EXPLICIT FORMULAE SHOWING THE EFFECTS OF TEXTURE ON ACOUSTOELASTIC COEFFICIENTS

Chi-Sing Man and Roberto Paroni
Department of Mathematics
University of Kentucky
Lexington, Kentucky 40506

INTRODUCTION

It is well known that crystallographic texture not only modifies the elastic constants of polycrystalline aggregates at (unstressed) natural states but also affects their acoustoelastic coefficients when the aggregates are stressed. While exact knowledge about the effects of texture on acoustoelastic coefficients has hitherto remained wanting, such effects are usually assumed to be negligible and are ignored in practical applications of acoustoelasticity (cf. [1] for example). Concerning this common practice, Thompson et al. [2] have urged caution:

> care must be taken when [this] assumption is made since the influence of texture on acoustoelastic constants is stronger than its influence on elastic moduli or velocities.

In this paper we shall show that this warning of Thompson et al. should be taken seriously. We have derived formulae by which we can quantitatively estimate the effects of texture on the acoustoelastic response of polycrystalline aggregates. As we shall illustrate below, there are situations in ultrasonic measurement of stress where ignoring the effects of texture on acoustoelastic coefficients could lead to huge error. Indeed prudence dictates that an error estimate be made in each case before the influence of texture on acoustoelastic coefficients could be safely ignored.

Throughout this paper we shall restrict our attention to an aggregate point X which consists of cubic crystallites of the same kind and is given in a configuration with an orthotropic texture and an initial stress $\overset{\circ}{\boldsymbol{T}}$. We shall only use the initially given configuration as the reference configuration for X. Moreover, our discussion below will always refer to a reference crystallite orientation and a spatial coordinate system chosen so that the coordinate planes agree with the planes of orthotropic texture symmetry and the coordinate axes coincide with the axes of cubic symmetry of the reference crystallite orientation.

MODEL-INDEPENDENT FORMULAE

Following usual practice, we describe the crystallographic texture at X by an orientation distribution function (ODF) w. Using the convention initiated by Roe [3], we expand w as an infinite series of the generalized spherical functions:

$$w(\psi,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{n=-l}^{l}W_{lmn}Z_{lmn}(\cos\theta)e^{-im\psi}e^{-in\phi}\,, \tag{1}$$

where Z_{lmn} are the normalized Jacobi polynomials and (ψ,θ,ϕ) are the Euler angles. From the normalization condition

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi}w(\psi,\theta,\phi)\sin\theta d\psi d\theta d\phi=1, \tag{2}$$

we always have

$$w(\psi,\theta,\phi)=\frac{1}{8\pi^2}+\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\sum_{n=-l}^{l}W_{lmn}Z_{lmn}(\cos\theta)e^{-im\psi}e^{-in\phi} \tag{3}$$

for any orientation distribution function w. We call those expansion coefficients W_{lmn} with $l\geq 1$ the texture coefficients. When all the texture coefficients vanish, the aggregate point X is without texture (i.e., the crystallites constituting X have random orientations) and $w=1/8\pi^2$.

Should the aggregate point X be stress-free and without texture at its reference configuration, its elastic behavior would be governed by the elasticity tensor

$$\boldsymbol{C}[\boldsymbol{E}]=\lambda(\mathrm{tr}\,\boldsymbol{E})\boldsymbol{I}+2\mu\boldsymbol{E}, \tag{4}$$

where $\boldsymbol{E}$ denotes the infinitesimal strain tensor, $\mathrm{tr}\,\boldsymbol{E}$ stands for the trace of $\boldsymbol{E}$, $\boldsymbol{I}$ is the identity tensor, and λ,μ are the Lamé constants of the aggregate. By adopting a suitable micromechanical model (e.g., the Voigt model, the Reuss model, etc.), we can express λ and μ in terms of the second-order elastic constants of the single crystal. Different models, however, will generally deliver different estimates of λ and μ. On the other hand, Eq. (4) with λ and μ unspecified is a general representation formula in continuum mechanics. As a general formula, Eq. (4) is model-independent and the Lamé constants are to be determined experimentally for each material in question.

