

SHARP QUANTITATIVE FABER-KRAHN INEQUALITIES AND THE ALT-CAFFARELLI-FRIEDMAN MONOTONICITY FORMULA

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ABSTRACT. The objective of this paper is two-fold. First, we establish new sharp quantitative estimates for Faber-Krahn inequalities on simply connected space forms. In these spaces, geodesic balls uniquely minimize the first eigenvalue of the Dirichlet Laplacian among all sets of a fixed volume. We prove that the gap between the first eigenvalue of a given set Ω and that of the ball quantitatively controls both the L^1 distance of this set from a ball *and* the L^2 distance between the corresponding eigenfunctions:

$$\lambda_1(\Omega) - \lambda_1(B) \gtrsim |\Omega \Delta B|^2 + \int |u_\Omega - u_B|^2,$$

where B denotes the nearest geodesic ball to Ω with $|B| = |\Omega|$ and u_Ω denotes the first eigenfunction with suitable normalization. On Euclidean space, this extends a result of Brasco-De Philippis-Velichkov; the eigenfunction control largely builds upon on new regularity results for minimizers of critically perturbed Alt-Caffarelli type functionals in our companion paper. On the round sphere and hyperbolic space, the present results are the first sharp quantitative results with respect to any distance; here the local portion of the analysis is based on new implicit spectral analysis techniques.

Second, we apply these sharp quantitative Faber-Krahn inequalities in order to establish a quantitative form of the Alt-Caffarelli-Friedman (ACF) monotonicity formula. A powerful tool in the study of free boundary problems, the ACF monotonicity formula is nonincreasing with respect to its scaling parameter for any pair of admissible subharmonic functions, and is constant if and only if the pair comprises two linear functions truncated to complementary half planes. We show that the energy drop in the ACF monotonicity formula from one scale to the next controls how close a pair of admissible functions is from a pair of complementary half-plane solutions. In particular, when the square root of the energy drop summed over all scales is small, our result implies the existence of tangents (unique blowups) of these functions.

1. INTRODUCTION

Let (M, g) be a smooth Riemannian n -manifold, and let Ω be an open bounded subset of M . The first Dirichlet eigenvalue, or principal frequency, of Ω is defined by

$$\lambda_1(\Omega) = \inf \left\{ \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2} : u \in C_0^\infty(\Omega) \right\}. \quad (1.1)$$

Here and in the sequel, $|\nabla u|^2 = \langle \nabla_g u, \nabla_g u \rangle_g$ and integration is with respect to the volume measure induced by the metric g . The infimum in (1.1) is achieved, and a minimizer u_Ω satisfies the equation

$$\begin{cases} -\Delta u_\Omega = \lambda_1(\Omega) u_\Omega & \text{in } \Omega \\ u_\Omega = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

where $\Delta = \Delta_g$ is the Laplacian with respect to g . Such a function u_Ω is called a first Dirichlet eigenfunction of Ω and $\lambda_1(\Omega)$ is the smallest number for which a nontrivial solution to this eigenvalue problem exists. If Ω is connected, then the first Dirichlet eigenfunction is unique up to constant multiples.

In Euclidean space, balls have the smallest principal frequency among subsets $\Omega \subset \mathbb{R}^n$ of a given volume. This fact, first established by Faber [25] and Krahn [35] in the 1920s, readily follows from the Polya-Szegö principle [42], which relies only on the on the coarea formula and the isoperimetry of balls. On all

simply connected space forms (Euclidean space, hyperbolic space and the round sphere), geodesic balls are isoperimetric sets for every volume. Consequently, on simply connected space forms, the Polya-Szegő principle holds and geodesic balls minimize the principal frequency among sets of a given volume. The resulting inequality is known as the *Faber-Krahn inequality*: letting B denote a geodesic ball and $|\cdot|$ denote the volume measure induced by the metric on either Euclidean space, hyperbolic space, or the round sphere, we have

$$\lambda_1(\Omega) \geq \lambda_1(B) \quad \text{whenever } |\Omega| = |B|.$$

It is known, moreover, that equality is achieved if and only if Ω is a geodesic ball, up to a set of capacity zero. On the round sphere, the Faber-Krahn inequality was first established by Sperner in [43] and is often called Sperner's inequality. We refer the reader to [10, 7, 20] for a discussion of spectral inequalities and symmetrization techniques in these settings.

Our first main theorem is a sharp quantitative stability result for the Faber-Krahn inequality on simply connected space forms. Broadly speaking, a quantitative stability result in this context states that

$$\lambda_1(\Omega) - \lambda_1(B) \geq c \operatorname{dist}(\Omega, B)^\alpha \tag{1.3}$$

for some $\alpha > 0$, where B is a geodesic ball with $|\Omega| = |B|$ and $\operatorname{dist}(\Omega, B)$ is a suitable notion of distance between Ω and the nearest ball B . In the statement below, we let u_Ω denote any first eigenfunction of Ω , i.e. solution of (1.2), that is normalized so that $u_\Omega \geq 0$ and $\int u_\Omega^2 = 1$, and extended by 0 to be defined on the entire space. We let $A\Delta B = (A \setminus B) \cup (B \setminus A)$ denote the symmetric difference between sets.

Theorem 1.1 (Sharp quantitative stability for Faber-Krahn inequalities). *Fix $n \geq 2$ and let (M^n, g) denote the round sphere, Euclidean space, or hyperbolic space. For any $v > 0$ (with $v < |S^n|$ in the case of the round sphere), there exists a constant $c = c(n, v)$ such that the following holds. For any open bounded set $\Omega \subset M$ with $|\Omega| = v$, we have*

$$\lambda_1(\Omega) - \lambda_1(B) \geq c \inf_{x \in M} \left(|\Omega \Delta B(x)|^2 + \int_M |u_\Omega - u_{B(x)}|^2 \right). \tag{1.4}$$

Here $B(x)$ is a geodesic ball centered at x with radius uniquely chosen so that $|B(x)| = v$, and B denotes such a ball centered at any point.

Remark 1.2 (Optimality of the quadratic power). Considering ellipsoidal perturbations of a geodesic ball (in normal coordinates in the case of the sphere or hyperbolic space) shows that the quadratic power $\alpha = 2$ is optimal for both of the terms on the right-hand side of (1.4), in the sense that the result is false if 2 is replaced by any $\alpha < 2$.

On Euclidean space, quantitative stability for the Faber-Krahn inequality was first considered by Hansen and Nadirashvili [30] and Melas [39]. Both of these papers restrict their attention to convex sets or simply connected planar sets, and consider “ L^∞ ” (Hausdorff) type set distances $\operatorname{dist}(\Omega, B)$. Among general open bounded sets, simple examples show that even qualitative stability fails with respect to such a distance. For this reason, the *asymmetry*, $\operatorname{dist}(\Omega, B) = \inf_{x \in \mathbb{R}^n} |\Omega \Delta B(x)|$, is a more natural notion of set distance to consider, and quantitative stability with respect to the asymmetry was conjectured in [12] and [41]. Non-sharp versions of this conjecture (i.e. with $\alpha > 2$ in (1.3)) were established in following years in [11] and [27]. All of these results are based on applying quantitative forms of the isoperimetric inequality to level sets of eigenfunctions.

The first sharp quantitative Faber-Krahn inequality on Euclidean space was established by Brasco, De Philippis, and Velichkov [16], stating

$$\lambda_1(\Omega) - \lambda_1(B) \geq c \inf_{x \in \mathbb{R}^n} |\Omega \Delta B(x)|^2 \tag{1.5}$$

for all open bounded sets Ω , where $|B(x)| = |\Omega|$. Their proof was based on a selection principle approach introduced by Cicalese-Leonardi [21] (see also [1]) using the regularity theory for Alt-Caffarelli type functionals. The selection principle is also the basis of the proof of Theorem 1.1, and we discuss these proofs further in Section 1.1 below. Following the work of [16], similar methods were used to establish sharp quantitative stability for the p -Laplacian [28] as well as for the capacity [24] and p -capacity [40] inequalities. We refer the reader to [15] for an overview on spectral inequalities in quantitative form.

A shortfall of the asymmetry as a measure of the distance between sets is that it only detects modifications up to sets of measure zero, while the eigenvalue sees modifications of a set up to sets of zero capacity. For instance, if we let $\Omega = B(0, 1) \setminus \{x_1 = 0\}$ be a slit domain in \mathbb{R}^2 , then the left-hand side of (1.5) is strictly positive while the right-hand side is zero. In contrast, the right-hand side of (1.4) is strictly positive in this example (and more generally, whenever the left-hand side is), and in this way, Theorem 1.1 nontrivially strengthens the result of [16] even in Euclidean space. This issue is discussed in the survey paper [15], and in particular Theorem 1.1 resolves [15, Open Problem 3].

On the sphere and hyperbolic space, significantly less progress has been made toward quantitative Faber-Krahn inequalities. These spaces lack the scaling invariance of Euclidean space, and certain computations that can be carried out explicitly on Euclidean space cannot be done in these spaces. In fact, the explicit forms of the first eigenvalue and eigenfunction of geodesic balls are not known in closed form! Ávila established analogues of Melas' aforementioned results for convex subsets of S^2 and H^2 in [9], following quantitative estimates for other spectral inequalities in [44]. Qualitative stability results for the Faber-Krahn inequality and other spectral inequalities on space forms is established in [8]. To our knowledge, Theorem 1.1 is the first (sharp or nonsharp) quantitative stability result for Faber-Krahn inequalities on the round sphere and hyperbolic space that is valid for all open bounded sets Ω .

It has been understood in recent years that eigenvalue optimization problems are closely linked to the study of free boundary problems. When optimizing functionals involving (linear combinations of) higher eigenvalues, or of the first eigenvalue with respect to additional constraints, one can no longer expect to explicitly characterize minimizers. Instead, one hopes to show that a minimizer exists in a weak class of sets, and then show that the boundary of a minimizer is regular at most points. The paper [17] considered the minimization of $\lambda_1(\Omega)$ among subsets Ω of some fixed bounded open container. Here the authors established regularity properties of minimizing sets by using the regularity theory of Alt and Caffarelli [5] for the one-phase free boundary problem. More recently, minimization problems for functionals that are linear combinations of Dirichlet eigenvalues have been recast as vectorial free boundary problems in [36, 37, 38, 19], and regularity of minimizers has been studied through this lens of free boundary regularity.

In these examples, regularity theory for free boundary problems is used as a tool in the study of eigenvalue optimization problems. In the next main result of this paper, the stream of tools will go in the opposite direction: we apply the quantitative Faber-Krahn inequality of Theorem 1.1 on the sphere in order to establish new quantitative estimates for the Alt-Caffarelli-Friedman monotonicity formula, a powerful tool in the study of various types of free boundary problems.

Fix $n \geq 2$ and let $B_r = B(0, r) \subset \mathbb{R}^n$ denote the ball of radius r centered at the origin in Euclidean space. Let $u_1, u_2 \in H^1(B_2)$ be two nonnegative continuous functions satisfying

$$\begin{cases} \Delta u_i \geq 0 & \text{in } \{u_i > 0\} =: \Omega_i \\ u_1 \cdot u_2 = 0 & \text{in } B_2 \end{cases} \quad (1.6)$$

and assume that $0 \in \partial\Omega_i$. The Alt-Caffarelli-Friedman monotonicity formula (hereafter denoted ACF formula) $J : (0, 1) \rightarrow \mathbb{R}$ is defined by

$$J(r) = J[u_1, u_2](r) = \frac{1}{r^4} \int_{B_r} |\nabla u_1|^2 \int_{B_r} |\nabla u_2|^2 \quad (1.7)$$

when $n = 2$ and

$$J(r) = J[u_1, u_2](r) = \frac{1}{r^4} \int_{B_r} \frac{|\nabla u_1|^2}{|x|^{n-2}} \int_{B_r} \frac{|\nabla u_2|^2}{|x|^{n-2}} \quad (1.8)$$

for $n \geq 3$. Heuristically thinking, one can view $J(r)$ as the product of the average Dirichlet energies of u_1, u_2 on B_r . This quantity has the important property of being nondecreasing for $r \in (0, 1)$. Note that this in particular guarantees the existence of the limit

$$J(0^+) := \lim_{r \rightarrow 0} J(r).$$

Furthermore, if $J(0^+)$ is strictly positive, then $J(r)$ is constant with respect to r if and only if

$$u_1 = \beta_1(x \cdot \nu)^+, \quad u_2 = \beta_2(x \cdot \nu)^-,$$

for some $\nu \in S^{n-1}$ and $\beta_1, \beta_2 > 0$. Here f^+ and f^- respectively denote the positive and negative parts of a function f . In other words, the ACF monotonicity formula is constant and positive if and only if u_1, u_2 are linear functions defined on complementary half spaces Ω_1 and Ω_2 .

Since its introduction in [6], the ACF monotonicity formula has proven useful in both free boundary problems and harmonic analysis. One application of the ACF monotonicity formula is the following. Suppose that a pair of functions u_1, u_2 satisfying (1.6) have $J(0^+) > 0$, and consider any sequence $r_k \rightarrow 0$. There exists a subsequence (which we do not relabel), constants $\beta_1, \beta_2 > 0$ and a direction $\nu \in S^{n-1}$ such that

$$\frac{u_1(r_k x)}{r_k} \rightarrow \beta_1(x \cdot \nu)^+, \quad \frac{u_2(r_k x)}{r_k} \rightarrow \beta_2(x \cdot \nu)^-.$$

However, it is important to note that these blowup limits depend on the sequence r_k and *does not* imply that u_1 and u_2 or their interfaces Ω_1, Ω_2 have unique tangents at the origin. In fact, this need not be the case! In [3] the first two authors gave an example of two harmonic functions defined on complementary subsets of B_2 , which have a slowly spiraling interface, where different sequences $r_k \rightarrow 0$ result in different ν in the limit.

