_M(\mathbf{v}') &\leq \mathrm{Per}_g(\mathcal{E}') \\
&\leq \frac{1}{2}\Bigg(\sum_{h\in\Lambda_1} \mathrm{Per}_g(\mathcal{E}_1(h)) + \sum_{h\in\Lambda_2} \mathrm{Per}_g(\mathcal{E}_2(h))\Bigg) \\
&\leq \frac{1}{2}\sum_{h\in\Lambda_1}\Big(\mathrm{Per}_g(\mathcal{E}(h)) + \mathrm{Per}_g\big(\mathrm{B}_g(p_h^1, r'_{\mathbf{v}}(h))\big)\Big) \\
&+ \frac{1}{2}\sum_{h\in\Lambda_2}\Big(\mathrm{Per}_g(\mathcal{E}(h)) + \mathrm{Per}_g\big(\mathrm{B}_g(p_h^2, r'_{\mathbf{v}}(h))\big)\Big) \\
&\leq \mathrm{Per}_g(\mathcal{E}) + C_2(n,k)\sum_{h=1}^N \left(\frac{|\mathbf{v}'(h) - \mathbf{v}(h)|}{v_0}\right)^{\frac{n-1}{n}} \\
&\leq \mathrm{I}_M(\mathbf{v}) + \epsilon + C_2(n,k)\sum_{h=1}^N \left(\frac{|\mathbf{v}'(h) - \mathbf{v}(h)|}{v_0}\right)^{\frac{n-1}{n}}.
\end{aligned}$$

Note that, once we prove $\mathbf{v} \in \mathrm{B}_{\mathbb{R}^N}(\mathbf{v}', R_{\mathbf{v}'})$, we can exchange $\mathbf{v}$ and $\mathbf{v}'$ to obtain the latter estimate with the modulus, i.e.,

$$|\mathrm{I}_M(\mathbf{v}) - \mathrm{I}_M(\mathbf{v}')| \leq C_2(n,k)\sum_{h=1}^N \left(\frac{|\mathbf{v}'(h) - \mathbf{v}(h)|}{v_0}\right)^{\frac{n-1}{n}}.$$

To this end, we note that $\mathrm{I}_M(\mathbf{v}') \leq \mathrm{I}_M(\mathbf{v}) + C_2(n,k)$ for $R_{\mathbf{v}}$ small enough and thus

$$\begin{aligned}
R_{\mathbf{v}}^{\frac{1}{n}} &\leq \frac{|\mathbf{v}(h)|}{\mathrm{I}_M(\mathbf{v}) + C_2(n,k)} \\
&\leq \frac{|\mathbf{v}'(h)| + R_{\mathbf{v}}}{\mathrm{I}_M(\mathbf{v}) + C_2(n,k)} \\
&\leq \frac{|\mathbf{v}'(h)|}{\mathrm{I}_M(\mathbf{v}) + C_2(n,k)} + \frac{R_{\mathbf{v}}}{\mathrm{I}_M(\mathbf{v}) + C_2(n,k)}.
\end{aligned}$$

Since, in particular, $R_{\mathbf{v}} < 1$, we obtain

$$\left(1 - \frac{1}{\mathrm{I}_M(\mathbf{v}) + C_2(n,k)}\right) R_{\mathbf{v}}^{\frac{1}{n}} \leq \frac{|\mathbf{v}'(h)|}{\mathrm{I}_M(\mathbf{v}) + C_2(n,k)}.$$