Here we are interested in the case where X consists of cubic crystallites and is given in a prestressed configuration with an orthotropic texture. The acoustoelastic properties of X are characterized [4, 5] by its incremental elasticity tensor $\boldsymbol{L}(w,\overset{\circ}{\boldsymbol{T}})[\boldsymbol{E}]$, where we have emphasized by notation the dependence of $\boldsymbol{L}$ on the orientation distribution function w and the initial stress $\overset{\circ}{\boldsymbol{T}}$. We have recently obtained [6] a general representation formula for $\boldsymbol{L}(w,\overset{\circ}{\boldsymbol{T}})[\boldsymbol{E}]$. Our formula is correct to within terms linear in W_{lmn}, $\overset{\circ}{T}_{ij}$, and their products $W_{lmn}\overset{\circ}{T}_{ij}$, where $\overset{\circ}{T}_{ij}$ are the components of $\overset{\circ}{\boldsymbol{T}}$.

For want of space, we shall not reproduce the general representation formula here. In this paper we would rather concentrate on some of its applications in acoustoelasticity. Nevertheless, to gain a deeper understanding of what we could get from the general formula, a glance at its specialized version for the case with $\overset{\circ}{\boldsymbol{T}} = \mathbf{0}$ might be helpful.

When $\overset{\circ}{\boldsymbol{T}} = \mathbf{0}$, the incremental elasticity tensor reduces to the elasticity tensor and our formula assumes the form [7]

$$\boldsymbol{L}(w, \mathbf{0})[\boldsymbol{E}] = \boldsymbol{C}(w)[\boldsymbol{E}] = \lambda(\operatorname{tr}\boldsymbol{E})\boldsymbol{I} + 2\mu\boldsymbol{E} + \alpha\,\boldsymbol{\Phi}(w)[\boldsymbol{E}]; \tag{5}$$

here α is a material constant; $\boldsymbol{\Phi}(w)[\cdot]$ is an orthotropic fourth-order tensor whose components, under the chosen spatial coordinate system and reference crystallite orientation, are given in the Voigt notation by the following formulae:

$$\begin{aligned}
(\Phi_{11}, \Phi_{22}, \Phi_{33}) &= -(a_2 + a_3, a_3 + a_1, a_1 + a_2),\\
(\Phi_{23}, \Phi_{13}, \Phi_{12}) &= (a_1, a_2, a_3),\\
(\Phi_{44}, \Phi_{55}, \Phi_{66}) &= (a_1, a_2, a_3),
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
a_1 &= -(16\sqrt{2}\pi^2/35)(W_{400} + \sqrt{5/2}\,W_{420}),\\
a_2 &= -(16\sqrt{2}\pi^2/35)(W_{400} - \sqrt{5/2}\,W_{420}),\\
a_3 &= (4\sqrt{2}\pi^2/35)(W_{400} - \sqrt{70}\,W_{440}).
\end{aligned} \tag{7}$$

Note that Eq. (5) has been derived previously under the Voigt model [8] and the Reuss model [9], where the parameters λ, μ, and α are expressed explicitly in terms of the second-order elastic constants of the single cubic crystal. The same equation, as it stands above, has a meaning different than its other self that results from micromechanical modelling. With the parameter λ, μ, and α unspecified, Eq. (5) is a general representation formula analogous to its counterpart (4) for the case without texture. That both the Voigt model and the Reuss model should lead to Eq. (5) with different estimates for λ, μ, and α corroborates the model-independence of the representation formula.

Our model-independent representation formula [6] for the incremental elasticity tensor $\boldsymbol{L}(w, \overset{\circ}{\boldsymbol{T}})[\boldsymbol{E}]$ contains 6 unspecified parameters (including λ and μ) in its isotropic part and another 6 undetermined coefficients (including α) in its anisotropic part. For $\boldsymbol{L}(w, \overset{\circ}{\boldsymbol{T}})[\boldsymbol{E}]$ to serve as a constitutive function in continuum mechanics, these 12 parameters are to be determined from experiments for each material in question. Under the chosen spatial coordinate system and reference crystallite orientation, only 7 texture coefficients, namely W_{4m0} ($m = 0, 2, 4$) and W_{6m0} ($m = 0, 2, 4, 6$), appear in the formula.

Acoustoelastic Birefringence

Let us now examine what we can learn from the general representation formula as regards the influence of texture on acoustoelastic coefficients. In this paper we consider only the case with $\overset{\circ}{\boldsymbol{T}} = \mathrm{diag}(\overset{\circ}{T}_{11}, \overset{\circ}{T}_{22}, \overset{\circ}{T}_{33})$, although our representation formula covers the general situation where the principal initial-stress directions need not coincide with the coordinate axes.