Toward understanding conditions that *do* guarantee the existence of unique tangents for functions u_1 and u_2 as in (1.6), we establish the following quantitative estimates for the Alt-Caffarelli-Friedman monotonicity formula. Roughly speaking, this theorem says that if $J(1) - J(0^+)$ is small, then u_1 and u_2 are quantitatively close to linear functions $\beta_i(x \cdot \nu)^\pm$. Note that if $J(0^+)$ is strictly positive, then up to rescaling we may assume that $J(0^+) = 1$.

Theorem 1.3 (Sharp quantitative stability for the ACF monotonicity formula). *Fix $n \geq 2$ and let $\rho \in [0, 1/2]$. There exists a constant $C = C(n) > 0$ such that the following holds. Suppose that u_1, u_2 satisfy (1.6). Then there exist constants $\beta_1, \beta_2 > 0$ and $\nu \in S^{n-1}$ such that*

$$\int_{B_1 \setminus B_\rho} (u_1 - \beta_1(x \cdot \nu)^+)^2 + (u_2 - \beta_2(x \cdot \nu)^-)^2 \leq C \log \left(\frac{J(1)}{J(\rho)} \right) \sum_{i=1}^2 \|u_i\|_{W^{1,2}(B_1)}^2.$$

Furthermore, there exists a constant $\epsilon_0 = \epsilon_0(n) > 0$ such that if $\log(J(1)/J(0^+)) < \epsilon_0$, then β_i and ν may be chosen independently of $\rho \in [0, 1/2]$.

This theorem is sharp, in the sense that the dependence on $\log \frac{J(1)}{J(\rho)}$ on the right cannot be improved; this can be seen from the examples constructed in [3, Section 2].

A direct consequence of Theorem 1.3 is the following corollary, which gives a criterion for naturally scaled blow-ups of the functions u_i at 0 to be unique. We believe this criterion is essentially sharp, for the same reasons that Theorem 1.3 is sharp. This resolves a question raised in [3].

Corollary 1.4 (Uniqueness of blowups). *Fix $n \geq 2$. Suppose that u_1, u_2 satisfy (1.6), assume $J(0^+) = 1$, and let $\omega(r) = J(r) - 1$. Assume in addition that*

$$\int_0^1 \frac{\sqrt{\omega(r)}}{r} dr < \infty.$$

Then there exists a number $\beta_1 > 0$ and a vector $\nu \in S^{n-1}$ such that

$$\lim_{r \searrow 0} \frac{1}{r^{n+2}} \int_{B_r} |u_1 - \beta_1(\nu \cdot x)_+|^2 = 0.$$

1.1. Proofs of the main theorems. Let us now discuss some aspects of the proofs of our main theorems. We begin by discussing Theorem 1.3, and particularly its connection to Theorem 1.1 in the case of the round sphere. The proof of Theorem 1.3 is accomplished by quantifying the following estimates when $J(1) - J(\rho)$ is small:

- (1) For a large set of $r \in [\rho, 1]$ and for $i = 1, 2$, the restriction of u_i to ∂B_r is close in an L^2 sense to the first Dirichlet eigenfunction of the spherical Laplacian on the set $\{u_i > 0\} \cap \partial B_r$.
- (2) For $i = 1, 2$, the function u_i is close in an L^2 sense to a one-homogeneous function in $B_1 \setminus B_\rho$.
- (3) For a large set of $r \in [\rho, 1]$ and for $i = 1, 2$, the first Dirichlet eigenvalue of the spherical Laplacian on the set $\{u_i > 0\} \cap \partial B_r$ is close to the first eigenvalue of a hemisphere in ∂B_r .

The one-homogeneous extension of the first eigenfunction of a hemisphere in ∂B_1 is a truncated linear function $(x \cdot \nu)^+$ for some $\nu \in S^{n-1}$. Theorem 1.3 is established by combining the estimates above to show that for $i = 1, 2$, the function u_i is close to a constant multiple $\beta_i > 0$ of this linear function, for suitably chosen ν .

In the case when $n = 2$, the proof of Theorem 1.3 is significantly simpler than in higher dimensions, fundamentally because the geometry of $S^1(r) = \partial B_r$ is sufficiently simple that much of the analysis can be done in an explicit manner. In dimension 2, the estimate (3) for the the first eigenvalue of $\{u_i > 0\} \cap \partial B_r$ immediately determines the length of the longest connected curve of $\{u_i > 0\} \cap \partial B_r$, as well as the explicit first eigenfunction on $\{u_i > 0\} \cap \partial B_r$.

When $n > 2$, on the other hand, the estimate (3) contains less immediate information. Knowing the first eigenvalue on $\{u_i > 0\} \cap \partial B_r$ does not determine the shape of $\{u_i > 0\} \cap B_r$, nor the eigenfunction on $\{u_i > 0\} \cap B_r$. It is here that we make crucial use of the sharp quantitative version of Sperner's inequality established in Theorem 1.1. Theorem 1.1 applied to $S^{n-1} = \partial B_1$, in conjunction with the estimate (3) allows us to deduce that the restriction of u_i to ∂B_1 is quantitatively close to the first eigenfunction of a spherical cap of the same volume. We emphasize that for this application, it is essential that we have established the improved form of stability that includes eigenfunction control in Theorem 1.1.

Let us now make some further remarks about Theorem 1.1 and its proof. We begin with some comments about the statement and optimality of the theorem.

Remark 1.5 (Dependence of constants on the volume). In Theorem 1.1, the constant c in the quantitative estimates depend on the volume. On Euclidean space, the scaling invariance can be used to state the Faber-Krahn inequality without a volume constraint as $\lambda_1(\Omega)|\Omega|^{2/n} \geq \lambda_1(B)|B|^{2/n}$ where $B = B_1$, and similarly

(1.4) can be restated in scale-invariant form as

$$\lambda_1(\Omega)|\Omega|^{2/n} - \lambda_1(B)|B|^{2/n} \geq c \left(\inf_{x \in \mathbb{R}^n} \frac{|\Omega \Delta B(x, r)|^2}{|\Omega|^2} + \inf_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |u_\Omega - u_{B(x, r)}|^2 \right)$$

where c is a constant depending on dimension only and r is chosen so that $|\Omega| = |B_r|$. On the sphere and hyperbolic space, the constant can clearly be taken to be uniform for any compact subset of volumes. We make no attempt to track the dependence of the constant $c = c(n, v)$ in (1.4) as $v \rightarrow 0$ or as $v \rightarrow |S^n|$ (in the case of the sphere) or $v \rightarrow \infty$ (in the case of hyperbolic space), though as $v \rightarrow 0$ one expects that the constants can be made uniform after the suitable rescaling.

Remark 1.6 (Optimality of the L^2 norm). It is natural to wonder if the L^2 distance between eigenfunctions in (1.4) can be strengthened to the H^1 distance. This is false, at least with the quadratic power. Indeed, let B be a ball of maximal overlap with Ω and suppose by way of contradiction that there exists $c > 0$ such that $\lambda_1(\Omega) - \lambda_1(B) \geq c \int |\nabla u_\Omega - \nabla u_B|^2$. On $B \setminus \Omega$, we have $\nabla u_\Omega = 0$ and $|\nabla u_B| \approx 1$. So,

$$\int |\nabla u_\Omega - \nabla u_B|^2 \geq \int_{B \setminus \Omega} |\nabla u_B|^2 \geq c |B \setminus \Omega| = \frac{c}{2} |B \Delta \Omega|.$$

This would imply that the eigenvalue deficit $\lambda_1(\Omega) - \lambda_1(B)$ controls the asymmetry *linearly*, which as we have already discussed is false. It is not hard to show that *qualitative* stability holds for the H^1 norm and we do not know if quantitative stability may hold with a different modulus of continuity, e.g. for some $\alpha > 2$ in (1.3).

Remark 1.7 (Quotient spaces). Naturally, the Polya-Szegő principle and thus the Faber-Krahn inequality hold on quotients of Euclidean space, hyperbolic space, and the round sphere when the volume of Ω is small. Theorem 1.1 also generalizes to quotient spaces for the volume v beneath a suitable threshold.

The proof of Theorem 1.1, like the proof of the quantitative Faber-Krahn inequality of [16], is based on a *selection principle* argument. This global-to-local reduction tool, introduced by Cicalese and Leonardi in [21] in the context of the isoperimetric inequality, is perhaps the most robust method to prove quantitative geometric inequalities and allows one to trade the task of proving stability in the entire class of objects for the task of proving stability among minimizers of some penalized functional, which typically enjoy strong regularity properties.

Let us sketch the basic scheme of the selection principle argument. Suppose by way of contradiction that the desired stability result (1.4) fails, and so there exist a sequence of open bounded sets Ω_j of a fixed volume v with $\lambda_1(\Omega_j) \rightarrow \lambda_1(B)$ and yet

$$\lambda_1(\Omega_j) - \lambda_1(B) \leq \frac{\text{dist}(\Omega_j, B)^2}{j}, \quad (1.9)$$

where we let $\text{dist}(\Omega, B)$ denote the distance on the right-hand side of (1.4). Now, consider a functional of the form

$$\mathcal{F}_j(\Omega) = \lambda_1(\Omega) + \tau \sqrt{c_j^2 + (c_j - \text{dist}(\Omega, B))^2}, \quad (1.10)$$

where $c_j = \text{dist}(\Omega_j, B)^2$ which is designed so that a minimizer Ω'_j of \mathcal{F}_j among sets of a fixed volume satisfies the same contradiction assumption (1.9). Theorem 1.1 will then follow once we establish the following two main ingredients for a selection principle:

- (1) (Regularity) Minimizers Ω'_j of the penalized functionals \mathcal{F}_j satisfy good ε -regularity properties. Combined with the (easily shown) fact that the Ω'_j converge to a ball B in a weak sense, this will guarantee that for j sufficiently large, the boundary of Ω'_j is a small $C^{2, \alpha}$ perturbation of the boundary of B .

- (2) (Local stability) The quantitative stability estimate (1.4) holds for sets that whose boundaries are small $C^{2,\alpha}$ perturbations of the boundary of B . This step is based on *linear stability*, i.e. strict positivity of the second variation of $\lambda_1(B)$ in non-translational directions.

In the context of Theorem 1.1, significant challenges arise in both of these two steps. Let us describe these challenges, and the new strategies introduced to deal with them.

Step (1): For an energy like (1.10) in which $\lambda_1(\Omega)$ (or the closely related torsional rigidity below) is the dominant term, the functional \mathcal{F}_j can be realized as a perturbation of the Alt-Caffarelli functional [5]. However, in contrast to [16], the presence of L^2 eigenvalue penalization in the distance makes this perturbation a critical one and leads to significant issues for regularity. Step (1) is carried out in our companion paper [4], where we consider a selection principle associated to the geometric inequality

$$\lambda_1(\Omega) + \mathfrak{T}\text{tor}(\Omega) \geq \lambda_1(B) + \mathfrak{T}\text{tor}(B)$$

for a small number $\mathfrak{T} > 0$. Here $\text{tor}(\Omega)$ is the (negative of the physical quantity) *torsional rigidity* of an open set $\Omega \subset \mathbb{R}^n$, defined and discussed further in Section 3.2. In Section 3.2, we will see how to connect this selection principle with the one we need to prove Theorem 1.1, using the so-called Kohler-Jobin inequality.

Step (2): In nearly all selection principle arguments, including the quantitative Faber-Krahn inequality of [16] in Euclidean space, Step (1) can be carried out using an *explicit* analysis of the second variation. On the round sphere and hyperbolic space, these computations can no longer be carried out explicitly. To overcome this challenge, we introduce a new technique, based on the maximum principle, to perform an *implicit spectral analysis*. We exploit the symmetry of the minimizers, which are balls, (thus indirectly using their explicit form), and use an ODE maximum principle argument to establish the desired spectral gap. To our knowledge, this is the first time this method has been used in a quantitative stability estimate or in a spectral analysis otherwise.

The remainder of the paper is organized in the following way. In Section 2 we prove the quantitative estimates for the Alt-Caffarelli-Friedman monotonicity formula of Theorem 1.3. We present the proof in the cases $n = 2$ and $n \geq 3$ separately because the two-dimensional case is simpler and provides a basic outline of the proof for higher dimensions. Corollary 1.4 is also proven in this section.

In Section 3, we prove Theorem 1.1. In the interest of clarity, we present the proof in the specific case of the round sphere, which is also the essential case in our application to Theorem 1.3. The proof carries over almost verbatim to Euclidean space and hyperbolic space, and we therefore only remark on the necessary modifications to the proof in these cases. As we have mentioned, the regularity estimates of Step (1), as well as the details of the selection principle, are shown in our companion paper [4]. Section 3 is thus largely dedicated to proving Step (2) in Section 3.1, with the regularity results of [4] recalled and the conclusion of Theorem 1.1 drawn in Section 3.2.

