A NOTE ON STRONG-FORM STABILITY FOR THE SOBOLEV INEQUALITY

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Abstract. In this note, we establish a strong form of the quantitive Sobolev inequality in Euclidean space for $p \in (1, n)$. Given any function $u \in \dot{W}^{1,p}(\mathbb{R}^n)$, the gap in the Sobolev inequality controls $\|\nabla u - \nabla v\|_p$, where v is an extremal function for the Sobolev inequality.

1. Introduction

Sobolev inequalities, broadly speaking, establish integrability or regularity properties of a function in terms of the integrability of its gradient. A fundamental example is the classical Sobolev inequality on Euclidean space, which states the following. Given $n \geq 2$ and $p \in (1, n)$, there exists a constant $S = S(n, p)$ such that

$$\|\nabla u\|_p \geq S\|u\|_{p^*}. \tag{1.1}$$

for any function $u \in \dot{W}^{1,p}(\mathbb{R}^n)$. Here, $p^* = np/(n-p)$, and $\dot{W}^{1,p}(\mathbb{R}^n)$ is the space of functions such that $u \in L^{p^*}(\mathbb{R}^n)$ and $|\nabla u| \in L^p(\mathbb{R}^n)$. Let us take S to be the largest possible constant for which (1.1) holds. Aubin [Aub76] and Talenti [Tal76] determined that equality is achieved in (1.1) for the function

$$\bar{v}(x) = \left(1 + |x|^{p'}\right)^{(p-n)/p},$$

as well as its translations, dilations, and constant multiples. Here and in the sequel, we let $p' = p/(p-1)$ denote the Hölder conjugate of p. In fact, these functions are the only such extremal functions for (1.1), and we will let

$$\mathcal{M} = \left\{ v \ : \ v(x) = c\,\bar{v}\left(\lambda(x-y)\right) \text{ for some } c \in \mathbb{R},\, \lambda \in \mathbb{R}_+,\, y \in \mathbb{R}^n \right\}$$

denote this $(n+2)$-dimensional space of extremal functions.

Brezis and Lieb raised the question of quantitative stability for the Sobolev inequality in [BL85], asking whether the deviation of a given function from attaining equality in (1.1) controls its distance to the family of extremal functions $\mathcal{M}$. The strongest notion of distance that one expects to control is the L^p norm between gradients. With this in mind, let us define the *asymmetry* of a function $u \in \dot{W}^{1,p}(\mathbb{R}^n)$ by

$$A(u) = \inf\left\{ \frac{\|\nabla u - \nabla v\|_p}{\|\nabla u\|_p} : v \in \mathcal{M} \right\}.$$

Note that $A(u)$ is invariant under the symmetries of the Sobolev inequality (translations, dilations, and constant multiples) and is equal to zero if and only if $u \in \mathcal{M}$. To quantify the deviation from equality in (1.1), we define the *deficit* of a function $u \in \dot{W}^{1,p}(\mathbb{R}^n)$ to be

$$\delta(u) = \frac{\|\nabla u\|_p^{p'} - S^{p'}\|u\|_{p^*}^{p'}}{\|u\|_{p^*}^{p'}} \qquad \text{if } p < 2$$

and

$$\delta(u) = \frac{\|\nabla u\|_p^{p} - S^{p}\|u\|_{p^*}^{p}}{\|u\|_{p^*}^{p}} \qquad \text{if } p \geq 2\,.$$

Like the asymmetry, the deficit is a non-negative functional that is invariant under translations, dilations, and constant multiples, and is equal to zero if and only if $u \in \mathcal{M}$.

By way of a concentration compactness argument as in [Lio85], one readily establishes the qualitative stability of (1.1). That is, if $\{u_i\}$ is a sequence of functions with $\delta(u_i) \to 0$, then $A(u_i) \to 0$. The first quantitative result was established in the case $p = 2$ in [BE91], where Bianchi and Egnell showed that there is a dimensional constant C such that