First we consider acoustoelastic birefringence. Let V_{IJ} be the phase velocity of shear waves propagating in the I-direction with polarization in the J-direction and $\bar{V}_T = (V_{31} + V_{32})/2$. One of the acoustoelastic birefringences is given by

$$\frac{V_{31} - V_{32}}{\bar{V}_T} = B_o + K_1 \overset{\circ}{T}_{11} + K_2 \overset{\circ}{T}_{22} + K_3 \overset{\circ}{T}_{33}, \tag{8}$$

where B_o is the birefringence at the (stress-free) relaxed state and K_i $(i = 1, 2, 3)$ are the acoustoelastic coefficients in question. From our general representation formula [6], we deduce that

$$\begin{aligned} \frac{V_{31} - V_{32}}{\bar{V}_T} &= \frac{16\sqrt{5}\pi^2\alpha}{35\mu} W_{420} + (\zeta_1 + \beta_1 W_{400} + \beta_3 W_{440} + \beta_4 W_{600} + \beta_6 W_{640})(\overset{\circ}{T}_{11} - \overset{\circ}{T}_{22}) \\ &+ (\beta_2 W_{420} + \beta_5 W_{620})(\overset{\circ}{T}_{11} + \overset{\circ}{T}_{22}) + (\gamma_2 W_{420} + \gamma_5 W_{620})\overset{\circ}{T}_{33}. \end{aligned} \tag{9}$$

Not all the coefficients in the preceding equation are independent. In addition to λ, μ and α, we may choose ζ_1 as one of the six parameters that define the isotropic part of $\boldsymbol{L}$ and select $\beta_1, \beta_2, \beta_3, \beta_4$ as among those which define the anisotropic part. The other parameters in (9) are then given by the formulae

$$\beta_5 = -\frac{4\sqrt{105}}{15}\beta_4, \qquad \beta_6 = \sqrt{14}\beta_4, \tag{10}$$

$$\gamma_2 = -\frac{\sqrt{10}}{2}\beta_1 + \beta_2 + \frac{\sqrt{7}}{2}\beta_3, \qquad \gamma_5 = \frac{8\sqrt{105}}{15}\beta_4. \tag{11}$$

A comparison of Eqs. (8) and (9) reveals how the orthotropic texture affects the acoustoelastic coefficients K_i. We note in passing that the familiar proportionality $B_o \propto W_{420}$ is model-independent.

Longitudinal Waves

Let V_{33}° be the phase velocity of longitudinal waves propagating in the 3-direction when $w = 1/8\pi^2$ and $\overset{\circ}{\boldsymbol{T}} = \mathbf{0}$ (i.e., when X is at a stress-free configuration without texture). Similar to Eq. (8), we have

$$\frac{V_{33} - V_{33}^{\circ}}{V_{33}^{\circ}} = \ell_o + \ell_1 \overset{\circ}{T}_{11} + \ell_2 \overset{\circ}{T}_{22} + \ell_3 \overset{\circ}{T}_{33}, \tag{12}$$

where ℓ_i $(i = 1, 2, 3)$ are the acoustoelastic coefficients in question. From our general representation formula, we obtain the expression

$$\begin{aligned} \frac{V_{33} - V_{33}^{\circ}}{V_{33}^{\circ}} &= \frac{\overset{\circ}{T}_{33}}{2(\lambda + 2\mu)} + \frac{2\sqrt{2}\pi^2\alpha}{35(\lambda + 2\mu)} W_{400} + (\zeta_2 + \eta_1 W_{400} + \eta_4 W_{600})(\overset{\circ}{T}_{11} + \overset{\circ}{T}_{22}) \\ &+ (\eta_2 W_{420} + \eta_5 W_{620})(\overset{\circ}{T}_{11} - \overset{\circ}{T}_{22}) + (\zeta_3 + \xi_1 W_{400} + \xi_4 W_{600})\overset{\circ}{T}_{33}. \end{aligned} \tag{13}$$