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2. QUANTITATIVE STABILITY FOR THE ALT-CAFFARELLI-FRIEDMAN MONOTONICITY FORMULA

This section is dedicated to the proof of Theorem 1.3 and Corollary 1.4. The cases $n = 2$ and $n \geq 3$ are carried out separately in Sections 2.1 and 2.2 respectively. Corollary 1.4 is shown in Section 2.3.

2.1. The two dimensional case. Throughout the section, we let $n = 2$ and let $u_1, u_2 : B_1 \rightarrow \mathbb{R}$ be two nonnegative functions satisfying (1.6). Let $J = J[u_1, u_2] : (0, 1) \rightarrow \mathbb{R}_+$ be the two dimensional Alt-Caffarelli-Friedman monotonicity formula defined in (1.7).

To begin, we prove the monotonicity of the functional $J(r)$ following [6, 18], along the way introducing some notation and outlining the main ideas of the proof of Theorem 1.3 when $n = 2$. We directly compute

$$\log(J(r))' = \frac{\int_{\partial B_r} |\nabla u_1|^2}{\int_{B_r} |\nabla u_1|^2} + \frac{\int_{\partial B_r} |\nabla u_2|^2}{\int_{B_r} |\nabla u_2|^2} - \frac{4}{r}. \quad (2.1)$$

For each fixed $r \in (0, 1]$ and for $i = 1, 2$, we denote by $\{\lambda_{k,r}^i\}$ and $\{Y_{k,r}^i\}$ the Dirichlet eigenvalues and eigenfunctions of the Laplacian $-\partial_{\tau\tau}$ on the set $\Omega_r^i = \{u_i > 0\} \cap \partial B_r$, with the eigenfunctions normalized so that $\|Y_{k,r}^i\|_{L^2(\Omega_r^i)} = 1$. Here and in the sequel, we let ∂_τ denote the tangential derivative on ∂B_r , so that in polar coordinates (r, θ) for \mathbb{R}^2 , we have $\partial_\tau = \frac{1}{r}\partial_\theta$. The positivity set Ω_r^i comprises a union of circular arcs in ∂B_r , and so the first eigenvalue and eigenfunction of this set are the (explicit) first eigenvalue and eigenfunction of the longest circular arc. More specifically, define $\theta_i(r) \in [0, 2\pi]$ so that the length of the longest circular arc in Ω_r^i is $r\theta_i(r)$. The first eigenvalue and eigenfunction of Ω_r^i are given by (up to the obvious rotation)

$$\lambda_{1,r}^i = \frac{\pi^2}{r^2\theta_i(r)^2}, \quad Y_{1,r}^i = \sqrt{\frac{2}{r\theta_i(r)}} \sin\left(r\sqrt{\lambda_{1,r}^i}\theta\right)^+. \quad (2.2)$$

Here we have written the eigenfunction in polar coordinates (r, θ) for \mathbb{R}^2 . It is worth noting that for a semicircle in ∂B_r , the first eigenfunction is $\lambda = 1/r^2$ and the first eigenfunction is $Y = \sqrt{2/\pi r} \sin(\theta)^+$, whose one-homogeneous extension is a linear function restricted to a half plane. Observe that $\theta_1(r) + \theta_2(r) \leq 2\pi$ because the supports of u_1 and u_2 are disjoint.

Now, we note that for $i = 1, 2$,

$$\begin{aligned} \int_{\partial B_r} |\nabla u_i|^2 &= \int_{\partial B_r} (\partial_\tau u_i)^2 + (\partial_r u_i^2) \\ &= \left(\int_{\partial B_r} (\partial_\tau u_i)^2 - \lambda_{1,r}^i u_i^2 \right) + \left(\int_{\partial B_r} \lambda_{1,r}^i u_i^2 - 2\sqrt{\lambda_{1,r}^i} u_i \partial_r u_i \right) + \left(\int_{\partial B_r} 2\sqrt{\lambda_{1,r}^i} u_i \partial_r u_i \right). \end{aligned}$$

Using now the subharmonicity and nonnegativity of u_i we have

$$\int_{B_r} |\nabla u_i|^2 \leq \int_{\partial B_r} u_i \partial_r u_i,$$

so that dividing by $\int_{\partial B_r} |\nabla u_i|^2$, summing in i , and subtracting $4/r$ we obtain from (2.1) and (2.2) that

$$\log(J(r))' \geq \delta_A(r) + \delta_B(r) + \delta_C(r), \quad (2.3)$$

where we define

$$\begin{aligned} \delta_A(r) &= \sum_{i=1}^2 \frac{1}{\int_{B_r} |\nabla u_i|^2} \left(\int_{\partial B_r} (\partial_\tau u_i)^2 - \lambda_{1,r}^i \int_{\partial B_r} u_i^2 \right), \\ \delta_B(r) &= \sum_{i=1}^2 \frac{1}{\int_{B_r} |\nabla u_i|^2} \left(\lambda_{1,r}^i \int_{\partial B_r} u_i^2 + \int_{\partial B_r} (\partial_r u_i)^2 - 2\sqrt{\lambda_{1,r}^i} \int_{\partial B_r} u_i \partial_r u_i \right), \\ \delta_C(r) &= \frac{2}{r} \left(\frac{\pi}{\theta_1(r)} + \frac{\pi}{\theta_2(r)} - 2 \right). \end{aligned}$$

The ACF functional $J(r)$ is shown to be monotone in [6, 18] by showing that each of $\delta_A(r)$, $\delta_B(r)$, and $\delta_C(r)$ is nonnegative; this is in particular a consequence of Lemma 2.1 below. The quantitative monotonicity formula

of Theorem 1.3 will be established by quantifying the loss in each inequality $\delta_{(\cdot)}(r) \geq 0$ and integrating with respect to r . Let us informally discuss the information contained in each $\delta_{(\cdot)}(r)$.

The quantity $\delta_A(r)$ is nonnegative by the Poincaré inequality on Ω_r^i , and equals zero for a given $r \in (0, 1]$ if and only if u_i restricted to ∂B_r is equal to the first eigenfunction $Y_{1,r}^i$ of Ω_r^i . By quantifying the positivity of $\delta_A(r)$ in (2.4) in Lemma 2.1 below, we see that $\delta_A(r)$ quantitatively controls the distance of u_i to the eigenfunction on this set.

The term in parentheses in the definition of $\delta_B(r)$ can be realized as a square as in (2.5) in Lemma 2.1 below. This immediately shows that $\delta_B(r)$ is nonnegative for all $r \in (0, 1)$. Furthermore, it provides quantitative control on how closely the restriction of u_i to ∂B_r behaves like a homogeneous function with homogeneity $\sqrt{\lambda_{1,r}^i}/r$.

The quantity $\delta_C(r)$ can be bounded below by square functions; see (2.6) in Lemma 2.1 below, which in particular shows that $\delta_C(r) \geq 0$. The estimate (2.6) shows that the longest circular arc in Ω_r^i (and thus Ω_r^i itself) is quantitatively close to a semicircle in ∂B_r , and the corresponding first eigenvalue $\lambda_{1,r}^i$ is quantitatively close to the first eigenvalue $1/r^2$ of a semicircle in ∂B_r .

The proof of Theorem 1.3 roughly goes as follows. By integrating $\delta_B(r)$ and $\delta_C(r)$ from ρ to 1, we show in Lemma 2.2 that each u_i is quantitatively close to the one-homogeneous function defined by $ru_i(1, \theta)$ in polar coordinates. Then, in Proposition 2.4, we apply the estimate (2.4) for $\delta_A(1)$ to deduce that $u_i(1, \theta)$ is almost equal to a multiple of first eigenfunction $Y_{1,1}^i$ of Ω_1^i , and applying the estimate (2.6) once again, we see that $Y_{1,1}^i$ is quantitatively close to the first eigenfunction of the semicircle. Finally, Theorem 1.3 follows from Proposition 2.4 once we guarantee that the semicircles corresponding to u_1 and u_2 do not have too much overlap.

The following lemma provides basic quantitative lower bounds for each of $\delta_A(r)$, $\delta_B(r)$, and $\delta_C(r)$ discussed above. Since each of the right-hand sides of (2.4)-(2.6) are nonnegative, Lemma 2.1 and (2.3) in particular imply the monotonicity of $J(r)$.

Lemma 2.1. *For each $r \in (0, 1]$ and $i = 1, 2$, let $Y_{i,r}^1$ be the eigenfunction defined in (2.2) and let $\beta_{1,r}^i = \int_{\partial B_r} u_i Y_{1,r}^i$. We have*

$$\delta_A(r) \geq \sum_{i=1}^2 \frac{(\lambda_{2,r}^i - \lambda_{1,r}^i)}{\int_{B_r} |\nabla u_i|^2} \int_{\partial B_r} (u_i - \beta_{1,r}^i Y_{1,r}^i)^2, \quad (2.4)$$

$$\delta_B(r) = \sum_{i=1}^2 \frac{1}{\int_{B_r} |\nabla u_i|^2} \int_{\partial B_r} \left(\sqrt{\lambda_{1,r}^i} u_i - \partial_r u_i \right)^2, \quad (2.5)$$

$$\delta_C(r) \geq \sum_{i=1}^2 \frac{2}{r} \frac{(\pi - \theta_i(r))^2}{\theta_i(r)(2\pi - \theta_i(r))} \geq \sum_{i=1}^2 \frac{1}{\sqrt{\lambda_{1,r}^i}} \left(\sqrt{\lambda_{1,r}^i} - \frac{1}{r} \right)^2. \quad (2.6)$$

Proof. We fix $r \in (0, 1]$, and for notational simplicity omit the subscripts r in the remainder of the proof. We first prove (2.4). For $i = 1, 2$, let $\beta_k^i = \int_{\partial B_r} u_i Y_k^i$, so, expanding the term in parentheses of $\delta_A(r)$ in this basis, we have

$$\begin{aligned} \int_{\partial B_r} (\partial_r u_i)^2 - \lambda_1^i \int_{\partial B_r} u_i^2 &= \sum_{k=1}^{\infty} \lambda_k^i (\beta_k^i)^2 - \lambda_1^i \sum_{k=1}^{\infty} (\beta_k^i)^2 \\ &\geq (\lambda_2^i - \lambda_1^i) \sum_{k=2}^{\infty} (\beta_k^i)^2 = (\lambda_2^i - \lambda_1^i) \int_{\partial B_r} (u_i - \beta_1^i Y_1^i)^2. \end{aligned}$$

Dividing this expression by $\int_{B_r} |\nabla u_i|^2$ and summing from $i = 1$ to 2 gives (2.4). Next, (2.5) is immediate by writing the integrand as a square. Finally, to prove (2.6), recall that $\theta_2(r) \leq 2\pi - \theta_1(r)$, and so

$$\begin{aligned} \delta_C(r) &\geq \frac{2}{r} \left(\frac{\pi}{\theta_1(r)} + \frac{\pi}{2\pi - \theta_1(r)} - 2 \right) \\ &= \frac{2}{r} \left(\frac{2\pi^2}{\theta_1(r)(2\pi - \theta_1(r))} - 2 \right) = \frac{4}{r} \frac{(\pi - \theta_1(r))^2}{\theta_1(r)(2\pi - \theta_1(r))}. \end{aligned}$$

The analogous expression holds with θ_2 in place of θ_1 . Summing these expressions and dividing by 2 gives the first estimate in (2.6). For the second, use (2.2) to see

$$\frac{4}{r} \frac{(\pi - \theta_1(r))^2}{\theta_1(r)(2\pi - \theta_1(r))} = \frac{1}{\sqrt{\lambda_1^1}} \frac{4\pi}{(2\pi - \theta_1(r))} \left(\sqrt{\lambda_1^1} - \frac{1}{r} \right)^2 \geq \frac{2}{\sqrt{\lambda_1^1}} \left(\sqrt{\lambda_1^1} - \frac{1}{r} \right)^2.$$

Again, the same expression holds with θ_2 in place of θ_1 . Summing these expressions and dividing by 2 gives the second estimate in (2.6). This concludes the proof of the lemma. \square

The following lemma makes use of the estimates for $\delta_B(r)$ and $\delta_C(r)$ in Lemma 2.1 and shows that u_1 and u_2 are quantitatively close to being 1-homogeneous.

Lemma 2.2. *Let u_1, u_2 be as above. Then for a universal constant $C > 0$ and any $\rho \in (0, 1)$,*

$$\int_{B_1 \setminus B_\rho} [ru_i(1, \theta) - u_i(r, \theta)]^2 \leq C \log \left(\frac{J(1)}{J(\rho)} \right) \|u_i\|_{W^{1,2}(B_1)}^2.$$

Proof. Fix $i \in \{1, 2\}$. For notational simplicity, we omit the sub and superscripts i in this proof.

