Existence of Extremals and Rigidity for Optimal Sobolev Inequalities

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Introduction. Sobolev inequalities, which relate the integrability or regularity of a function to the integrability of its derivatives, are a fundamental tool across analysis and geometry. A classical example is the optimal Sobolev inequality on Euclidean space: for $n \geq 2$ and $p \in (1, n)$ fixed, any function $u \in C_0^\infty(\mathbb{R}^n)$ satisfies

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} \geq S_{n,p} \|u\|_{L^{p^\star}(\mathbb{R}^n)} \,. \tag{1}$$

The critical exponent $p^\star = np/(n-p)$ is the unique value making the left- and right-hand sides of (1) scale the same way under dilations $u(x) \mapsto u(x/\alpha)$ for $\alpha > 0$. Optimal Sobolev constants often encode geometric information about their underlying domain or manifold. For example, Ledoux showed in [6] that if a complete Riemannian manifold (M^n, g) with non-negative Ricci curvature admits a Sobolev inequality of the form (1) with optimal constant S_g, then $S_g \leq S_{n,p}$, with $S_g = S_{n,p}$ if and only if (M^n, g) is isometric to Euclidean space. A stability result corresponding to this geometric comparison theorem was shown in [13]; for a comparison theorem for optimal log-Sobolev constants, a different type of stability was shown in [7] for manifolds with (almost) non-negative scalar curvature.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with C^1 boundary. The Sobolev inequality (1) holds for any $u \in C_c^\infty(\Omega)$, with the same optimal constant $S_{n,p}$. This is just one slice of the global picture, though; functions that do not vanish on the boundary also enjoy a Sobolev inequality once a term with a trace norm on $\partial\Omega$ is included. The optimal form of such a Sobolev inequality is defined through a family of variational problems with two critical constraints: for $T \geq 0$, let

$$\Phi_\Omega(T) = \inf \left\{ \left(\int_\Omega |\nabla u|^p \right)^{1/p} : \int_\Omega |u|^{p^\star} = 1, \int_{\partial\Omega} |u|^{p^\sharp} = T^{p^\sharp} \right\} . \tag{2}$$

Here $p^\sharp = (n-1)p/(n-p)$ is the critical exponent making this norm scale the same way as the other two; the critical scaling of the norms makes $\Phi_\Omega(T)$ invariant under dilations as well as translations of the domain Ω. By definition, $\Phi_\Omega(T)$ is the optimal constant in a family of Sobolev inequalities:

$$\|\nabla u\|_{L^p(\Omega)} \geq \Phi_\Omega(T) \|u\|_{L^{p^\star}(\Omega)} \qquad \text{whenever} \qquad \|u\|_{L^{p^\sharp}(\partial\Omega)} \big/ \|u\|_{L^{p^\star}(\Omega)} = T. \tag{3}$$

The particular slice $T = 0$ is the inequality (1), i.e. $\Phi_\Omega(0) = S_{n,p}$. A constant test function shows that $\Phi_\Omega(\mathrm{I}(\Omega)^{1/p^\sharp}) = 0$, where we set $\mathrm{I}(\Omega) := \mathrm{Per}(\Omega)/|\Omega|^{\frac{n-1}{n}}$.

In [10], Maggi and Villani proved a geometric comparison theorem for the optimal Sobolev constants $\Phi_\Omega(T)$, showing that balls have the *worst* optimal Sobolev constant. More precisely, letting $B = \{|x| < 1\} \subset \mathbb{R}^n$, they showed that

$$\Phi_\Omega(T) \geq \Phi_B(T) \qquad \text{for all } T \in [0, \mathrm{I}(B)^{1/p^\sharp}] \,. \tag{4}$$

They also proved existence and characterization of minimizers for the variational problem $\Phi_B(T)$ for all T in this parameter range. Scaling shows that half spaces have the *best* optimal Sobolev constant, i.e. $\Phi_\Omega(T) \le \Phi_H(T)$ for all $T > 0$, where $H = \{x \cdot e_n > 0\} \subset \mathbb{R}^n$. In [8], Maggi and the author established the existence and characterization of minimizers of $\Phi_H(T)$ for all $T > 0$ (see also [3] when $p = 2$).

Main Results. Two main open problems about $\Phi_\Omega(T)$ motivate the paper [9].