We may choose ζ_2, ζ_3 and ξ_1 to be three of the remaining four independent parameters. In particular, ξ_1 and α, β_1, β_2, β_3, β_4 complete a list of six independent parameters that define the anisotropic part of $\boldsymbol{L}$. The remaining coefficients in Eq. (13) are given by the formulae

$$\eta_1 = \frac{\mu}{\lambda + 2\mu}\left(-\frac{1}{2}\beta_1 + \frac{3\sqrt{10}}{10}\beta_2 + \frac{\sqrt{70}}{20}\beta_3\right) - \frac{1}{2}\xi_1 \tag{14}$$

$$\eta_2 = \frac{\mu}{\lambda + 2\mu}\left(\frac{3\sqrt{10}}{4}\beta_1 - \frac{1}{2}\beta_2 - \frac{5\sqrt{7}}{28}\beta_3\right) + \frac{\sqrt{10}}{4}\xi_1 \tag{15}$$

$$\eta_4 = -\frac{2\mu}{\lambda + 2\mu}\beta_4, \qquad \eta_5 = \frac{4\sqrt{105}\mu}{15(\lambda + 2\mu)}\beta_4, \qquad \xi_4 = \frac{4\mu}{\lambda + 2\mu}\beta_4. \tag{16}$$

Formulae (9) and (13) are model-independent, but they carry a total of eleven undetermined coefficients. Like other material constants in constitutive equations of continuum mechanics, these coefficients can be determined from suitable experiments. Through micromechanical modelling, we can also express these coefficients in terms of the second- and third-order elastic constants of the single crystal, to which discussion we now turn.

A SIMPLE MICROMECHANICAL MODEL

We have recently proposed [5] a simple micromechanical model by which explicit formulae for all the components of the incremental elasticity tensor $\boldsymbol{L}$ can be derived. In our model, we make two basic assumptions:

1. Every crystallite pertaining to the aggregate point X carries the same homogeneous initial stress $\overset{\circ}{\boldsymbol{T}}$ at the initial configuration.
2. When a transplacement with deformation gradient $\boldsymbol{F}(X)$ is superimposed on the initial configuration of X, every crystallite pertaining to X undergoes a homogeneous transplacement with deformation gradient $\boldsymbol{F}$ with respect to its own initial configuration.

Note that if $\overset{\circ}{\boldsymbol{T}} = \boldsymbol{0}$, our model will reduce to that of Voigt for textured aggregates. Should $\overset{\circ}{\boldsymbol{T}}$ arise as a result of an elastic deformation from an unstressed configuration of the aggregate, our Assumption 1 would be nothing but the familiar assumption of Reuss as applied to the pre-deformation. Here we are really interested in the general situation that the given configuration of the polycrystalline aggregate is prestressed, the aggregate need not have an unstressed configuration, and the initial stress $\overset{\circ}{\boldsymbol{T}} \neq \boldsymbol{0}$ does not arise from an elastic pre-deformation of the aggregate.

With our model, all the unspecified parameters in the constitutive function $\boldsymbol{L}(w, \overset{\circ}{\boldsymbol{T}})[\boldsymbol{E}]$ can be expressed in terms of the second- and third-order elastic constants of the single cubic crystal. For instance, we have [5]

$$\mu = (c_{11} - c_{12} + 3c_{44})/5, \tag{17}$$

$$\zeta_1 = \frac{1}{70\mu}\Big((c_{11} - c_{12})(8\,s_{11} - 8\,s_{12} + 3\,s_{44}) + 3\,c_{44}(4\,s_{11} - 4\,s_{12} + 5\,s_{44}) + 2(c_{111} - 3\,c_{112} + 2\,c_{123})(s_{11} - s_{12}) + 9\,c_{456}\,s_{44} + 6(c_{155} - c_{144})(s_{11} - s_{12} + s_{44})\Big), \tag{18}$$

$$\beta_4 = \frac{8\sqrt{26}\pi^2}{1001\mu}\Big(2(c_{11} - c_{12} - 2\,c_{44})(2\,s_{11} - 2\,s_{12} - s_{44}) + (c_{111} - 3\,c_{112} + 2\,c_{123})(s_{11} - s_{12}) + 8\,c_{456}\,s_{44} - 4(c_{155} - c_{144})(s_{11} - s_{12} + s_{44})\Big); \tag{19}$$

here the c_{ij}, c_{ijk}, and s_{ij} are the second-order stiffnesses, third-order stiffnesses, and second-order compliances of the single cubic crystal at its reference orientation. If the relevant elastic constants of the single crystal are known, we can use the explicit formulae in our model to estimate all the twelve unknown parameters in the constitutive function $\boldsymbol{L}$.