If needed, increasing the constant C_2 we have that $\frac{1}{2}R_{\mathbf{v}}^{\frac{1}{n}} \leq \frac{|\mathbf{v}'(h)|}{\mathrm{I}_M(\mathbf{v}) + C_2(n,k)}$. Furthermore, it is a matter of readjusting the constant and, if necessary, reducing $R_{\mathbf{v}}$ again to obtain that $R_{\mathbf{v}} \leq R_{\mathbf{v}'}$ which ensures $\mathbf{v} \in \mathrm{B}_{\mathbb{R}^N}(\mathbf{v}', R_{\mathbf{v}'})$. Now, take $\mathbf{v}_1, \mathbf{v}_2 \in \mathrm{B}_{\mathbb{R}^N}\left(\mathbf{v}, \frac{1}{2}R_{\mathbf{v}}\right)$, then $|\mathbf{v}_1 - \mathbf{v}_2| \leq |\mathbf{v}_1 - \mathbf{v}| + |\mathbf{v}_2 - \mathbf{v}| \leq R_{\mathbf{v}}$ and, by the conclusions above $\min\{R_{\mathbf{v}_1}, R_{\mathbf{v}_2}\} \geq R_{\mathbf{v}}$, we prove $\mathbf{v}_1 \in \mathrm{B}_{\mathbb{R}^N}(\mathbf{v}_2, R_{\mathbf{v}_2})$ and $\mathbf{v}_2 \in \mathrm{B}_{\mathbb{R}^N}(\mathbf{v}_1, R_{\mathbf{v}_1})$ which concludes the proof of the theorem. □

2.3. **The structure lemma for minimizing sequences and the generalized existence theorem.** Sequences of clusters satisfying the hypothesis of Lemma 2 are called either **minimizing sequences of N-clusters for $\mathbf{v} \in \mathbb{R}^N_+$** or **minimizing sequences for the multi-isoperimetric problem**. In Lemma 1 we only assumed that (M^n, g) has bounded geometry and the components of the vector volumes and the perimeters of the sequence of the clusters were uniformly bounded. However, if we force the sequence of clusters to be a minimizing sequence for the multi-isoperimetric problem, we can show what is the limit of the sequence of the perimeters instead of the simply existence of it provided by item (iii) of the Lemma 1. Moreover, this stronger assumption on the sequence of clusters put us in position to prove that the number of pieces that we split the clusters is finite, i.e. $J \in \mathbb{N}$.

Lemma 2 (Structure Lemma for Minimizing Sequences of Clusters)**.** *Assume that (M^n, g) has bounded geometry. Let $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ be a sequence of N-clusters in (M, g) with $\mathbf{v}_g(\mathcal{E}_k) = \mathbf{v} \in \mathbb{R}^N_+$, $\forall k \in \mathbb{N}$, and $\operatorname{Per}_g(\mathcal{E}_k) \to \mathrm{I}_M(\mathbf{v})$. Then there exists $J \in \mathbb{N}$ such that, for all $j \in \{1, ...J\}$ and $h \in \{1, ...N\}$, there exist a sequence of points $(p^h_{jk})_{k\in\mathbb{N}}$, a sequence of radii $R^h_{jk} \to +\infty$ as k goes to $+\infty$ and a volume vector $\mathbf{v}_j \in \mathbb{R}^N$, such that, if we set $\mathcal{E}^0_k(h) = \mathcal{E}_k(h) \cap \mathrm{B}_g\big(p^h_{0k}, R^h_{0k}\big)$,*

$$\mathcal{E}^j_k(h) = \mathcal{E}_k(h) \cap \mathrm{B}_g\left(p^h_{jk}, R^h_{jk}\right) \setminus \mathring{\bigcup}_{i=0}^{j-1} \mathcal{E}^i_k(h),$$

for $j \geq 1$, and the N-cluster $\mathcal{E}'_k = \{\cup_{j=0}^J \mathcal{E}^j_k(h)\}_{h=1}^N$, the following properties hold

- (i) : $\mathbf{v}_j(h) = \lim_{k\to+\infty} \operatorname{Vol}_g\left(\mathcal{E}^j_k(h)\right)$, *for each* $h \in \{1, ..., N\}$,
- (ii) : $\mathbf{v}_g(\mathcal{E}'_k) \to \sum_{j=0}^J \mathbf{v}_j = \mathbf{v}$,
- (iii) : $\lim_{k\to+\infty} \operatorname{Per}_g(\mathcal{E}'_k) = \lim_{k\to+\infty} \operatorname{Per}_g(\mathcal{E}_k) = \mathrm{I}_M(\mathbf{v})$.