$$A(u)^2 \leq C\,\delta(u)\,.$$

This result, in addition to being optimal in the strength of the distance controlled, is sharp in the sense that the exponent 2 cannot be replaced by a smaller one. The proof relies strongly on the fact that $W^{1,2}(\mathbb{R}^n)$ is a Hilbert space, and in the absence of this structure, the case when $p \neq 2$ has proven much more difficult to treat. Nevertheless, in [CFMP09], Cianchi, Fusco, Maggi, and Pratelli established a quantitative stability result in which the deficit controls the distance of a function to $\mathcal{M}$ in terms of the L^{p^*} norm; see Theorem 2.1 below for a precise statement. The argument combines symmetrization arguments in the spirit of [FMP08] with a mass transportation argument in one dimension. More recently, in [FN19], Figalli and the author strengthened this result in the case $p \geq 2$ by showing that the deficit of a function controls a power of $A(u)$. The main idea there was to view $W^{1,p}(\mathbb{R}^n)$ as a weighted Hilbert space and to establish a spectral gap for the linearized operator in the second variation as in [BE91]. However, bounding the difference between the deficit and the second variation required the use of the main result of [CFMP09].

In this note, we establish a reduction theoreom that, paired with [CFMP09], allows us to deduce a strong-form quantitative stability result in which the deficit of a function controls a power of $A(u)$. For $p \geq 2$, this recovers the main result of [FN19] with a simpler proof, while in the case $p \in (1, 2)$, it provides the first known quantitative estimate for (1.1) at the level of gradients.

Theorem 1.1. *Fix $n \geq 2$ and $p \in (1, n)$. There exist constants $C_1(n, p)$ and $C_2(n, p)$ such that the following holds. For any $u \in \dot{W}^{1,p}(\mathbb{R}^n)$ and for any $v \in \mathcal{M}$ with $\|u\|_{p^*} = \|v\|_{p^*}$, we have*

$$\left(\frac{\|\nabla u - \nabla v\|_p}{\|u\|_{p^*}}\right)^{\alpha} \leq C_1\,\delta(u) + C_2\frac{\|u - v\|_{p^*}}{\|u\|_{p^*}}\,. \tag{1.2}$$

Here, $\alpha = p'$ if $p \in (1, 2)$ and $\alpha = p$ if $p \in [2, n)$.

Pairing Theorem 1.1 with the main result of [CFMP09] (Theorem 2.10 below), we establish the following quantitative estimate.

Corollary 1.2. *Fix $n \geq 2$ and $p \in (1, n)$. There exist constants $C = C(n, p)$ and $\beta = \beta(n, p)$ such that the following holds. For any $u \in \dot{W}^{1,p}(\mathbb{R}^n)$, we have*

$$A(u)^{\beta} \leq C\,\delta(u)\,. \tag{1.3}$$

The value of β in Corollary 1.2 is given by

$$\beta = \begin{cases} p'\left(p^*\left(3 + 4p - \frac{3p+1}{n}\right)\right)^2 & \text{if } p \in (1,2) \\ p\left(p^*\left(3 + 4p - \frac{3p+1}{n}\right)\right)^2 & \text{if } p \in [2,n)\,. \end{cases}$$

The proof of Theorem 1.1 is elementary and at its core relies on the convexity of the function $t \mapsto t^p$. It is inspired by the recent paper [HS], in which Hynd and Seuffert give a qualitative description of extremal functions in (a certain form of) Morrey's inequality. Interestingly, they are able to establish a quantitative stability result, even without knowing the explicit form of extremal functions.

Quantitive stability for Sobolev-type inequalities has been a topic of interest in recent years. Closely related to the main results here, a strong-form quantitative stability result was shown for the Sobolev inequality (1.1) with $p = 1$ in [FMP13], following [FMP07, Cia06]. Quantitative stability results have also been shown for (a different form of) Morrey's inequality [Cia08], the log-Sobolev inequality [IM14, BGRS14, FIL16], the higher order Sobolev inequality [BWW03, GW10], the fractional Sobolev inequality [CFW13], Gagliardo-Nirenberg-Sobolev inequalities [CF13, DT13, DT16, Seu, Ngu], and Strichartz inequalities [Neg].

More broadly, strong-form stability estimates (in which the gap in a given inequality controls the strongest possible norm, typically involving the oscillation of a set or function) have been studied for various functional and geometric inequalities. For instance, such results have been shown for isoperimetric inequalities in Euclidean space [FJ14], on the sphere [BDF17], and in hyperbolic space [BDS15], as well as for anisotropic [Neu16] and Gaussian [Eld15, BBJ17] isoperimetric inequalities.