We first note that

$$\begin{aligned} \int_{B_1 \setminus B_\rho} [ru(1, \theta) - u(r, \theta)]^2 &\leq 2 \int_{B_1 \setminus B_\rho} [ru(1, \theta) - A(r)u(1, \theta)]^2 + 2 \int_{B_1 \setminus B_\rho} [A(r)u(1, \theta) - u(r, \theta)]^2 \\ &= (I) + (II), \end{aligned}$$

where we define $A(r) := e^{-\int_r^1 \sqrt{\lambda(t)} dt}$ and $\lambda(t) = \lambda_{1,t}$. We bound (II) in the following manner. Using that

$$u(1, \theta)A(r) - u(r, \theta) = \int_r^1 [u_r(s, \theta) - \sqrt{\lambda(s)}u(s, \theta)] e^{-\int_r^s \sqrt{\lambda(t)} dt} ds$$

as well as that $|e^{-\int_r^s \sqrt{\lambda(t)} dt}| \leq 1$ we have

$$|u(1, \theta)A(r) - u(r, \theta)|^2 \leq \int_r^1 |u_r(s, \theta) - \sqrt{\lambda(s)}u(s, \theta)|^2 ds. \quad (2.7)$$

Now to continue with (II) we have

$$\begin{aligned}
\int_{B_1 \setminus B_\rho} [u(r, \theta) - A(r)u(1, \theta)]^2 &= \int_\rho^1 \int_{\partial B_1} r [u(r, \theta) - A(r)u(1, \theta)]^2 d\theta dr \\
&\leq \int_\rho^1 \int_{\partial B_1} r \int_r^1 |u_r(s, \theta) - \sqrt{\lambda(s)}u(s, \theta)|^2 ds d\theta dr \\
&\leq \int_\rho^1 \int_{\partial B_1} \int_r^1 s |u_r(s, \theta) - \sqrt{\lambda(s)}u(s, \theta)|^2 ds d\theta dr \\
&\leq \int_\rho^1 \int_{\partial B_1} \int_\rho^1 s |u_r(s, \theta) - \sqrt{\lambda(s)}u(s, \theta)|^2 ds d\theta dr \\
&\leq \int_\rho^1 \int_\rho^1 \delta_B(s) \left(\int_{B_s} |\nabla u|^2 \right) ds dr \\
&\leq \left(\int_\rho^1 \delta_B(s) ds \right) \int_{B_1} |\nabla u|^2.
\end{aligned}$$

In order to bound (I) we first note that if $\sqrt{\lambda(s)} \geq 2/s$, then from (2.6) we have that $\sqrt{\lambda(s)} \leq 4\delta_C(s)$, so that $\max\{\sqrt{\lambda(s)}, 2/s\} \leq 4\delta_C(s) + 2/s$. Now using that e^{-x} is 1-Lipschitz for $x \geq 0$ we have

$$\begin{aligned}
|A(r) - r| &= \left| e^{-\int_r^1 \sqrt{\lambda(s)} ds} - e^{-\int_r^1 1/s ds} \right| \\
&\leq \left| \int_r^1 \sqrt{\lambda(s)} - 1/s ds \right| \\
&\leq \left(\int_r^1 \frac{(\sqrt{\lambda(s)} - 1/s)^2}{\sqrt{\lambda(s)}} ds \right)^{1/2} \left(\int_r^1 \sqrt{\lambda(s)} ds \right)^{1/2} \\
&\leq \left(\int_r^1 \delta_C(s) ds \right)^{1/2} \left(\int_r^1 \max\{\sqrt{\lambda(s)}, 2/s\} ds \right)^{1/2} \\
&\leq \left(\int_r^1 \delta_C(s) ds \right)^{1/2} \left(\int_r^1 4\delta_C(s) + 2/s ds \right)^{1/2} \\
&\leq \left(\int_r^1 \delta_C(s) ds \right)^{1/2} (4 - 2 \ln r)^{1/2},
\end{aligned}$$

so that

$$|A(r) - r|^2 \leq \left(\int_r^1 \delta_C(s) ds \right) (4 - 2 \ln r). \quad (2.8)$$

To continue with (I), we use the trace inequality on B_1 and the fact that $r \mapsto r(4 - 2 \ln r)$ is bounded for $r \in (0, 1]$ to find

$$\begin{aligned} \int_{B_1 \setminus B_\rho} [ru(1, \theta) - A(r)u(1, \theta)]^2 &= \left(\int_{\partial B_1} u^2(1, \theta) d\theta \right) \int_\rho^1 r[r - A(r)]^2 dr \\ &\leq C \|u\|_{W^{1,2}(B_1)}^2 \int_\rho^1 r \left(\int_r^1 \delta_C(s) ds \right) (4 - 2 \ln r) dr \\ &\leq C \|u\|_{W^{1,2}(B_1)}^2 \int_\rho^1 \left(\int_\rho^1 \delta_C(s) ds \right) dr \\ &\leq C \|u\|_{W^{1,2}(B_1)}^2 \int_\rho^1 \delta_C(s) ds. \end{aligned}$$

Recalling (2.3) concludes the proof. \square

This next lemma is a straightforward computation for sine functions that will be used in the proofs of Proposition 2.4 and Theorem 1.3.

Lemma 2.3. *There exists a universal constant $C > 0$ such that for any $\hat{\theta} \in [0, 2\pi)$ and any $\lambda > 0$ with $|1 - \sqrt{\lambda}| \leq 1/2$,*

$$\int_{S^1} [\sin(\theta + \hat{\theta})^+ - \sin(\theta)^+]^2 \leq C\hat{\theta}^2 \quad (2.9)$$

$$\int_{S^1} [\sin(\sqrt{\lambda}\theta)^+ - \sin(\theta)^+]^2 \leq C(1 - \sqrt{\lambda})^2. \quad (2.10)$$

Proof. To estimate (2.9), from the bound $|\sin'(\theta)| \leq 1$, one has $(\sin(\theta + \hat{\theta}) - \sin(\theta))^2 \leq C\hat{\theta}^2$. If $\sin(\theta + \hat{\theta}) > 0$ and $\sin(\theta) \leq 0$, then $(\sin(\theta + \hat{\theta}))^2 \leq \hat{\theta}^2$. This is sufficient to conclude (2.9).

To estimate (2.10) we break up the integral over S^1 into the regions $(0, \pi) \cup (\pi, \bar{\theta})$ where we let $\bar{\theta} = \pi/\sqrt{\lambda}$. If $\sqrt{\lambda} > 1$, then it suffices to only consider the interval $(0, \pi)$. We first have

$$\begin{aligned} \int_0^\pi [\sin(\sqrt{\lambda}\theta)^+ - \sin(\theta)^+]^2 &\leq \int_0^\pi [\sin(\sqrt{\lambda}\theta) - \sin(\theta)]^2 \\ &= \pi - \frac{\sin((\sqrt{\lambda} - 1)\pi)}{\sqrt{\lambda} - 1} - \frac{\sin(2\sqrt{\lambda}\pi)}{4\sqrt{\lambda}} + \frac{\sin((\sqrt{\lambda} + 1)\pi)}{\sqrt{\lambda} + 1} \end{aligned}$$

which is a smooth function in $\sqrt{\lambda}$ and has a minimum of zero at $\sqrt{\lambda} = 1$, so there exists a universal constant C such that

$$\int_0^\pi [\sin(\sqrt{\lambda}\theta)^+ - \sin(\theta)^+]^2 \leq C(1 - \sqrt{\lambda})^2$$

provided that $|\sqrt{\lambda} - 1| \leq 1/2$. Furthermore, using a change of variables and a Taylor expansion of $\sin^2(\cdot)$ at $\theta = 0$, we have

$$\int_\pi^{\bar{\theta}} [\sin(\sqrt{\lambda}\theta)^+ - \sin(\theta)^+]^2 = \int_\pi^{\bar{\theta}} \sin^2(\sqrt{\lambda}\theta) = \int_0^{\bar{\theta} - \pi} \sin^2(\sqrt{\lambda}\theta) \leq C(1 - \sqrt{\lambda})^3$$

for a universal constant C when $|\sqrt{\lambda} - 1| \leq 1/2$. We conclude (2.10). \square

The next proposition contains the core of the proof of Theorem 1.3 and shows that u_1 and u_2 are each well-approximated by a linear function in a quantitative sense. From Lemma 2.2, we know that u_1 and u_2 are each well-approximated by one-homogeneous functions. In this proposition, we show that these one-homogeneous functions can be well-approximated by linear functions.

Proposition 2.4. *Assume $\log(J(1)/J(\rho)) \leq 1$ with $\rho \in [0, 1/2]$. For $i = 1, 2$, let $\hat{\beta}_i = \beta_{1,1}^i = \int_{\partial B_1} u_i Y_{1,1}^i$ and let $\beta_i = \sqrt{2}\hat{\beta}_i/\sqrt{\pi}$. There exists $\theta_i \in [0, 2\pi]$ such that if we let v_i be the linear function defined in polar coordinates by $v_i = r \sin(\theta + \theta_i)$, then*

$$\int_{B_1 \setminus B_\rho} [u_i - \beta_i v_i^+]^2 \leq C \log \left(\frac{J(1)}{J(\rho)} \right) \|u_i\|_{W^{1,2}(B_1)}^2.$$

Proof. Let

$$\varepsilon = \log \left(\frac{J(1)}{J(\rho)} \right) = \int_\rho^1 \log(J(r))' dr.$$

Since the integrand is nonnegative, it follows from Chebyshev's inequality that $\log(J(r))' \leq 10\varepsilon$ outside a set of measure $\varepsilon/10$. In particular, $\log(J(r))' \leq 10\varepsilon$ for some $r \in (1 - \varepsilon/5, 1]$. So, by scaling we may assume that

$$\log(J(1))' \leq \varepsilon. \quad (2.11)$$

Fix $i \in \{1, 2\}$. For the remainder of the proof, we omit subscripts i for notational simplicity. After a rotation, we assume that the longest arc on which $u \cap \partial B_1$ is positive is $(0, \theta(1))$. After this rotation, we let $v = r \sin(\theta)$. Applying the estimate (2.6) for $\delta_C(r)$ with $r = 1$, we deduce that there exist universal constants c and C such that

$$|\pi - \theta(1)| \leq C\sqrt{\varepsilon}, \quad |\sqrt{\lambda} - 1| \leq C\sqrt{\varepsilon}, \quad \lambda_2 - \lambda \geq c. \quad (2.12)$$

Here and in the remainder of the proof, we let $\lambda = \lambda_{1,1}$, i.e. λ is the first Dirichlet eigenvalue of Ω_1 . Likewise, λ_2 denotes the second Dirichlet eigenvalue of Ω_1 . We now apply the triangle inequality to split the integral on the left-hand side into three pieces as follows:

$$\begin{aligned} \int_{B_1 \setminus B_\rho} (u - \beta v^+)^2 &\leq 4 \int_{B_1 \setminus B_\rho} (u - ru(1, \theta))^2 \\ &\quad + 4 \int_{B_1 \setminus B_\rho} (ru(1, \theta) - r\hat{\beta}Y_{1,1})^2 \\ &\quad + 4 \int_{B_1 \setminus B_\rho} (\hat{\beta}rY_{1,1} - \beta r \sin(\theta)^+)^2. \end{aligned} \quad (2.13)$$

We apply Lemma 2.2 to the first term on the right-hand side of (2.13) to find that for a universal constant C , we have

$$4 \int_{B_1 \setminus B_\rho} (u - ru(1, \theta))^2 \leq C\varepsilon \|u\|_{W^{1,2}(B_1)}^2. \quad (2.14)$$

Next, we use the estimate (2.4) for $\delta_A(r)$ with $r = 1$ to bound the second term on the right-hand side of (2.13). Indeed, by (2.4) and the uniform eigenvalue gap (2.12), we see that

$$\begin{aligned} 4 \int_{B_1 \setminus B_\rho} (ru(1, \theta) - \hat{\beta}rY_{1,1})^2 &\leq C \int_{\partial B_1} (u(1, \theta) - \hat{\beta}Y_{1,1})^2 \\ &\leq C \log(J(1))' \|\nabla u\|_{L^2(B_1)}^2 \leq C\varepsilon \|\nabla u\|_{L^2(B_1)}^2 \end{aligned} \quad (2.15)$$

for a universal constant C . In the final inequality we have made use of (2.11).

Finally, the third term in (2.13) can be estimated using the eigenvalue estimate in (2.12) and the estimate (2.10) from Lemma 2.3. Indeed, noting that $\hat{\beta} \leq \|u\|_{L^2(\partial B_1)} \leq C\|u\|_{W^{1,2}(B_1)}$, we conclude that

$$\begin{aligned}
4 \int_{B_1 \setminus B_\rho} (\hat{\beta} r Y_{1,1} - \beta r \sin(\theta)^+)^2 &\leq \hat{\beta}^2 \int_{\partial B_1} \left[Y_{1,1} - \sqrt{\frac{2}{\pi}} \sin(\theta)^+ \right]^2 \\
&\leq C \hat{\beta}^2 \int_{\partial B_1} \left[\sqrt{\frac{2}{\theta(1)}} \sin(\sqrt{\lambda}\theta)^+ - \sqrt{\frac{2}{\pi}} \sin(\theta)^+ \right]^2 \\
&\leq C \hat{\beta}^2 \int_{\partial B_1} \left[\sin(\sqrt{\lambda}\theta)^+ - \sin(\theta)^+ \right]^2 + C \hat{\beta}^2 \left(\frac{1}{\sqrt{\theta(1)}} - \frac{1}{\sqrt{\pi}} \right)^2 \int_{\partial B_1} \left[\sin(\sqrt{\lambda}\theta)^+ \right]^2 \\
&\leq C \hat{\beta}^2 (1 - \sqrt{\lambda})^2 + C \hat{\beta}^2 (\pi - \theta(1))^2 \leq C \varepsilon \|u\|_{W^{1,2}(B_1)}^2.
\end{aligned} \tag{2.16}$$

By combining (2.13), (2.14), (2.15) and (2.16), we conclude the proof of the proposition. \square

We are now ready to prove Theorem 1.3 in the case $n = 2$.