Remark. *We notice that item (ii) shows that the new sequence of clusters $\mathcal{E}'_k$ is not needed to be minimizing for the multi-isoperimetric problem since we just guaranteed the convergence of the vector volume sequence $\mathbf{v}_g(\mathcal{E}'_k)$ to the vector volume $\mathbf{v}$.*

Proof. The unique part that is not a particular case of Lemma 1 is item (iii). Aiming to prove (iii), once we prove that J is finite, we can use that the multi-isoperimetric profile of M coincides with the multi-isoperimetric profile of a formal disjoint union of pointed limits of M, the proof of this fact is done at the end, and we use the continuity of I_M given by Theorem 5 to get that

$$\mathrm{I}_M(\mathbf{v}) = \lim_{k\to+\infty} \mathrm{I}_M(\mathbf{v}_g(\mathcal{E}'_k)) \leq \lim_{k\to+\infty} \operatorname{Per}_g(\mathcal{E}'_k).$$

Furthermore, item (iii) of Lemma 1 ensures the reverse inequality. To see that $J < +\infty$, we proceed by contradiction. As in the second proof of Theorem 2 in [Nar14], we can prove the existence of a constant M, which does not depend on J, such that

$$M \geq \frac{\mathrm{I}_M(\mathbf{v}'_J)}{\|\mathbf{v}'_J\|}, \quad \mathbf{v}'_J = \mathbf{v} - \sum_{j=0}^J \mathbf{v}_j, \tag{15}$$

whenever J is sufficiently large and thus $\mathbf{v}'_J$ has small norm. Now, we want to obtain a lower bound of $\mathrm{I}_M(\mathbf{v}'_J)$, possibly depending on $\mathbf{v}'_J$, in order to lead (15) into a contradiction. By Proposition 1, and the fact that $\operatorname{Per}_g(A) + \operatorname{Per}_g(B) \geq \operatorname{Per}_g(A \cup B)$ for any Caccioppoli sets A and B, we have

$$\begin{aligned}\mathrm{I}_M(\mathbf{v}'_J) &\geq \inf\left\{\frac{1}{2}\sum_{h=1}^{N}\operatorname{Per}_g(\mathcal{E}(h)) : \mathcal{E} \text{ is a N-cluster with } \mathbf{v}_g(\mathcal{E}) = \mathbf{v}'_J\right\}\\ &\geq \frac{1}{2}\inf\left\{\operatorname{Per}_g\left(\bigcup_{h=1}^{N}\mathcal{E}(h)\right) : \mathcal{E} \text{ is a N-cluster with } \mathbf{v}_g(\mathcal{E}) = \mathbf{v}'_J\right\},\end{aligned}$$

Using the latter inequality we now obtain

$$\begin{aligned}\mathrm{I}_M(\mathbf{v}'_J) &\geq \frac{1}{2}\inf\left\{\operatorname{Per}_g\left(\bigcup_{h=1}^{N}\mathcal{E}(h)\right) : \mathcal{E} \text{ is a N-cluster with } \mathbf{v}_g(\mathcal{E}) = \mathbf{v}'_J\right\}\\ &\overset{(*)}{\geq} \frac{1}{2C}\inf\left\{\operatorname{Vol}_g\left(\bigcup_{h=1}^{N}\mathcal{E}(h)\right)^{\frac{n-1}{n}} : \mathcal{E} \text{ is a N-cluster with } \mathbf{v}_g(\mathcal{E}) = \mathbf{v}'_J\right\}\\ &= \frac{1}{2C}\inf\left\{\left(\sum_{h=1}^{N}\operatorname{Vol}_g(\mathcal{E}(h))\right)^{\frac{n-1}{n}} : \mathcal{E} \text{ is a N-cluster with } \mathbf{v}_g(\mathcal{E}) = \mathbf{v}'_J\right\}\\ &= \frac{1}{2C}\left(\sum_{h=1}^{N}\mathbf{v}'_J(h)\right)^{\frac{n-1}{n}},\end{aligned}$$

where in $(*)$ we take J big enough to apply the Isoperimetric Inequality (Proposition 3 below) for small volume, i.e. $\|\mathbf{v}'_J\|$ sufficiently small. The last chain of inequalities and (15) provide