Let us now consider a concrete example, namely that of aluminum at 25 °C. We adopt the following values for the second-order [10] and third-order stiffnesses [11] of single-crystal aluminum: $c_{11} = 106.75$, $c_{12} = 60.41$, $c_{44} = 28.34$, $c_{111} = -1224$, $c_{112} = -373$, $c_{123} = 25$, $c_{144} = -64$, $c_{155} = -368$, and $c_{456} = -27$, all of which are in units of GPa. Equations (9) and (13) now become

$$\frac{V_{31} - V_{32}}{\bar{V}_T} = -3.97W_{420} + (-.043 - .47\,W_{400} + .04\,W_{440} + .92\,W_{600} + 3.45\,W_{640})(\mathring{T}_{11} - \mathring{T}_{22}) - (.07\,W_{420} + 2.52\,W_{620})(\mathring{T}_{11} + \mathring{T}_{22}) + (.72\,W_{420} + 5.03\,W_{620})\mathring{T}_{33}, \tag{20}$$

$$\frac{V_{33} - V_{33}^{\circ}}{V_{33}^{\circ}} = .005\,\mathring{T}_{33} + .595\,W_{400} + (.008 - .11\,W_{400} - .44\,W_{600})(\mathring{T}_{11} + \mathring{T}_{22}) + (-.02\,W_{420} + .60\,W_{620})(\mathring{T}_{11} - \mathring{T}_{22}) + (-.056 + .30\,W_{400} + .87\,W_{600})\mathring{T}_{33}, \tag{21}$$

respectively, where the stresses are in units of GPa.

COMPARISON WITH EXPERIMENTAL DATA

Our further discussion on the effects of texture will rely on the accuracy of the predictions of our micromechanical model. Before we venture further, let us compare some predictions of this model with experimental data.

When the effects of texture on acoustoelastic coefficients are ignored, the commonly used birefringence formula is

$$\frac{V_{31} - V_{32}}{\bar{V}_T} = B_o + C_a(\mathring{T}_{11} - \mathring{T}_{22}), \tag{22}$$

Table 1. Acoustoelastic coefficients of longitudinal waves in 7039-T6 aluminum.

Propagation Direction	Loading Direction	Measured Value (TPa^{-1})	Predicted Value (TPa^{-1})
Normal	Rolling	11.0	15.0
Normal	Transverse	7.7	9.7
Transverse	Normal	8.6	8.1
Transverse	Rolling	7.6	5.1
Rolling	Transverse	10.4	12.6
Rolling	Normal	3.9	2.1

where C_a is usually called the acoustoelastic constant. By setting $W_{lmn} = 0$ in the acoustoelastic coefficients K_i, we observe from Eq. (9) that $C_a = \zeta_1$.

Fukuoka and Toda [12] conducted acoustoelastic measurements on polycrystalline samples of 99.5% pure aluminum. From their experimental data, they obtained an average value of $C_a = -4.42 \times 10^{-5}$ MPa^{-1}. Presumably their experiments were performed at room temperature, which "is kept within 0.1 °C of fluctuation". Kobori and Iwashimizu [13] obtained the experimental value of $C_a = -4.62 \times 10^{-5}$ MPa^{-1} for an aluminum alloy (with 97.5% aluminum) at 20 °C. In both these papers the authors assumed in their data analysis that their samples were without texture. From Eq. (18) and the single-crystal elastic constants listed above, we obtain $\zeta_1 = -4.32 \times 10^{-5}$ MPa^{-1} for polycrystalline aluminum at 25 °C.

Johnson [14] measured six acoustoelastic coefficients for longitudinal waves propagating in a 7036-T6 aluminum sample plate. The relevant texture coefficients of the sample are: $W_{400} = -0.0090$, $W_{420} = -0.0008$, $W_{440} = 0.0039$, $W_{600} = 0.0062$, $W_{620} = -0.0025$, $W_{640} = 0.0024$, and $W_{660} = 0.0076$, which pertain to the coordinate system with the 1-, 2-, and 3-axis coinciding with the rolling, normal, and transverse direction of the plate, respectively. Table 1 gives a comparison of the measured acoustoelastic coefficients with the corresponding values predicted by our micromechanical model. In our calculations of the predicted values, we have appealed to Eq. (21) and to the transformation formula (cf. equation (17) of Roe [3]) for the W_{lmn} coefficients under a change of spatial coordinate system. For instance, substituting the given set of texture coefficients into Eq. (21) yields the predicted values of 5.1 and 8.1 TPa^{-1} for the acoustoelastic coefficients ℓ_1 and ℓ_2 in Eq. (12), respectively.