Proof of Theorem 1.3 for $n = 2$. If $\log(J(1)/J(\rho)) \geq 1$, then we may choose $\beta_i = \|u\|_{L^2(B_1)}$ and $\nu = e_1$, and the result immediately follows for large enough C depending only on n . If $\log(J(1)/J(\rho)) < 1$, then we may choose for $i = 1, 2$, the constants β_1^i and θ_i be as in Proposition 2.4. In order to conclude Theorem 1.3, the only issue is that the functions $\sin(\theta + \theta_1)^+$ and $\sin(\theta + \theta_2)^+$ may not have disjoint support. However, from (2.12) we point out that the overlap may not exceed $C\sqrt{\varepsilon}$. Therefore, by Lemma 2.3, we may rotate both functions to have disjoint support and thus obtain the result. \square

2.2. The n -dimensional case. We now move to the proof of Theorem 1.3 in higher dimensions. Throughout this section, let $n \geq 3$, let u_1, u_2 be as in (1.6) and let $J = J[u_1, u_2] : (0, 1] \rightarrow \mathbb{R}$ be the n -dimensional Alt-Caffarelli-Friedman monotonicity formula defined in (1.8). The proof of Theorem 1.3 follows the same basic scheme as the proof in the two dimensional setting, wherein we estimate the positivity of $\log(J(r))'$ on each sphere ∂B_r . Whereas in the two dimensional case the positivity sets $\partial B_r \cap \{u_i > 0\}$ were unions of circular arcs, in higher dimensions the geometry of the positivity sets may be more complicated. In particular, we can no longer hope to write the first eigenfunction and eigenvalue in an explicit way. The key tool to overcome this difficulty is Theorem 1.1. In this section only, we will use the notation \mathcal{B} to denote an $(n-1)$ -dimensional spherical cap (i.e. geodesic ball in S^{n-1}), so as not to be confused with the notation B used for a ball in n -dimensional Euclidean space.

To begin the n -dimensional setting, let us recall the proof from [6, 18] that the ACF formula (1.8) is monotone. We directly compute

$$\log(J(r))' = \frac{r^{2-n} \int_{\partial B_r} |\nabla u_1|^2}{\int_{B_r} \frac{|\nabla u_1|^2}{|x|^{n-2}}} + \frac{r^{2-n} \int_{\partial B_r} |\nabla u_2|^2}{\int_{B_r} \frac{|\nabla u_2|^2}{|x|^{n-2}}} - \frac{4}{r}. \tag{2.17}$$

For each $r \in (0, 1)$ and $i = 1, 2$, set $\Omega_r^i = \{u_i > 0\} \cap \partial B_r$. Let $\lambda_{1,r}^i$ and $\tilde{u}_{i,r}$ respectively denote the first Dirichlet eigenvalue and eigenfunction $-\Delta_\tau$ on Ω_r^i , where $\tilde{u}_{i,r}$ is normalized so that $\|\tilde{u}_{i,r}\|_{L^2(\Omega_r^i)} = 1$. Here Δ_τ denotes the Laplacian of ∂B_r . For $n > 2$, we may no longer write $\lambda_{1,r}^i$ explicitly as we did in (2.2) in the two dimensional case. For a hemisphere in ∂B_r , the first eigenvalue is $\lambda_1 = (n-1)/r^2$ with corresponding eigenfunction given by the restriction a linear function on \mathbb{R}^n to this set. (By symmetry, one can see that the first eigenfunction of a hemisphere is the restriction of the first spherical harmonic that vanishes on the boundary of the hemisphere, which up to a rotation is the second spherical harmonic.)

An important quantity in this setting is the characteristic constant $\alpha = \alpha_i(r) \in (0, \infty)$ of Ω_i^r defined by

$$\alpha_i(r)^2 + (n-2)\alpha_i(r) - r^2\lambda_{1,r}^i = 0, \quad (2.18)$$

see [26]. The characteristic constant is defined in such a way so that the $\alpha = \alpha_i(r)$ homogeneous extension $w(x) = r^\alpha \tilde{u}_{i,r}$ of $\tilde{u}_{i,r}$ is harmonic in the cone generated by Ω_i^r . Like the first Dirichlet eigenvalue, the characteristic constant is monotone decreasing under set inclusion, and among sets of a fixed volume is minimized by a spherical cap. A hemisphere in ∂B_r has characteristic constant $\alpha = 1$.

Since u_i is subharmonic and $u_i(0) = 0$ we have [18] that

$$\int_{B_r} \frac{|\nabla u_i|^2}{|x|^{n-2}} \leq r^{2-n} \int_{\partial B_r} u_i(u_i)_r + \frac{n-2}{2} \frac{u_i^2}{r}. \quad (2.19)$$

From (2.18), we see that we can express $\lambda_{1,r}^i$ as $\lambda_{1,r}^i = r^{-2}\alpha_i(r)^2 + (n-2)r^{-2}\alpha_i(r)$. With this in mind, we now write $|\nabla u_i|^2$ in the following way for any point, where for a point $x \in \partial B_r$, we let ∇_τ denote the projection of the gradient onto the tangent space of ∂B_r at x .

$$\begin{aligned} |\nabla u_i|^2 &= |\nabla_\tau u_i|^2 + (\partial_r u_i)^2 = |\partial_\tau u_i|^2 - \lambda_1^i u_i^2 + \lambda_1^i u_i^2 + (\partial_r u_i)^2 \\ &= |\partial_\tau u_i|^2 - \lambda_1^i u_i^2 + \frac{\alpha_i^2}{r^2} u_i^2 + 2\frac{\alpha_i}{r} \frac{n-2}{2r} u_i^2 + (\partial_r u_i)^2 \\ &= |\partial_\tau u_i|^2 - \lambda_1^i u_i^2 \\ &\quad + \frac{\alpha_i^2}{r^2} u_i^2 - 2\frac{\alpha_i u_i}{r} \partial_r u_i + (\partial_r u_i)^2 \\ &\quad + 2\frac{\alpha_i}{r} u_i \partial_r u_i + 2\frac{\alpha_i}{r^2} \frac{n-2}{2} u_i^2. \end{aligned}$$

Using this expression in the numerator of (2.17), as well as the fact that $|x|^{n-2} \leq r^{n-2}$ on B_r and (2.19), we see that $\log(J(r))' \geq \delta_A(r) + \delta_B(r) + \delta_C(r)$ where

$$\begin{aligned} \delta_A(r) &= \sum_{i=1}^2 \frac{1}{\int_{B_r} |\nabla u_i|^2} \left(\int_{\partial B_r} (\partial_\tau u_i)^2 - \lambda_1^i \int_{\partial B_r} u_i^2 \right) \\ \delta_B(r) &= \sum_{i=1}^2 \frac{1}{\int_{B_r} |\nabla u_i|^2} \left(\int_{\partial B_r} \frac{\alpha_i(r)^2}{r^2} u_i^2 + (\partial_r u_i)^2 - 2\frac{\alpha_i(r)}{r} u_i \partial_r u_i \right) \\ \delta_C(r) &= \frac{2}{r} (\alpha_1(r) + \alpha_2(r) - 2). \end{aligned}$$

In [6, 18], the ACF functional $J(r)$ is shown to be monotone by showing that each of $\delta_A(r)$, $\delta_B(r)$, and $\delta_C(r)$ is nonnegative, which we will see is a consequence of Lemma 2.5 below. As in the two dimensional case, the quantitative monotonicity formula of Theorem 1.3 will be established by quantifying the loss in each inequality $\delta_{(\cdot)}(r) \geq 0$ and integrating with respect to r . Let us discuss the information captured by each $\delta_{(\cdot)}(r)$.

The quantity $\delta_A(r)$ is the direct analogue of $\delta_A(r)$ in the two dimensional setting. It is nonnegative by the Poincaré inequality on Ω_i^r , and equals zero for a given $r \in (0, 1]$ if and only if u_i restricted to ∂B_r is a multiple of the first eigenfunction $\tilde{u}_{i,r}$ of Ω_i^r . As in the case $n = 2$, $\delta_A(r)$ quantitatively controls the distance of u_i to the eigenfunction on this set; see (2.20) in Lemma 2.5 below.

The term in parentheses in the definition of $\delta_B(r)$ can be realized as a square as in (2.21) in Lemma 2.5 below, and so $\delta_B(r)$ is nonnegative for all $r \in (0, 1)$. It quantitatively controls how closely the restriction of u_i to ∂B_r behaves like a homogeneous function with homogeneity $\alpha_i(r)$.

The nonnegativity of $\delta_C(r)$ comes from the Friedland-Hayman inequality [26]. In (2.22) of Lemma 2.5 below, we show that $\delta_C(r)$ provides quantitative control over how close the characteristic constant $\alpha_i(r)$ of Ω_i^r is to the characteristic constant $\alpha = 1$ of a hemisphere in ∂B_r .

Let us outline the ingredients of the proof of Theorem 1.3 that make up the rest of this section. By integrating $\delta_B(r)$ and $\delta_C(r)$ from ρ to 1, we show in Lemma 2.6 that each u_i is quantitatively close to the one-homogeneous function defined by $ru_i(1, \theta)$ in polar coordinates $(r, \theta) \in \mathbb{R}_+ \times S^{n-1}$ for \mathbb{R}^n . Lemmas 2.7 and Lemma 2.8 provide quantitative and qualitative information about the proximity of Ω_i^1 to a spherical cap when $\delta_C(1)$ is assumed to be small. Lemma 2.9 compares the eigenfunctions of spherical caps of different radii. In Proposition 2.10, we apply the estimate (2.20) for $\delta_A(1)$ to deduce that $u_i(1, \theta)$ is almost equal to a multiple of first eigenfunction $\tilde{u}_{1,1}^i$ of Ω_i^1 , and applying Lemma 2.9 and Theorem 1.1, we see that $\tilde{u}_{1,1}^i$ is quantitatively close to the first eigenfunction of a hemisphere. Finally, Theorem 1.3 follows from Proposition 2.10 once we guarantee that the hemispheres corresponding to u_1 and u_2 do not have too much overlap.

The following lemma is the higher dimensional analogue of Lemma 2.1 and contains the quantitative lower bounds for each of $\delta_A(r)$, $\delta_B(r)$, and $\delta_C(r)$ discussed above. Since each of the right-hand sides of (2.20)-(2.22) are nonnegative, Lemma 2.5 in particular implies the monotonicity of $J(r)$.

Lemma 2.5. *For each $r \in (0, 1]$ and $i = 1, 2$, let $\tilde{u}_{i,r}$ be the eigenfunction of Ω_i^r as above and let $\beta_{1,r}^i = \int_{\partial B_r} u_i \tilde{u}_{i,r}$. We have*

$$\delta_A(r) \geq \sum_{i=1}^2 \frac{(\lambda_{2,r}^i - \lambda_{1,r}^i)}{\int_{B_r} |\nabla u_i|^2} \int_{\partial B_r} (u_i - \beta_{1,r}^i \tilde{u}_{i,r})^2 \quad (2.20)$$

$$\delta_B(r) = \sum_{i=1}^2 \frac{1}{\int_{B_r} |\nabla u_i|^2} \int_{\partial B_r} \left(\frac{\alpha_i(r)}{r} u_i - \partial_r u_i \right)^2 \quad (2.21)$$

$$\delta_C(r) \geq \sum_{i=1}^2 c(n) \frac{|\alpha_i(r) - 1|^2}{r \max\{\alpha_i(r), 2\}}, \quad (2.22)$$

with $c(n)$ a dimensional constant.

Proof. The proof of (2.20) is exactly the same as that of (2.4) of Lemma 2.1, and as in (2.5), we recognize the integrand in $\delta_B(r)$ above as a square to see (2.21).

Observe that the inequality in (2.22) is straightforward when $\delta_C(r)$ is large. Therefore, it suffices to consider the case when $\delta_C(r)$ is small. It also suffices to consider the case when Ω_1^r and Ω_2^r are both spherical caps. Indeed, let $\bar{\alpha}_1$ and $\bar{\alpha}_2$ denote the characteristic constants of disjoint spherical caps of the same volume as Ω_1^r and Ω_2^r respectively. Then by Sperner's inequality (recall (2.18)), the corresponding $\delta_C(r)$ can only decrease if by replacing Ω_1^r and Ω_2^r by spherical caps, and also using the Friedland-Hayman inequality, we see

$$\delta_C(r) = \frac{1}{r} (|\alpha_1 - \bar{\alpha}_1| + |\alpha_2 - \bar{\alpha}_2| + \bar{\alpha}_1 + \bar{\alpha}_2 - 2) \geq \frac{1}{r} (|\alpha_1 - \bar{\alpha}_1| + |\alpha_2 - \bar{\alpha}_2|). \quad (2.23)$$

That is, the deficit $\delta_C(r)$ controls $|\alpha_1 - \bar{\alpha}_1|$ linearly. Moreover, because the characteristic constant is monotone with respect to set inclusion, we may assume without loss of generality that we have two complementary caps.

So, define the function $\hat{\alpha} : (0, 1) \rightarrow \mathbb{R}$ by letting $\hat{\alpha}(t)$ be the characteristic constant of the spherical cap of radius $r = 2\pi t$. Let $\hat{\delta}(h) = \hat{\alpha}(1/2 + h) + \hat{\alpha}(1/2 - h) - 2$, so that by the Friedland-Hayman inequality we have $\hat{\delta}(h) \geq 0$.