$$2MC \geq \frac{\left(\sum_{h=1}^{N}\mathbf{v}'_J(h)\right)^{\frac{n-1}{n}}}{\|\mathbf{v}'_J\|} \overset{(*)}{\geq} \frac{K(n,N)\|\mathbf{v}'_J\|^{\frac{n-1}{n}}}{\|\mathbf{v}'_J\|} = \frac{K(n,N)}{\|\mathbf{v}'_J\|^{\frac{1}{n}}} \longrightarrow +\infty, \tag{16}$$

as J goes to $+\infty$, where in $(*)$ the constant $K(n,N)$, which does not depend on J, appears from the equivalence of all norms in Euclidean spaces (specifically $\mathbb{R}^N$). Equation (16) states the contradiction that we needed. So, J has to be finite. Now, to prove the assertion at the beginning, it is easy to see that $\mathrm{I}_M \geq \mathrm{I}_{M_\infty}$ where $(M_\infty, g_\infty) = \sqcup_{i=1}^{J}\lim_{i\to+\infty}(M, x_i, g)$, note that we already know that J is finite, and the limit is taken with respect to the $C^{m,\alpha}$ topology as defined and explained in the introductory section. To prove the reverse inequality, we fix $\mathbf{v} \in \mathrm{B}(0, \operatorname{Vol}_g(M))$ and $\mathcal{E}_\infty(h) \subset M_\infty$ any domain of volume $\mathbf{v}(h)$. Due to the $C^{m,\alpha}$ convergence, we set $\mathcal{E}_i(h) = F_i^{-1}(\mathcal{E}_\infty(h)) \subset M$ with $\mathbf{v}_i(h) = \operatorname{Vol}_g(\mathcal{E}_i(h))$ satisfying $\mathbf{v}_i(h) \to \mathbf{v}(h)$ and $\operatorname{Per}_g(\mathcal{E}_i(h)) \to \operatorname{Per}_{g_\infty}(\mathcal{E}_\infty(h))$ as i goes to $+\infty$. Now, we suppose that $\mathrm{I}_M(\mathbf{v}) > \mathrm{I}_{M_\infty}(\mathbf{v})$, in particular,

$$\mathrm{I}_M(\mathbf{v}) > \operatorname{Per}_{g_\infty}(\mathcal{E}_\infty) \geq \mathrm{I}_{M_\infty}(\mathbf{v}), \tag{17}$$

where $\mathcal{E}_\infty = \{\mathcal{E}_\infty(h)\}_{h=1}^{N}$. Applying the readjustment of volumes process given in [Nar14, Lemma 2.7] to each chamber of the limit cluster $\mathcal{E}_\infty$, we directly obtain a new limit cluster $\mathcal{E}_\infty^i = \{\mathcal{E}_\infty^i(h)\}_{h=1}^{N}$ with

$$\operatorname{Per}_{g_\infty}(\mathcal{E}_\infty^i) \leq \operatorname{Per}_{g_\infty}(\mathcal{E}_\infty) + c|\mathbf{v}^i|,$$

where $\mathbf{v}^i \to 0$, and we set $\mathcal{G}_i = \{F_i(\mathcal{E}_\infty^i(h))\}_{h=1}^{N}$ with $\operatorname{Vol}_g(\mathcal{G}_i(h)) = \mathbf{v}(h)$. In short, we obtained a cluster $\mathcal{G}$ in M such that $\mathbf{v}_g(\mathcal{G}) = \mathbf{v}$ and $\operatorname{Per}_g(\mathcal{G}_i) \to \operatorname{Per}_{g_\infty}(\mathcal{E}_\infty)$ which is a contraction, since by (17), for i large enough, we would have $\operatorname{Per}_g(\mathcal{G}_i) < \mathrm{I}_M(\mathbf{v})$. □

Finally, we are in position to prove Theorem 2 which states the existence of minimizing clusters for the multi-isoperimetric problem.

Proof of Theorem 2. The proof turns out to be a simple application of Theorem 1 and Lemma 2. ☐

2.4. **Boundedness of isoperimetric clusters.** We recall the statement of the classical isoperimetric inequality in the Riemannian setting which can be consulted in Lemma 3.2 of Emmanuel Hebey work ([Heb00]), it requires that E is open and has smooth boundary. Nevertheless, we state the inequality for finite perimeter sets.