Johnson reported that his sample contains 6.2–8.2% of alloying elements (mainly zinc and magnesium), which could be a source of discrepancy between the measured and the predicted values. Allowing for this fact and also for the usual margin of error in the determination of texture coefficients, we can say that the predictions square rather well with the measured values.

DISCUSSIONS

Clearly Eq. (22), the formula often used in practical applications of birefringence measurements, need not be an acceptable approximation to Eq. (9). Indeed a particularly acute situation could arise if $\overset{\circ}{T}_{11} = \overset{\circ}{T}_{22} \neq 0$. There the ζ_1 term drops out of formula (9), but the initial stress will contribute to the birefringence if $W_{420} \neq 0$ or $W_{620} \neq 0$. Take, for instance, the sample of 7039-T64 aluminum plate studied by

Johnson and Springer [15]. Under a coordinate system whose 1-, 2-, and 3-axis coincide with the normal, transverse, and rolling direction of the plate, respectively, the relevant texture coefficients of their sample are: $W_{400} = -0.0017$, $W_{420} = 0.0043$, $W_{440} = -0.0039$, $W_{600} = -0.0037$, $W_{620} = -0.0055$, $W_{640} = 0.0012$, and $W_{660} = -0.00057$. Substituting these values into Eq. (20), we obtain for this sample the birefringence formula

$$\frac{V_{31} - V_{32}}{\bar{V}_T} = -0.0171 - 0.028\,\overset{\circ}{T}_{11} + 0.055\,\overset{\circ}{T}_{22} - 0.024\,\overset{\circ}{T}_{33}, \tag{23}$$

where the stresses are in units of GPa. For $\overset{\circ}{T}_{11} = \overset{\circ}{T}_{22} = 70$ MPa and $\overset{\circ}{T}_{33} = 0$, Eq. (23) gives the "stress-induced birefringence" a value of 0.19%. Using the value of $C_a = \zeta_1 = -4.32 \times 10^{-5}$ MPa^{-1}, we shall then draw the totally erroneous conclusion that $\overset{\circ}{T}_{11} - \overset{\circ}{T}_{22} = -44$ MPa. If $\overset{\circ}{T}_{11} = \overset{\circ}{T}_{22} = 70$ MPa and $\overset{\circ}{T}_{33} = -70$ MPa, then the estimate by formula (22) will be further worsened to $\overset{\circ}{T}_{11} - \overset{\circ}{T}_{22} = -83$ MPa. This example shows that in ultrasonic measurement of stress the effects of texture on acoustoelastic coefficients should not be ignored without analysis.

ACKNOWLEDGMENTS

The research reported here was supported in part with funds from the National Science Foundation (Grant No. CMS-9522829).

REFERENCES

1. M. Hirao, H. Ogi and H. Fukuoka, Res. Nondestr. Eval. 5, 211 (1994).
2. R.B. Thompson, W.-Y. Lu and A.V. Clark, in *SEM Monograph on Techniques for Residual Stress Measurement*, eds. J. Lu and M.R. James (Society for Experimental Mechanics, in press).
3. R.-J. Roe, J. Appl. Phys. 36, 2024 (1965).
4. C.-S. Man and W.Y. Lu, J. Elasticity 17, 159 (1987).
5. C.-S. Man and R. Paroni, J. Elasticity, in press.
6. R. Paroni and C.-S. Man, in preparation.
7. C.-S. Man, Arch. Rational Mech. Anal., submitted.
8. C.M. Sayers, J. Phys. D 15, 2157 (1982).
9. M. Hirao, K. Aoki and H. Fukuoka, J. Acoust. Soc. Am. 81, 1434 (1987).
10. J.F. Thomas, Phys. Rev. 175, 955 (1968).
11. V.P.N. Sarma and P.J. Reddy, Phys. Stat. Sol. (a) 10, 563 (1972).
12. H. Fukuoka and H. Toda, Arch. Mech. 29, 673 (1977).
13. O. Kobori and Y. Iwashimizu, in *Elastic Waves and Ultrasonic Nondestructive Evaluation*, eds. S.K. Datta, J.D. Achenbach and Y.S. Rajapakse (Elsevier, Amsterdam, 1990), p. 339.
14. G.C. Johnson, in *Review of Progress in QNDE*, Vol. 4, eds. D.O. Thompson and D.E. Chimenti (Plenum, New York, 1985), p. 1043.
15. G.C. Johnson and W.C. Springer, Int. J. Solids Structures 25, 609 (1989).