We claim that $\hat{\delta}(h) \geq ch^2$ for a universal constant c . It is known [18, 31] that $\hat{\alpha}$ is convex with $\hat{\alpha}''(1/2) = 2c > 0$, where c is a universal constant, and therefore there exists h_0 such that $\hat{\alpha}''(1/2 + h) \geq c$ for all h with $|h| \leq h_0$. For the moment, we assume that $|h| \leq h_0$. So, a Taylor expansion shows that

$$\hat{\alpha}(1/2 + h) \geq \hat{\alpha}(1/2) + \hat{\alpha}'(1/2)h + \frac{c}{2}h^2$$

for any h with $|h| \leq h_0$. Replacing h by $-h$ and summing the corresponding terms, we conclude that $\hat{\delta}(h) \geq ch^2$ for all $|h| \leq h_0$. By the convexity of α , it follows that $\hat{\delta}(h_1) \geq \hat{\delta}(h_0)$ for any $h_1 \geq h_0$. Therefore, by assuming that δ_C is small enough, we also have necessarily that $|h| < |h_0|$.

Finally, since $\hat{\alpha}$ is a locally Lipschitz function, we have $|\alpha_i(r) - 1| \leq h$, and arrive at (2.22). This concludes the proof. \square

The following lemma is the higher dimensional analogue of Lemma 2.2. Using the estimates for $\delta_B(r)$ and $\delta_C(r)$ in Lemma 2.5, this lemma shows that u_1 and u_2 are quantitatively close to being 1-homogeneous.

Lemma 2.6. *Let u_1, u_2 be as above. Then*

$$\int_{B_1 \setminus B_\rho} [ru_i(1, \theta) - u_i(r, \theta)]^2 \leq C \log \left(\frac{J(1)}{J(\rho)} \right) \|u\|_{W^{1,2}(B_1)}^2.$$

Proof. Fix $i \in \{1, 2\}$. For notational simplicity, we omit the sub and superscripts i in this proof. The proof follows the proof of Lemma 2.2 using Lemma 2.5 but with $A(r)$ now defined as $A(r) := e^{-\int_r^1 \frac{\alpha(t)}{t} dt}$. \square

In Lemma 2.5, we established a quantitative lower bound for $\delta_C(r)$ in term of the characteristic constants of the regions Ω_1^r and Ω_2^r . In the following lemma, we prove some further quantitative estimates for $\delta_C(1)$ under the assumption that $\delta_C(1)$ is bounded above by one, which will be applied in the proof of Proposition 2.4. Recall that we let \mathcal{B} denote an $(n-1)$ -dimensional spherical cap.

Lemma 2.7. *Assume that $\delta_C(1) \leq 1$ and fix $i = 1, 2$. Let $\bar{\mathcal{B}}_i \subset \partial B_1$ be a spherical cap such that $|\bar{\mathcal{B}}_i| = |\Omega_i^1|$, and let $\bar{\lambda}^i$ denote the first eigenvalue of $\bar{\mathcal{B}}_i$. There exists a constant $C(n)$ such that*

$$\delta_C(1) \geq C \left(|\lambda_{1,1}^i - \bar{\lambda}^i| + |\bar{\lambda}^i - (n-1)|^2 \right). \quad (2.24)$$

Proof. We fix i and suppress the dependence on i and $r = 1$ in the remainder of the proof. Let $\bar{\alpha}$ denote the characteristic constant of $\bar{\mathcal{B}}$. Note that if $\delta_C(1) \leq 1$ then by definition, $\alpha, \bar{\alpha} \leq 3$. We already showed in (2.23) in the proof of Lemma 2.5 that $|\alpha - \bar{\alpha}|$ is linearly controlled by $\delta_C(1)$. This combined with (2.18) shows the first estimate in (2.24):

$$|\lambda - \bar{\lambda}| \leq (n-2)|\alpha - \bar{\alpha}| + |\alpha^2 - \bar{\alpha}^2| \leq C|\alpha - \bar{\alpha}| \leq C\delta_C(1). \quad (2.25)$$

Similarly, it follows from the proof of Lemma 2.5 that $|\bar{\alpha} - 1|$ is quadratically controlled by $\delta_C(1)$, and using (2.18) in the same way, we conclude the second estimate in (2.24) holds. \square

The next lemma shows that for $\delta_C(1)$ sufficiently small, the sets Ω_1^1 and Ω_2^1 enjoy a uniform spectral gap.

Lemma 2.8. *Fix $\eta > 0$. There exists $\varepsilon_0 = \varepsilon_0(\eta, n)$ such that if $\delta_C(1) \leq \varepsilon_0$, then $\lambda_2(\Omega_i^1) - \lambda_1(\Omega_i^1) \geq \eta$ for $i = 1, 2$.*

Proof. We argue by contradiction. Suppose we may find a sequence of functions $(u_{1,k}, u_{2,k})$ satisfying (1.6) such that $\delta_{C,k}(1) \rightarrow 0$ and $\Omega_k := \{u_{1,k} > 0\} \cap \partial B_1$ has $|\lambda_2(\Omega_k) - \lambda_1(\Omega_k)| \rightarrow 0$.

By Lemma 2.7, we see that $|\lambda_1(\Omega_k) - (n-1)| \rightarrow 0$ and $|\lambda_1(\mathcal{B}_k) - (n-1)| \rightarrow 0$, where \mathcal{B}_k is a spherical cap with $|\mathcal{B}_k| = |\Omega_k|$. The first fact shows that $|\lambda_1(\Omega_k) - (n-1)| \rightarrow 0$, and the second ensures that $|\Omega_k| \rightarrow |\mathcal{B}_+|$, where $\mathcal{B}_+ \subset S^{n-1}$ denotes a hemisphere.

Now, for each k , let $\tilde{u}_{k,1}$ and $\tilde{u}_{k,2}$ respectively denote first and second eigenfunctions of Ω_k , normalized to have L^2 norm equal to one and extended by zero to be defined on the entire sphere. So, there exist $\tilde{u}_{\infty,1}, \tilde{u}_{\infty,2} \in W^{1,2}(S^{n-1})$ such that after passing to a subsequence,

$$\begin{aligned} \tilde{u}_{k,1} &\rightarrow \tilde{u}_{\infty,1} && \text{weakly in } W^{1,2}(S^{n-1}), && \text{strongly in } L^2(S^{n-1}), && \text{pointwise a.e.}, \\ \tilde{u}_{k,2} &\rightarrow \tilde{u}_{\infty,2} && \text{weakly in } W^{1,2}(S^{n-1}), && \text{strongly in } L^2(S^{n-1}), && \text{pointwise a.e.}, \end{aligned}$$

and $\int \tilde{u}_{\infty,1} \cdot \tilde{u}_{\infty,2} = 0$.

Let $\Omega = \{\tilde{u}_{1,\infty} > 0\}$. By lower semicontinuity of the norm, $\lambda_1(\Omega) \leq n - 1$. By Fatou's lemma, $|\Omega| \leq \liminf |\{\tilde{u}_{k,1} > 0\}| = \lim |\Omega_k| = |\mathcal{B}_+|$. From Sperner's inequality and the monotonicity of λ_1 with respect to set inclusion, we see that up to a translation, $\Omega = \mathcal{B}_+$ and $\tilde{u}_{\infty,1}$ is the first eigenfunction of \mathcal{B}_+ .

Now, let $\Omega_2 = \{\tilde{u}_{2,\infty} > 0\}$. The same argument and the uniqueness of the first eigenfunction show that $\tilde{u}_{\infty,1} = \tilde{u}_{\infty,2}$, contradicting their orthogonality. This completes the proof. \square

Lemma 2.9. *Fix a compact interval $[r_1, r_2] \subset (0, \pi)$, let $r_1 \leq r, \bar{r} \leq r_2$, and let $\mathcal{B}, \bar{\mathcal{B}} \subset \partial B_1$ denote spherical caps of radius r, \bar{r} centered at the north pole. Let λ and $\bar{\lambda}$ denote the first eigenvalue of \mathcal{B} and $\bar{\mathcal{B}}$ respectively, and let v and \bar{v} denote the corresponding first eigenfunctions extended by zero to be defined on the whole sphere and normalized so that $\|v\|_{L^2(S^{n-1})} = \|\bar{v}\|_{L^2(S^{n-1})} = 1$. There exist constants $C = C(r_1, r_2)$ and $\bar{c} = \bar{c}(r, \bar{r})$ such that*

$$\int_{S^{n-1}} |\bar{c}v - \bar{v}|^2 \leq C(\lambda - \bar{\lambda})^2.$$

Proof. The functions v and \bar{v} are radially symmetric, and so with a slight abuse of notation we let v, \bar{v} denote the radial part of the functions. These functions can be extended to $(r, \pi), (\bar{r}, \pi)$ to satisfy

$$\begin{aligned} v''(\phi) + (n-2)\frac{\cos\phi}{\sin\phi}v'(\phi) &= -\lambda v(\phi) \text{ for } \phi \in (0, \pi), \\ \bar{v}''(\phi) + (n-2)\frac{\cos\phi}{\sin\phi}\bar{v}'(\phi) &= -\bar{\lambda}\bar{v}(\phi) \text{ for } \phi \in (0, \pi), \end{aligned}$$

with $v(r) = \bar{v}(\bar{r}) = 0, v'(0) = \bar{v}'(0) = 0$ and $v, \bar{v} \geq 0$ on $(0, r)$ and $(0, \bar{r})$ respectively. Setting $f_\lambda(t) = \bar{c}v(\cos^{-1}(t))$, where \bar{c} is a constant to be chosen later in the proof, we see that f_λ satisfies

$$(1-t^2)f_\lambda''(t) - (n-1)tf_\lambda'(t) + \lambda f_\lambda(t) = 0,$$

and likewise for $f_{\bar{\lambda}}(t) = \bar{v}(\cos^{-1}(t))$. The solutions to the above equation are Legendre functions and are well known. Expanding $v = \sum_{k=0}^{\infty} a_k(\lambda)(1-t)^k$ as a series solution about $t = 1$ we obtain the recursion formula

$$a_{k+1}(\lambda) = a_k(\lambda) \frac{k^2 + (n-2)k - \lambda}{(k+1)(2k+n-1)}. \quad (2.26)$$

The series converges absolutely on the interval $(-1, 3)$. We normalize by choosing the zero order coefficient \bar{a}_0 for $f_{\bar{\lambda}}$ to satisfy the constraint $\|\bar{v}\|_{L^2(S^{n-1})} = 1$ and by choosing the zero order coefficient a_0 of f_λ so that $a_0 = \bar{a}_0$, in this way determining the constant \bar{c} in the statement of the lemma. Then

$$a_{k+1}(\lambda) = \prod_{j=0}^k \frac{j^2 + (n-2)j - \lambda}{(j+1)(2j+n-1)}. \quad (2.27)$$

Taking the derivative of $\log a_{k+1}$ with respect to λ we obtain

$$|a'_{k+1}(\lambda)| = |a_{k+1}(\lambda)| \sum_{j=0}^k \frac{1}{(j+1)(2j+n-1)} \leq M|a_{k+1}(\lambda)|.$$

Furthermore, from (2.26), we see that for any compact interval $I \in (0, \infty)$, there is a constant $C = C(I)$ such that $|a_{k+1}(\lambda)| \leq C/2^k$ for all $\lambda \in I$. Therefore, $|a_{k+1}(\lambda) - a_{k+1}(\bar{\lambda})| \leq C|\lambda - \bar{\lambda}|/2^k$ for any $\lambda, \bar{\lambda} \in I$. Then summing over k we obtain for any $t \in [0, t_0]$ with $t_0 < \pi$ that

$$|f_\lambda(t) - f_{\bar{\lambda}}(t)| \leq |\lambda - \bar{\lambda}| \sum_{k=0}^{\infty} \frac{C}{2^k} < C|\lambda - \bar{\lambda}|.$$

The conclusion then follows after changing variables back to ϕ and integrating over ∂B_1 . \square

The next proposition is the higher dimensional analogue of Proposition 2.4. It shows that each of the u_i , $i = 1, 2$, is well approximated in L^2 by a linear function.

Proposition 2.10. *There exist $\varepsilon_0 = \varepsilon_0(n)$ and $C = C(n)$ such that the following holds. Assume $\log\left(\frac{J(1)}{J(\rho)}\right) \leq \varepsilon_0$ and $0 \leq \rho \leq 1/2$. Let v be the first eigenfunction of the spherical cap of radius $\pi/2$ in ∂B_1 , extended by 0 to be defined on ∂B_1 . For $i = 1, 2$ there exist $\beta_i = \beta_i(u_i)$ such that, up to a rotation, we have*

$$\int_{B_1 \setminus B_\rho} (r\beta_i v - u_i)^2 \leq C \log\left(\frac{J(1)}{J(\rho)}\right) \|u_i\|_{W^{1,2}(B_1)}^2.$$

Proof. Let $\varepsilon = \log(J(1)/J(\rho))$. Since $\varepsilon = \int_\rho^1 \log(J(r))' dr$ and the integrand is positive, it follows that $\log(J(r))' \leq 10\varepsilon$ outside a set of measure $\varepsilon/10$. In particular, $\log(J(r))' \leq 10\varepsilon$ for some $r \in (1 - \varepsilon/5, 1]$. By scaling we assume that $\log(J(r))' \leq \varepsilon$ at $r = 1$.