Lemma 3 (Isoperimetric Inequality)**.** *Let (M^n, g) be a complete Riemannian manifold with bounded geometry. There exist positive constants C and η depending only on n, k and v_0 (Definition 2) such that, for any finite perimeter set E of (M^n, g) with $\mathrm{Vol}_g(E) \leq \eta$, we have*

$$\mathrm{Vol}_g(E)^{\frac{n-1}{n}} \leq C\,\mathrm{Per}_g(E).$$

Proof. The proof of this inequality is a standard approximation argument using Lemma 3.2 of [Heb00] and Lemma 2.4 of [FN14]. ☐

The recent work of Antonelli, Pasqualetto, and Pozzetta, [APP21], generalized this well known result to a more general context which included the aforementioned RCD-metric spaces.

The proof of the boundedness of isoperimetric clusters goes back to ideas given by Frank Morgan on Chapter 13 of [Mor09] in the Euclidean setting. We adapted these ideas to the context of Riemannian manifolds with bounded geometry.

Proof of Theorem 3. Let $\mathcal{E}$ be an isoperimetric N-cluster of (M^n, g) and fix $p \in M$. Let us define the function $V : (0, +\infty) \to (0, +\infty)$ which measures the amount cluster's volume outside of large balls as follows

$$V(r) = \sum_{h=1}^{N} \mathrm{Vol}_g(\mathcal{E}(h) \setminus \mathrm{B}_g(p, r)).$$

By the coarea formula, we get that

$$V'(r) = -\sum_{h=1}^{N} \mathrm{Per}_g(\mathrm{B}_g(p, r), \mathcal{E}(h)). \tag{18}$$

If we set $A_h(r) = \mathrm{Per}_g(\mathcal{E}(h), M \setminus \mathrm{B}_g(p, r))$, equation (18), standard arguments with Caccioppoli sets and operations with the perimeter measure yield

$$|V'(r)| + \sum_{h=1}^{N} A_h(r) = \sum_{h=1}^{N} \mathrm{Per}_g(\mathcal{E}(h) \cap (M \setminus \mathrm{B}_g(p, r))), \tag{19}$$

for almost all r big enough. From standard Riemannian comparison geometry techniques, we can easily see that $V(r)$ is decreasing and tends to 0 as r goes to $+\infty$. Then, for almost all r sufficiently large, we can apply Lemma 3, (19) and the inequality $a^{\frac{n-1}{n}} + b^{\frac{n-1}{n}} \geq (a+b)^{\frac{n-1}{n}}, a, b \geq 0$ to obtain

$$
\begin{aligned}
|V'(r)| + \sum_{h=1}^{N} A_h(r) &\geq C \sum_{h=1}^{N} \mathrm{Vol}_g \left(\mathcal{E}(h) \cap (M \setminus \mathrm{B}_g(p,r))\right)^{\frac{n-1}{n}} \\
&\geq C \left(\sum_{h=1}^{N} \mathrm{Vol}_g \left(\mathcal{E}(h) \cap (M \setminus \mathrm{B}_g(p,r))\right) \right)^{\frac{n-1}{n}} \\
&\geq C \mathrm{Vol}_g \left(\left(\bigcup_{h=1}^{N} \mathcal{E}(h) \right) \cap (M \setminus \mathrm{B}_g(p,r)) \right)^{\frac{n-1}{n}} \\
&\overset{(*)}{=} C \left(\sum_{h=1}^{N} \mathrm{Vol}_g \left(\mathcal{E}(h) \cap (M \setminus \mathrm{B}_g(p,r))\right) \right)^{\frac{n-1}{n}} \\
&= C V(r)^{\frac{n-1}{n}}.
\end{aligned} \tag{20}
$$

where, by the definition of clusters (Definition 13), $(*)$ follows from the fact that the chambers do not overlap in the measure-theoretic sense (i.e. the intersection has volume zero). By [Mor09, Lemma 13.5], which holds in our setting by a straightforward adaptation of the proof, Proposition 1, and using the fact that the cluster is isoperimetric, we have

$$
|V'(r)| + cV(r) \geq \mathrm{Per}_g \left(\mathcal{E}, M \setminus \mathrm{B}_g(p,r)\right) = \frac{1}{2} \sum_{h=0}^{N} A_h(r),
$$

which leads to

$$
2|V'(r)| + 2cV(r) \geq \sum_{h=0}^{N} A_h(r) \geq \sum_{h=1}^{N} A_h(r). \tag{21}
$$