Apart from their innate interest from a variational perspective, quantitative stability estimates have found applications in the study of geometric problems [FM11, CS13, KM14] and PDE [CF13, DT16]. Certain applications, such as those in [FMM18, CNT], necessitate strong-form quantitative estimates of the type established here.

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2. Proofs of Theorem 1.1 and Corollary 1.2

In the proof of Theorem 1.1, we will make use of the following version of Clarkson's inequalities for vector-valued functions, which state the following. Let $F, G : \mathbb{R}^n \to \mathbb{R}^n$ with

$|F|, |G| \in L^p(\mathbb{R}^n)$. Then

$$\left\|\frac{F+G}{2}\right\|_p^{p'} + \left\|\frac{F-G}{2}\right\|_p^{p'} \leq \left(\frac{1}{2}\|F\|_p^p + \frac{1}{2}\|G\|_p^p\right)^{p'/p} \tag{2.1}$$

if $p \in (1,2)$, and

$$\left\|\frac{F+G}{2}\right\|_p^{p} + \left\|\frac{F-G}{2}\right\|_p^{p} \leq \frac{1}{2}\|F\|_p^p + \frac{1}{2}\|G\|_p^p. \tag{2.2}$$

if $p \geq 2$. These inequalities were shown for scalar- and complex-valued functions in [Cla36], and were extended to functions mapping from $\mathbb{R}$ to $\mathbb{R}^n$ in [Boa40]. Though Clarkson's inequalities have been generalized in a number of directions, we could not locate a reference for the precise form of (2.1) and (2.2), so in Section 3 we prove (2.2) and show how to deduce (2.1) from its scalar-valued analogue.

Proof of Theorem 1.1. We first consider the case $p \in (1,2)$. Applying (2.1) with $F = \nabla u$ and $G = \nabla v$, we find that

$$\left\|\frac{\nabla u - \nabla v}{2}\right\|_p^{p'} \leq \left(\frac{1}{2}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla v\|_p^p\right)^{p'/p} - \left\|\frac{\nabla u + \nabla v}{2}\right\|_p^{p'} \tag{2.3}$$

Next, the Sobolev inequality (1.1) implies that

$$\|\nabla v\|_p^p \leq \|\nabla u\|_p^p, \tag{2.4}$$

and

$$\|\nabla u + \nabla v\|_p^{p'} \geq S^{p'} \, \|u+v\|_{p^*}^{p'} \, . \tag{2.5}$$

In (2.4) we have used the assumption that $\|u\|_{p*} = \|v\|_{p^*}$. Together (2.3), (2.4), and (2.5) imply that

$$\left\|\frac{\nabla u - \nabla v}{2}\right\|_p^{p'} \leq \|\nabla u\|_p^{p'} - S^{p'} \left\|\frac{u+v}{2}\right\|_{p^*}^{p'} . \tag{2.6}$$

Finally, we claim that

$$\left\|\frac{u+v}{2}\right\|_{p^*}^{p'} \geq \|u\|_{p^*}^{p'} - p'\|u\|_{p^*}^{p'-1} \left\|\frac{u-v}{2}\right\|_{p^*} . \tag{2.7}$$

Indeed, Minkowski's inequality implies that

$$\left\|\frac{u+v}{2}\right\|_{p^*}^{p'} \geq \left(\|u\|_{p^*} - \left\|\frac{u-v}{2}\right\|_{p^*}\right)^{p'} . \tag{2.8}$$

Then, convexity of the function $t \mapsto t^{p'}$ implies that

$$\left(\|u\|_{p^*} - \left\|\frac{u-v}{2}\right\|_{p^*}\right)^{p'} \geq \|u\|_{p^*}^{p'} - p'\|u\|_{p^*}^{p'-1} \left\|\frac{u-v}{2}\right\|_{p^*} . \tag{2.9}$$

Together (2.8) and (2.9) imply (2.7). Finally, combining (2.6) and (2.7) and dividing through by $\|u\|_{p^*}^{p'}$ establishes the proof of (1.2) with $C_1 = 2^{p'}$ and $C_2 = p' 2^{p'-1} S^{p'}$.