Fix $i \in \{1, 2\}$. For the remainder of the proof, we omit subscripts i for notational simplicity. Let $\Omega = \partial B_1 \cap \{u > 0\}$ and let \tilde{u} denote the first eigenfunction of Ω , normalized so that $\|\tilde{u}\|_{L^2(\Omega)} = 1$ and extended by zero to be defined everywhere on ∂B_1 . Let $\bar{B} \subset \partial B_1$ denote a spherical cap centered at the north pole such that $|\bar{B}| = |\Omega|$, and let \bar{v} denote its first eigenfunction with the normalization $\|\bar{v}\|_{L^2(\bar{B})} = 1$ and extended by zero to be defined on the whole ∂B_1 . Up to a rotation, we may assume that the infimum over $x \in S^{n-1}$ in (1.4) of Theorem 1.1 is achieved at the north pole. Lemmas 2.7 and 2.8 guarantee that for dimensional constants c and C ,

$$|\lambda_1 - \lambda(\bar{B})| \leq C\varepsilon, \quad |\lambda_1(\bar{B}) - (n-1)| \leq C\sqrt{\varepsilon}, \quad \lambda_2 - \lambda_1 \geq c, \quad (2.28)$$

Let $\hat{\beta} = \beta_1 = \int_{\partial B_1} u \tilde{u}$ be as in (2.20), and let \bar{c} be as in Lemma 2.9 applied to $r = \pi/2$ and the radius \bar{r} of \bar{B} . Let $\beta = \bar{c}\hat{\beta}$. We use the triangle inequality to see that

$$\begin{aligned} \int_{B_1 \setminus B_\rho} (u - \beta r v)^2 &\leq 4 \int_{B_1 \setminus B_\rho} (u - r u(1, \theta))^2 + 4 \int_{B_1 \setminus B_\rho} (r u(1, \theta) - \hat{\beta} r \tilde{u})^2 \\ &\quad + 4\hat{\beta}^2 \int_{B_1 \setminus B_\rho} (r \tilde{u} - r \bar{v})^2 + 4\hat{\beta}^2 \int_{B_1 \setminus B_\rho} (r \bar{v} - r \bar{c} v)^2. \end{aligned} \quad (2.29)$$

We can estimate the first two terms on the right-hand side of (2.29) precisely the way we estimated their two-dimensional analogues in the proof of Proposition 2.4. Indeed, by Lemma 2.6, for the first term we have

$$4 \int_{B_1 \setminus B_\rho} (u - r u(1, \theta))^2 \leq C\varepsilon \|u\|_{W^{1,2}(B_1)}^2. \quad (2.30)$$

For the second term on the right-hand side of (2.29), we use the estimate (2.20) for $\delta_A(1) \leq \varepsilon$ and the uniform eigenvalue gap of (2.28) to see that for a dimensional constant C we have

$$4 \int_{B_1 \setminus B_\rho} (ru(1, \theta) - \hat{\beta}r\tilde{u})^2 \leq C \int_{\partial B_1} (u(1, \theta) - \hat{\beta}\tilde{u})^2 \leq C\varepsilon \|u\|_{W^{1,2}(B_1)}^2. \quad (2.31)$$

To estimate the third term on the right-hand side of (2.29), we call upon the quantitative form of Sperner's inequality established in Theorem 1.1:

$$4 \int_{B_1 \setminus B_\rho} r^2(\tilde{u} - \bar{v})^2 \leq C\hat{\beta}^2 \int_{\partial B_1} (\tilde{u} - \bar{v})^2 \leq C\hat{\beta}^2 (\lambda_1 - \lambda_1(\bar{\mathcal{B}})) \leq C\varepsilon \|u\|_{W^{1,2}(B_1)}^2, \quad (2.32)$$

where in the final inequality we recall (2.28) and that $\hat{\beta} \leq \|u\|_{L^2(\partial B_1)} \leq C\|u\|_{W^{1,2}(B_1)}$.

Finally, we estimate the fourth term in (2.29). Applying Lemma 2.9 with $r = \pi/2$ and \bar{r} the radius of $\bar{\mathcal{B}}$ followed by (2.28), we see that

$$4\hat{\beta}^2 \int_{B_1 \setminus B_\rho} (r\bar{v} - r\bar{v})^2 \leq C\hat{\beta}^2 \int_{\partial B_1} (\bar{v} - c\bar{v})^2 \leq C\hat{\beta}^2 (\lambda_1(\bar{\mathcal{B}}) - (n-1))^2 \leq C\varepsilon \|u\|_{W^{1,2}(B_1)}^2. \quad (2.33)$$

Combining (2.30)–(2.33), we conclude the proof of the proposition. \square

We now prove Theorem 1.3 in the case $n \geq 3$.

Proof of Theorem 1.3 for $n \geq 3$. Let ε_0 be as in Proposition 2.10. If $\log(J(1)/J(\rho)) > \varepsilon_0$, then we simply choose $\beta_i = \|u_i\|_{L^2(B_1)}$ and $\nu = e_1$, and the result follows trivially for large enough C . Let now $\varepsilon = \log(J(1)/J(\rho))$ and assume $\varepsilon \leq \varepsilon_0$. As in the proof of the two-dimensional case, Theorem 1.3 almost follows from Proposition 2.10, but it remains to show that the truncated linear functions obtained for each of u_1, u_2 in Proposition 2.10 can be taken to have disjoint supports. We also repeat the following notation from Proposition 2.10, now emphasizing the dependence on $i = 1, 2$: let $\Omega_i = \{u_i > 0\} \cap \partial B_1$, let $\bar{\mathcal{B}}_i \subset \partial B_1$ be the spherical cap of the same volume as Ω^i with center achieving the infimum over $x \in S^{n-1}$ in (1.4) of Theorem 1.1, and \mathcal{B}_i the hemisphere with the same center. We first show that

$$|\mathcal{B}_1 \Delta \mathcal{B}_2| \leq C\varepsilon^{1/2}. \quad (2.34)$$

Indeed, we repeatedly apply the triangle inequality to find

$$|\mathcal{B}_1 \Delta \mathcal{B}_2| \leq (|\mathcal{B}_1 \Delta \bar{\mathcal{B}}_1| + |\mathcal{B}_2 \Delta \bar{\mathcal{B}}_2|) + (|\bar{\mathcal{B}}_1 \Delta \Omega_1| + |\bar{\mathcal{B}}_2 \Delta \Omega_2|) + |\Omega_1 \Delta \Omega_2|.$$

Since the supports of u_1 and u_2 are disjoint, $|\Omega_1 \Delta \Omega_2| = 0$. Next, we can apply Theorem 1.1 to Ω_1 and Ω_2 and use (2.28) to see that

$$|\bar{\mathcal{B}}_1 \Delta \Omega_1| + |\bar{\mathcal{B}}_2 \Delta \Omega_2| \leq C (\lambda_1(\Omega_1) - \lambda_1(\bar{\mathcal{B}}_2))^{1/2} + (\lambda_1(\Omega_2) - \lambda_1(\bar{\mathcal{B}}_1))^{1/2} \leq C\varepsilon^{1/2}.$$

Next, using (2.28), we see that

$$|\mathcal{B}_1 \Delta \bar{\mathcal{B}}_1| + |\mathcal{B}_2 \Delta \bar{\mathcal{B}}_2| \leq C \left(\left| r_1 - \frac{\pi}{2} \right| - \left| r_2 - \frac{\pi}{2} \right| \right) \leq C (|\lambda_1(\bar{\mathcal{B}}_1) - (n-1)| + |\lambda_1(\bar{\mathcal{B}}_2) - (n-1)|) \leq C\varepsilon^{1/2}.$$

Together these estimates prove the claim (2.34).

Without loss of generality we may assume that \mathcal{B}_2 is centered at the south pole. Let x_0 be the center of \mathcal{B}_1 ; from (2.34) it follows that $d(x_0, o) \leq C\varepsilon^{1/2}$ where o is the north pole and $d(\cdot, \cdot)$ denotes the distance on the sphere. Let w_1 be the truncated linear function corresponding to u_1 provided by Proposition 2.10 (so

w is defined in polar coordinates by $w(r, \theta) = \beta r v(\theta)$ and let \tilde{w}_1 be its rotation so that $\{\tilde{w}_1 > 0\} \cap \partial B_1$ is the hemisphere centered at the north pole. Since $|w_1| \leq C \|u_1\|_{W^{1,2}(B_1)}$, we see that

$$\begin{aligned} \int_{B_1 \setminus B_\rho} |\tilde{w}_1 - u_1|^2 &\leq C\varepsilon \|u\|_{W^{1,2}(B_1)}^2 + \int_{B_1 \setminus B_\rho} |\tilde{w}_1 - w_1|^2 \\ &\leq C\varepsilon \|u\|_{W^{1,2}(B_1)}^2 + C \|u\|_{W^{1,2}(B_1)}^2 d(x_0, o)^2 \leq C \|u\|_{W^{1,2}(B_1)}^2 \varepsilon. \end{aligned}$$

So, choosing \tilde{w}_1 and the truncated linear function w_2 corresponding to u_2 provided by Proposition 2.10 completes the proof of the theorem. \square

2.3. Uniqueness of blow-ups when the ACF formula decays fast. Here we sketch the proof of Corollary 1.4. The argument gives a precise modulus for the convergence of the blow-ups as well.

Proof of Corollary 1.4. We omit the subscripts 1, 2 in u_1, u_2 below to simplify notation; any statement involving u or β is valid for both $u = u_1$ and $u = u_2$, up to replacing $(\nu \cdot x)^+$ by $(\nu \cdot x)^-$ if $i = 2$.

For $k = 0, 1, 2, \dots$, consider the rescalings $u^k(x) = \frac{u(2^{-k}x)}{2^{-k}}$. Recall that $\omega(r) = J(r) - J(0^+)$. We will show, by induction on k , that there are numbers $\beta^k > 0$ and unit vectors $\nu_k \in S^{n-1}$ such that

$$\int_{B_1 \setminus B} |u^k - \beta^k (\nu_k \cdot x)^+|^2 \leq C_1 \omega(2^{-k}) \quad (2.35)$$

and

$$|\beta^k - \beta^{k-1}|^2 + |\nu_k - \nu_{k-1}|^2 \leq C_2 \omega(2^{-k-1}), \quad (2.36)$$

where the constants C_1 and C_2 depend on n and $\|u_i\|_{W^{1,2}(B_1)}$ but are independent of k . For each k , the vector ν_k for $i = 1$ will equal to $-\nu_k$ for $i = 2$. For $k = 0$ this follows from a single application of Theorem 1.3.

Assume (2.35) and (2.36) hold for all $j \leq k-1$. Then we may apply Theorem 1.3 (with $\rho = 0$) to u^k , as $J[u_1^k, u_2^k](r) = J[u_1, u_2](2^{-k}r) < 1 + \varepsilon$ by assumption:

$$\int_{B_1} |u^k - \beta^k (\nu_k \cdot x)^+|^2 \leq C \omega(2^{-k}) \sum_{i=1}^2 \|u_i^k\|_{W^{1,2}(B_1)}^2.$$

The main point now is to estimate $\|u^k\|_{W^{1,2}(B_1)}$ in a manner independent of k , and then show that β^k is close to β^{k-1} (and similarly for ν_k).

We apply the Caccioppoli inequality to u^k to give

$$\|u^k\|_{W^{1,2}(B_1)} \leq C \|u^k\|_{L^2(B_2)},$$

recalling they are subharmonic. From (2.35) with $k-1$, we have that

$$\begin{aligned} \|u^k\|_{L^2(B_2)} &= 2^{n/2+1} \|u^{k-1}\|_{L^2(B_1)} \\ &\leq 2^{n/2+1} \|u^{k-1} - \beta^{k-1} (\nu_{k-1} \cdot x)^+ \|_{L^2(B_1)} + C \beta_i^{k-1} \\ &\leq C \left[\sqrt{C_1 \omega(2^{-k-1})} + \beta^{k-1} \right] \\ &\leq C \left[\sqrt{C_1 \varepsilon} + \beta^{k-1} \right]. \end{aligned}$$

Now applying (2.36) for every $j \leq k-1$, we have

$$\begin{aligned} \beta^{k-1} &\leq \beta^0 + \sum_{j=1}^{k-1} |\beta^j - \beta^{j-1}| \\ &\leq C \|u\|_{W^{1,2}(B_1)} + \sqrt{C_2} \sum_{j=1}^{k-1} \sqrt{\omega(2^{-j-1})} \leq C \left(1 + \int_0^{2^{-k-1}} \frac{\sqrt{\omega(r)}}{r} dr\right) \leq C. \end{aligned}$$

Choosing C_1 large enough and then ε small in terms of C_1 gives

$$\|u^k\|_{W^{1,2}(B_1)}^2 \leq C[C_1\varepsilon + 1] \leq \frac{C_1}{2}.$$

This implies (2.35) for k .

To control β^k , we change variables:

$$\int_{B_{1/2}} |u^{k-1} - \beta^k(\nu_k \cdot x)^+|^2 = 2^{-n-2} \int_{B_1} |u^k - \beta^k(\nu_k \cdot x)^+|^2 \leq C\omega(2^{-k}).$$

Then

$$\int_{B_{1/2}} |\beta^k(\nu_k \cdot x)^+ - \beta^{k-1}(\nu_{k-1} \cdot x)^+|^2 \leq C\omega(2^{-k-1})$$

by combining with (2.35) with $k-1$. Direct evaluation leads to

$$|\beta^k - \beta^{k-1}|^2 + |\nu_k - \nu_{k-1}|^2 \leq C_2\omega(2^{-k-1}),$$

and so (2.36) holds for k .