Adding inequalities (20) and (21) furnishes

$$
3|V'(r)| \geq CV(r)^{\frac{n-1}{n}} - 2cV(r) \geq \frac{C}{4} V(r)^{\frac{n-1}{n}}, \tag{22}
$$

where V being decreasing ensures the last inequality. Finally, if we suppose that one of the chambers of $\mathcal{E}$ is unbounded, we obtain that $V(r) > 0$ for all r and hence, by (22), we get

$$
\left(V^{\frac{1}{n}}\right)' = \frac{V'}{nV^{\frac{n-1}{n}}} \leq -\frac{C}{12n} < 0.
$$

Finally, by the mean value theorem and since $\frac{C}{12n}$ is negative and independent of r, the last equation contradicts that V is decreasing and positive. □

3. Application to the classical existence of isoperimetric clusters

To this aim, let us contextualize the classical setting that we mentioned before.

Definition 16. *We say that (M^n, g) is C^0-locally asymptotically a space form, if it has C^0-bounded geometry and for every diverging sequence of points (p_k) we have*

$$(M, g, p_k) \to (\mathbb{M}^n_k, g_{standard}, x)$$

in the C^0-topology, where $\mathbb{M}^n_k$ is a n-dimensional space form of curvature k and x is any point in $\mathbb{M}^n_k$.

Proof of Theorem 4. By Lemma 2, Theorem 2 and Definition 16, we get that at most one of the limit chamber's pieces $\mathcal{E}^j_k(h)$ lives at the limit manifold $\mathbb{M}^n_k$. If all limit chamber's pieces live in M, we have nothing to do and the proof is done. However, if one of the limit chamber's pieces lives at $\mathbb{M}^n_k$, we start recalling that

$$(\mathcal{E}_\infty(h), g, p^h_{j\infty}) = \bigcup_{j=0}^{J} \lim_{k\to+\infty} (\mathcal{E}^j_k(h), g, p^h_{jk}),$$

where the limit is taken in the C^0-topology. Set $\lim_{k\to+\infty}(\mathcal{E}^j_k(h), g, p^h_{jk}) = (\mathcal{E}^j_\infty(h), g, p^h_{j\infty})$, the set $\Lambda_1 = \{h \in \{1, \cdots, N\} : \mathcal{E}^{j_h}_\infty(h)$ lives at $\mathbb{M}^n_k$ for some $j_h\}$ and $\Lambda_2 = \{1, \cdots, N\} \setminus \Lambda_1$. Note that j_h is unique since at most one limit chamber's pieces lives at the space form and for those $h \in \Lambda_2$ we set $j_h := J$. Denote the volume of each $\mathcal{E}^j_\infty(h)$ by $\mathbf{v}^j(h)$. Then, we choose a metric ball $\mathrm{B}_g(p_h, r_h)$ in M with volume $\mathbf{v}^{j_h}(h)$ and positive distance from $\cup_{j=0, j\neq j_h}^{J} \mathcal{E}^j_\infty(h)$, what we can do thanks to the boundedness of the isoperimetric clusters in its ambient space which can be a formal disjoint union of manifolds, Theorem 3. We have to introduce two new N-cluster to simplify further equations, even though they will be used only in this proof, define

$$\mathcal{A} = \{\mathring{\bigcup}_{j=0, j\neq j_h}^{J} \mathcal{E}^j_\infty(h)\}_{h=1}^N,$$
$$\mathcal{B} = \{\mathcal{E}^{j_h}_\infty(h)\}_{h=1}^N.$$