Next, the proof for the case $p \geq 2$ is completely analogous. Indeed, applying Clarkson's inequality (2.2) followed by the Sobolev inequality (1.1), and then (2.7) (with p replacing p'), we find that

$$\begin{aligned}\left\|\frac{\nabla u - \nabla v}{2}\right\|_p^p &\leq \frac{1}{2}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla v\|_p^p - \left\|\frac{\nabla u + \nabla v}{2}\right\|_p^p \\ &\leq \|\nabla u\|_p^p - S^p \left\|\frac{u+v}{2}\right\|_{p^*}^p \\ &\leq \|\nabla u\|_p^p - S^p \|u\|_{p^*}^p + S^p p \|u\|_{p^*}^{p-1} \left\|\frac{u-v}{2}\right\|_{p^*} .\end{aligned}$$

Dividing by $\|u\|_{p^*}^p$ establishes (1.2) with $C_1 = 2^p$ and $C_2 = p 2^{p-1} S^p$. □

Now, let us recall the main result from [CFMP09]. The notion of L^{p^*} asymmetry considered there is

$$\lambda(u) = \inf \left\{ \frac{\|u - v\|_{p^*}}{\|u\|_{p^*}} : v \in \mathcal{M},\ \|v\|_{p^*} = \|u\|_{p^*} \right\}$$

Theorem 2.1 (Cianchi, Fusco, Maggi, Pratelli)**.** *Fix $n \geq 2$ and $p \in (1, n)$. There exists a constant $C = C(n, p)$ such that the following holds. For any $u \in \dot{W}^{1,p}(\mathbb{R}^n)$,*

$$\lambda(u)^\beta \leq C\, \frac{\|\nabla u\|_p - S\|u\|_{p^*}}{\|u\|_{p^*}}, \tag{2.10}$$

Here $\beta = \left(p^* \left(3 + 4p - \frac{3p+1}{n}\right)\right)^2$.

We now prove Corollary 1.2 by combining Theorems 1.1 and 2.1.

Proof of Corollary 1.2. First, let us note that Theorem 2.1 implies that $\lambda(u)^\beta \leq C\delta(u)$, because

$$\frac{\|\nabla u\|_p - S\|u\|_{p^*}}{\|u\|_{p^*}} \leq \delta(u)\,. \tag{2.11}$$

To see this, note that for any $\alpha \geq 1$, the function $t \mapsto t^\alpha - t$ is increasing for $t \geq 1$. In particular, if $a \geq b \geq 1$, we have

$$a^\alpha - b^\alpha \geq a - b\,. \tag{2.12}$$

Let $a = \|\nabla u\|_p / S\|u\|_{p^*}$ and $b = 1$. Then applying (2.12) with $\alpha = p'$ for $p \in (1, 2)$ and $\alpha = p$ for $p \in [2, n)$ establishes (2.11).

Now, note $A(u) \leq 1$, and thus Corollary 1.2 holds trivially when $\delta(u) \geq 1$. We therefore assume that $\delta(u) \leq 1$. Let $v \in \mathcal{M}$ be a function such that $\|v\|_{p^*} = \|u\|_{p^*}$ and $\|u - v\|_{p^*} / \|u\|_{p^*} \leq 2\lambda(u)$. Then, again letting $\alpha = p'$ for $p \in (1, 2)$ and $\alpha = p$ for $p \in [2, n)$, we apply the Sobolev inequality, Theorem 1.1, and Theorem 2.1 respectively to find that

$$\begin{aligned} A(u)^\alpha \leq \left(\frac{\|\nabla u - \nabla v\|_p}{\|\nabla u\|_p}\right)^\alpha &\leq S^{-\alpha}\left(\frac{\|\nabla u - \nabla v\|_p}{\|u\|_{p^*}}\right)^\alpha \\ &\leq C_1\delta(u) + C_2\frac{\|u-v\|_{p^*}}{\|u\|_{p^*}} \\ &\leq C_1\delta(u) + C\delta(u)^{1/\beta'} \leq C\delta(u)^{1/\beta'}. \end{aligned}$$

The final inequality follows because $\delta(u) \leq 1$ and $1/\beta' \leq 1$. Taking the power β' of the left- and right-hand sides, we establish Corollary 1.2. □