(A) When do minimizers of the variational problem (2) exist? Equivalently, when do extremal functions exist in the sharp Sobolev inequality (3)?

(B) Does rigidity hold in the geometric comparison theorem (4)?

From scaling and the characterization of extremal functions of (1) due to Aubin [2] and Talenti [12], it is easily shown that for $T = 0$, minimizers of (2) cannot exist unless $\Omega = \mathbb{R}^n$. For $T > 0$, in [9] we use the characterization of $\Phi_H(T)$ from [8] to prove that if existence fails on an open bounded domain Ω with C^1 boundary, it can only occur because a minimizing sequence concentrates at exactly one point located on $\partial\Omega$. Assuming further regularity of $\partial\Omega$ and restricting the dimension, we rule out this possibility and prove the following existence theorem.

Theorem 1. *Fix $p > 1$ and $n > 2p$. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with boundary of class C^2. For every $T \in (0, \infty)$, a minimizer of (2) exists.*

Question (B) was posed as an open problem in [10], and the proof of (4) implies the following *rigidity criterion*: If Ω is connected and $\Phi_\Omega(T) = \Phi_B(T)$ for some $T \in (0, \mathrm{I}(B)^{1/p^\sharp})$, *and additionally a minimizer of (2) exists for this T*, then Ω is a ball. The connectedness assumption is necessary for rigidity to hold; consider the union of a ball and any other domain. Thanks to this rigidity criterion, we obtain an affirmative answer to Question (B) under the assumptions of Theorem 1.

Corollary 2. *Fix $p > 1$ and $n > 2p$. Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, connected domain with boundary of class C^2. If $\Phi_\Omega(T) = \Phi_B(T)$ for some $T \in (0, I(B)^{1/p^\sharp}]$, then Ω is a ball.*

Finally, we obtain the following weak rigidity theorem without additional restrictions on n or $\partial\Omega$.

Theorem 3. *Fix $n \ge 2$ and $p \in (1, n)$. Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, connected domain with boundary of class C^1. If there exists $T_* > 0$ such that $\Phi_\Omega(T) = \Phi_B(T)$ for all $T \in (0, T_*)$, then Ω is a ball.*

The proof of Theorem 3 again uses the rigidity criterion above; based on the characterization of minimizers of $\Phi_H(T)$ from [8] and an analysis of the Euler-Lagrange equation asymptotically satisfied by a concentrating minimizing sequence, we prove that minimizers of (2) exist for $T > 0$ sufficiently small under the assumptions of Theorem 3.

Open problems. There are quite a few open problems related to this program. First, can one show Theorem 1 in all dimensions? In [9], we build an explicit ("Aubin-type" [1]) test function and expand its energy to rule out concentration. The dimension restriction comes from the tail decay rate of extremals of $\Phi_H(T)$; this issue is familiar from the Yamabe problem and one may hope to construct a "Schoen-type" [11] global test function to show existence in low dimensions.

Second, under the assumptions of Corollary 2, is the comparison theorem (4) *stable*, i.e. if $\Phi_\Omega(T) \approx \Phi_B(T)$ for some $T \in (0, \mathrm{I}(B)^{1/p^\sharp}]$, then is Ω close to a ball in a suitable sense? A starting point here is to analyze the mass transportation proof of (4) from [10] and to show that the optimal transport map taking a minimizer of $\Phi_\Omega(T)$ to a minimizer of $\Phi_B(T)$ is close to the identity.

Finally, when $p = 2$, the variational problem (2) is related to the Yamabe problem for manifolds with boundary [4, 5], where one seeks a conformal metric of constant scalar curvature and constant mean curvature boundary on a given Riemannian manifold with boundary. In the conformally flat case $(M, g) = (\Omega, g_{\mathrm{euc}})$, this is equivalent to showing the existence of critical points of the energy $\int_\Omega |\nabla u|^2 + c_n \int_{\partial\Omega} h\, u^2$ in the same constraint space as in (2). Here h is the mean curvature of $\partial\Omega$. Can our analysis in [9] be refined to produce a one-parameter family $\{g_T\}_{T>0}$ of Yamabe metrics with the prescribed ratio $\mathrm{vol}(\partial M, g_u)/\mathrm{vol}(M, g_u) = T$?

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