We make this definitions to avoid working with M-clusters where $M \neq N$. Let us explain what are these sets, for the h's that belong to Λ_1, we choose the respective chamber's piece $\mathcal{E}^{j_h}_\infty(h)$ who lives in the space form in the definition of $\mathcal{B}$ and the remaining chamber's pieces $\mathcal{E}^j_\infty(h), j \neq j_h$, we take the union and define as a chamber of $\mathcal{A}$. The h's in Λ_2, we only take an arbitrary piece of $\mathcal{E}_\infty(h)$ (which was $\mathcal{E}^J_\infty(h)$) in the definition of $\mathcal{B}$ to have that $\mathcal{B}$ is an N-cluster. By definition of the multi-isoperimetric profile, Theorem 2 and Proposition 1, we have that

$$\begin{aligned}
\mathrm{I}_M(\mathbf{v}) &= \mathrm{Per}_{g_\infty}(\mathcal{E}_\infty) \\
&= \mathrm{Per}_g(\mathcal{A}) + \mathrm{Per}_{g_\infty}(\mathcal{B}) \\
&= \mathrm{Per}_g(\mathcal{A}) + \mathrm{Per}_{g_{standard}}(\mathcal{B}) \\
&\geq \mathrm{Per}_g(\mathcal{A}) + \frac{1}{2}\sum_{h\in\Lambda_2} \mathrm{Per}_{g_{standard}}(\mathcal{B}(h)) + \frac{1}{2}\sum_{h\in\Lambda_1} \tilde{\mathrm{I}}_{\mathbb{M}^n_k}\left(\mathbf{v}^J(h_0)\right),
\end{aligned}$$

where $\tilde{\mathrm{I}}_{\mathbb{M}^n_k}$ denotes the isoperimetric profile of the space form $\mathbb{M}^n_k$. By the latter inequality and recalling that balls are the isoperimetric regions in space forms, we get that

$$
\begin{aligned}
\mathrm{I}_M(\mathbf{v}) &\geq \mathrm{Per}_g(\mathcal{A}) + \frac{1}{2}\sum_{h\in\Lambda_2} \mathrm{Per}_{g_{standard}}(\mathcal{B}(h)) + \frac{1}{2}\sum_{h\in\Lambda_1} \mathrm{Per}_{g_{standard}}\left(\mathrm{B}_{M_k^n}(\mathbf{v}^J(h))\right) \\
&\overset{(*)}{\geq} \mathrm{Per}_g(\mathcal{A}) + \frac{1}{2}\sum_{h\in\Lambda_2} \mathrm{Per}_{g_{standard}}(\mathcal{B}(h)) + \frac{1}{2}\sum_{h\in\Lambda_1} \mathrm{Per}_g\left(\mathrm{B}_g(p_h, r_h)\right)
\end{aligned} \tag{23}
$$

where $\mathrm{B}_{M_k^n}(\mathbf{v}^J(h))$ denotes a ball of volume $\mathbf{v}^J(h)$ in $\mathbb{M}_k^n$, and $(*)$ is due to equation (2) of the proof of Proposition 3.2 in [MN16]. Again by Proposition 1, the definition of $\mathcal{A}$, and equation (23), we obtain that

$$
\mathrm{I}_M(\mathbf{v}) \geq \frac{1}{2}\sum_{h\in\Lambda_1} \mathrm{Per}_g\left(\mathrm{B}_g(p_h, r_h) \mathring{\cup} \left(\mathring{\bigcup}_{j=0 j\neq j_h}^{J} \mathcal{E}_\infty^j(h)\right)\right) + \frac{1}{2}\sum_{h\in\Lambda_2} \mathrm{Per}_g\left(\mathcal{E}_\infty(h)\right).
$$

Finally, we set the N-cluster $\mathcal{E}$ as follows

$$
\mathcal{E} = \{\mathcal{E}_\infty(h)\}_{h\in\Lambda_2} \cup \left\{\mathrm{B}_g(p_h, r_h) \mathring{\cup} \left(\mathring{\bigcup}_{j=0, j\neq j_h}^{J} \mathcal{E}_\infty^j(h)\right)\right\}_{h\in\Lambda_1},
$$

□

which is in M and satisfies the sought conditions.

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