3. Clarkson's inequalities for vector valued functions on $\mathbb{R}^n$

For $p \in (1,2)$, Clarkson [Cla36] established the following inequality for, in particular, real numbers a and b:

$$|a+b|^{p'} + |a-b|^{p'} \leq 2(|a|^p + |b|^p)^{p'/p}. \tag{3.1}$$

Let us see how to deduce (2.1) from (3.1). We make use of the reverse Minkowski inequality: if $s \in (0,1)$, then for $(a_1,\dots,a_n) \subset \mathbb{R}^n$ and $(b_1,\dots,b_n) \subset \mathbb{R}^n$ we have

$$\left(\sum |a_i|^s\right)^{1/s} + \left(\sum |b_i|^s\right)^{1/s} \leq \left(\sum |a_i + b_i|^s\right)^{1/s} \tag{3.2}$$

This inequality follows from the concavity of the function $t \mapsto t^s$. We take $s = 2/p'$ and let $a_i = |F_i + G_i|^{p'}$ and $b_i = |F_i - G_i|^{p'}$ for $i = 1,\dots,n$. Here F_i denotes the ith component of F in some fixed basis. Then, applying (3.2) followed by (3.1), we find that

$$\begin{aligned} |F-G|^{p'} + |F+G|^{p'} &\leq \left(\sum(|F_i+G_i|^{p'} + |F_i - G_i|^{p'})^{2/p'}\right)^{p'/2} \\ &\leq 2\left(\sum(|F_i|^p + |G_i|^p)^{2/p}\right)^{p'/2}. \end{aligned} \tag{3.3}$$

On the left-hand side, we have used $|F|$ to denote the Euclidean norm. Next, applying the usual form of Minkowski's inequality with $r = 2/p$ to $(|a_i|^p)$ and $(|b_i|^p)$, we find

$$\left(\sum(|a_i|^p + |b_i|^p)^{2/p}\right)^{1/2} \leq \left(\left(\sum |a_i|^2\right)^{p/2} + \left(\sum |b_i|^2\right)^{p/2}\right)^{1/p}.$$

Pairing this with (3.3), we find that

$$|F+G|^{p'} + |F-G|^{p'} \leq 2\left(|F|^p + |G|^p\right)^{p'/p}. \tag{3.4}$$

Finally, we make use of the integral form of (3.2): for $s \in (0,1)$, we have

$$\|h_1\|_{L^s(\mathbb{R}^n)} + \|h_2\|_{L^s(\mathbb{R}^n)} \leq \|h_1 + h_2\|_{L^s(\mathbb{R}^n)}. \tag{3.5}$$

We apply (3.5) with $s = p/p'$ and with $h_1 = |F+G|^{p'}$ and $h_2 = |F-G|^{p'}$, and then apply (3.4), in order to find that

$$\|F+G\|_p^{p'} + \|F-G\|_p^{p'} \le \left(\int \left(|F+G|^{p'} + |F-G|^{p'} \right)^{p/p'} \right)^{p'/p} \tag{3.6}$$

$$\le 2 \left(\int |F|^p + |G|^p \right)^{p'/p} . \tag{3.7}$$

This establishes (2.1). The corresponding inequality (2.2) for $p \ge 2$ is straightforward. Note that $a^{p/2} + b^{p/2} \le (a+b)^{p/2}$ for $p \ge 2$. Applying this property and then expanding the squares, we have

$$\begin{aligned} |F+G|^p + |F-G|^p &= \left(\sum |F_i + G_i|^2 \right)^{p/2} + \left(\sum |F_i - G_i|^2 \right)^{p/2} \\ &\le \left(\sum (|F_i + G_i|^2 + |F_i - G_i|^2) \right)^{p/2} \\ &= \left(2 \left(|F|^2 + |G|^2 \right) \right)^{p/2} . \end{aligned} \tag{3.8}$$

Finally, convexity of the function $t \mapsto t^{p/2}$ implies that

$$\begin{aligned} \left(2 \left(|F|^2 + |G|^2 \right) \right)^{p/2} &= 2^p \left(\frac{|F|^2}{2} + \frac{|G|^2}{2} \right)^{p/2} \\ &\le 2^{p-1} (|F|^p + |G|^p). \end{aligned} \tag{3.9}$$

We combine (3.8) and (3.9) and integrate to conclude the proof of (2.2).

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