We are now in a position to conclude. From (2.36) the numbers β^k and vectors ν^k have

$$\sum_{k=0}^{\infty} |\beta^k - \beta^{k-1}| + |\nu_k - \nu_{k-1}| \leq C(\varepsilon + \int_0^1 \frac{\sqrt{\omega(r)}}{r} dr) < \infty,$$

so the sequences converge to some β_i, v . Set

$$\tilde{\omega}(r) = \omega(r) + \int_0^{2r} \frac{\sqrt{\omega(r)}}{r} dr,$$

which a nondecreasing function tending to 0 at 0. Then from (2.35),

$$\int_{B_1} |u^k - \beta(v \cdot x)^+|^2 \leq C\tilde{\omega}(2^{-k}),$$

and so changing variables,

$$2^{(n+2)k} \int_{B_{2^{-k}}} |u - \beta(v \cdot x)^+|^2 \leq C\tilde{\omega}(2^{-k}).$$

This may be rewritten to give

$$\frac{1}{r^{n+2}} \int_{B_r} |u - \beta(v \cdot x)^+|^2 \leq C\tilde{\omega}(2r) \rightarrow 0,$$

concluding the proof. \square

3. QUANTITATIVE STABILITY FOR THE FABER-KRAHN INEQUALITY

We now turn toward the proof of the quantitative Faber-Krahn inequality, Theorem 1.1. As we discussed in the introduction, the proof is essentially identical in the cases of the round sphere, Euclidean space, and hyperbolic space. To keep the notation from becoming unreasonably heavy, we will therefore present the proof in case of the round sphere, and include remarks throughout the proof guiding the reader through any modifications needed to generalize the proof to Euclidean or hyperbolic spaces.

All three simply connected space forms share the property that balls centered at *any* point minimize the first eigenvalue. It is clear, then, that any stability statement must involve the distance to a nearby ball to Ω . In (1.4), we chose to “mod out” by this symmetry by taking the infimum over all x in the space of the distance to balls $B(x)$ centered at x . It turns out to be more convenient to prove the slightly stronger statement given below as Theorem 3.1, in which we get rid of the translational invariance by choosing a particular “set center” of a given set Ω , defined in the following way.

Consider the standard embedding of the round sphere $S^n(1) \subset \mathbb{R}^{n+1}$ and let $y_\Omega = \int_\Omega y \, d\text{vol}_g(y) \in \mathbb{R}^{n+1}$. Provided $y_\Omega \neq 0$, we define the *set center* x_Ω of Ω by

$$x_\Omega = \frac{y_\Omega}{|y_\Omega|}$$

It is worth noting that there are other possible ways that one may define a notion of “barycenter” of a set $\Omega \subset S^n$. Another possible notion is given in [13].

Throughout this section, given a set $\Omega \subset S^n$, we let u_Ω denote its first eigenfunction extended by 0 to be defined on all of S^n and normalized so that $\|u_\Omega\|_{L^2(\Omega)} = 1$.

Theorem 3.1. *Fix $n \geq 1$ and $v \in (0, |S^n|)$. There exists a constant $c = c(n, v)$ such that the following holds. Let $\Omega \subset S^n$ with $|\Omega| = v$ be a set for which $x_\Omega \in S^n$ is defined. Letting B denote the spherical cap centered at x_Ω with $|B| = v$, we have*

$$\lambda_1(\Omega) - \lambda_1(B) \geq c \left(|\Omega \Delta B|^2 + \int_{S^n} |u_\Omega - u_B|^2 \right). \quad (3.1)$$

Remark 3.2 (Sets with no barycenter). The statement of Theorem 3.1 may look restrictive compared to Theorem 1.1. However, if Ω is a set for which the barycenter is not defined, then necessarily $\lambda_1(\Omega) - \lambda_1(B) \geq c$, and so (1.4) holds trivially in this case. In particular, Theorem 3.1 implies Theorem 1.1 in the case of the round sphere.

Remark 3.3 (Set centers on Euclidean space and hyperbolic space). On Euclidean space, an open set $\Omega \subset \mathbb{R}^n$ has a uniquely defined barycenter $x_\Omega = \int_\Omega x \, dx$. For an open set $\Omega \subset H^n$ in hyperbolic space, the set center $x_\Omega = \text{argmin}_{x_0 \in H^n} \int_\Omega d(x, x_0)^2 \, d\text{vol}(x)$ is well defined. The analogues of Theorem 3.1 hold in each of these two spaces with x_Ω defined in this way; the proof will be identical up to the obvious modifications. Once more, the analogues of Theorem 3.1 for Euclidean and hyperbolic space imply Theorem 1.1 on these space forms.

We will work in spherical coordinates $(\theta, \phi) \in S^{n-1} \times (0, \pi)$ on S^n , in which the round metric takes the form $g_{S^n} = d\phi^2 + \sin^2(\phi)d\theta^2$, where $d\theta^2$ is used to denote the round metric on S^{n-1} . A point $(\theta, \phi) \in S^n$ has geodesic distance ϕ to the north pole $o \in S^n$. The Laplacian of a function f in spherical coordinates (θ, ϕ) is given by

$$\Delta_{S^n} f(\theta, \phi) = (\sin \phi)^{1-n} \frac{\partial}{\partial \phi} \left((\sin \phi)^{n-1} \frac{\partial f}{\partial \phi} \right) + (\sin \phi)^{-2} \Delta_{S^{n-1}} f. \quad (3.2)$$

Remark 3.4 (Coordinates on hyperbolic space and Euclidean space). On hyperbolic space, we similarly use polar coordinates (θ, ϕ) , so that the metric is expressed in these coordinates by $g_{hyp} = d\phi^2 + \sinh^2(\phi)d\theta^2$, where once more we let $d\theta^2$ denote the round metric on S^{n-1} . A point $(\theta, \phi) \in H^n$ has geodesic distance ϕ from the distinguished point from which the polar coordinates are centered. The Laplacian of a function $f : H^n \rightarrow \mathbb{R}$ in these coordinates is given by

$$\Delta_{H^n} f(\theta, \phi) = (\sinh \phi)^{1-n} \frac{\partial}{\partial \phi} \left((\sinh \phi)^{n-1} \frac{\partial f}{\partial \phi} \right) + (\sinh \phi)^{-2} \Delta_{S^{n-1}} f.$$

On Euclidean space we use standard polar coordinates (θ, ϕ) , where ϕ is the distance to the origin and the Laplacian of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is given by .

$$\Delta_{\mathbb{R}^n} f(\theta, \phi) = \phi^{1-n} \frac{\partial}{\partial \phi} \left(\phi^{n-1} \frac{\partial f}{\partial \phi} \right) + \phi^{-2} \Delta_{S^{n-1}} f.$$

Throughout this section, we will use the shorthand $B_\rho = B(o, \rho)$ to denote the spherical cap of radius ρ centered at the north pole. We will frequently make the identification of the boundary of a geodesic ball, $\partial B(o, \rho)$, with the sphere S^{n-1} , noting that the induced metric on $\partial B(o, \rho)$ is $\sin^2(\rho)d\theta^2$, where $d\theta^2$ is the round metric on S^{n-1} .

We proceed in two steps: in Section 3.1, we establish Theorem 3.1 for sets that are small perturbations of a spherical cap, and in Section 3.2, we call upon results from our companion paper [4] to conclude Theorem 3.1 in the general case.

3.1. Stability for nearly spherical sets. The main goal of this section is to establish Theorem 3.6 below, which is Theorem 3.1 restricted to a class of sets in S^n that are small $C^{2,\alpha}$ perturbations of spherical caps.

The class of *nearly spherical sets* given in the following definition will be our main objects of interest in this section.

Definition 3.5. *Given $\rho \in (0, \pi)$ and $\xi \in C^{2,\alpha}(\partial B_\rho)$ with $\|\xi\|_{C^{2,\alpha}(\partial B_\rho)} \leq \varepsilon$, we define the following:*

- (1) *A set $\Omega \subset S^n$ is parametrized by ξ over B_ρ if in spherical coordinates $\partial\Omega$ takes the form*

$$\partial\Omega = \{(\theta, (1 + \xi(\theta))\rho) : (\theta, \rho) \in \partial B_\rho\}.$$

- (2) *A set $\Omega \subset S^n$ is a nearly spherical set if Ω is parametrized by ξ over B_ρ such that $|\Omega| = |B_\rho|$ and the barycenter x_Ω of Ω is the north pole o .*

Note that the barycenter of a set parametrized over B_ρ is always defined provided that ε is taken to be sufficiently small depending on n and ρ . The following theorem establishes quantitative stability for Sperner's inequality among nearly spherical sets.

Theorem 3.6 (Quantitative stability for nearly spherical sets). *Fix $n \geq 2$ and $\gamma \in (0, \pi/2)$. There exist positive constants c and ε depending only on n and γ such that the following holds. Let $|\rho - \pi/2| < \gamma$ and let Ω be a nearly spherical set parametrized by ξ over B_ρ with $\|\xi\|_{C^{2,\alpha}(\partial B_\rho)} \leq \varepsilon$. Then*

$$\lambda_1(\Omega) - \lambda_1(B_\rho) \geq c \left(|\Omega \Delta B_\rho|^2 + \int_{S^n} |u_\Omega - u_{B_\rho}|^2 \right). \quad (3.3)$$

The basic idea of the proof of Theorem 3.6 is the following. In Section 3.1.1, we show that the deficit $\lambda_1(\Omega) - \lambda_1(B_\rho)$ of a nearly spherical set is equivalent to the second variation of $\lambda_1(B_\rho)$, up to an error that can be made arbitrarily small by taking $\|\xi\|_{C^{2,\alpha}(\partial B_\rho)}$ to be small; see Theorem 3.7. Section 3.1.2 contains the core of proof of Theorem 3.6: in Theorem 3.11 we establish a gap in the spectrum for the second variation operator. As we discuss in the introduction, this spectral analysis is carried out through

an implicit argument based on the maximum principle. Theorem 3.11 is then applied in Theorem 3.14 to show that the deficit $\lambda_1(\Omega) - \lambda_1(B_\rho)$ of a nearly spherical set controls the $H^{1/2}$ norm of the parametrizing function ξ . Finally, in Proposition 3.15 below of Section 3.1.3, we show that the right-hand side of (1.4) is controlled linearly by $\|\xi\|_{H^{1/2}(\partial B)}^2$.

Recall that the $H^{1/2}$ seminorm of a function $\xi : \partial B_\rho \rightarrow \mathbb{R}$ is defined by $\|\xi\|_{H^{1/2}(\partial B_\rho)}^2 = \int_{B_\rho} |\nabla h_\xi|^2$, where h_ξ is the harmonic extension of ξ , i.e. the unique solution to

$$\begin{cases} \Delta h_\xi = 0 & \text{in } B_\rho \\ h_\xi = \xi & \text{on } \partial B_\rho. \end{cases} \quad (3.4)$$

The $H^{1/2}$ norm of a function $\xi : \partial B_\rho \rightarrow \mathbb{R}$ is defined by $\|\xi\|_{H^{1/2}(\partial B_\rho)}^2 = \|\xi\|_{L^2(\partial B_\rho)}^2 + \|\xi\|_{H^{1/2}(\partial B_\rho)}^2$. Among functions $\xi \in H^{1/2}(\partial B_\rho)$ with $\int_{\partial B_\rho} \xi = 0$, the $H^{1/2}$ norm and seminorm are equivalent.

3.1.1. The deficit and the second variation. The first eigenfunction u_{B_ρ} of a spherical cap B_ρ is radially symmetric and decreasing. We let $u_\rho(\phi) = u_{B_\rho}(\theta, \phi)$ and note that $|\nabla u_{B_\rho}(\theta, \phi)| = -u'_\rho(\phi) = |u'_\rho(\phi)|$. In the sequel, we will use the shorthand $|u'_\rho|$ to refer to number $|u'_\rho(\rho)| > 0$ and will let $\lambda_\rho = \lambda_1(B_\rho)$.

Given a function $\xi \in H^{1/2}(\partial B_\rho)$ with $\int_{\partial B_\rho} \xi = 0$, let $w_\xi : B_\rho \rightarrow \mathbb{R}$ be the solution to

$$\begin{cases} -\Delta w_\xi = \lambda_\rho w_\xi & \text{in } B_\rho \\ w_\xi = |u'_\rho| \xi & \text{on } \partial B_\rho \\ \int_{B_\rho} w_\xi u_{B_\rho} = 0. \end{cases} \quad (3.5)$$

Such a solution exists and is unique by the Fredholm Alternative and is continuous by [29]. For such a function ξ , the second variation $\delta^2 \lambda_1(B_\rho)[\xi, \xi]$ of λ_1 at B_ρ in the direction $[\xi, \xi]$ is given by

$$\delta^2 \lambda_1(B_\rho)[\xi, \xi] = 2 \int_{B_\rho} (|\nabla w_\xi|^2 - \lambda_\rho w_\xi^2) dx + \mathcal{H}_{B_\rho} |u'_\rho|^2 \int_{\partial B_\rho} \xi^2 d\sigma. \quad (3.6)$$

Here \mathcal{H}_{B_ρ} is the scalar mean curvature of ∂B_ρ with the sign convention that $\mathcal{H}_{B_\rho} > 0$ for $\rho < \pi/2$ and $\mathcal{H}_{B_\rho} < 0$ for $\rho \in (\pi/2, \pi)$. On Euclidean space and hyperbolic space, the second variation takes the analogous form.

Let $L_\rho : H^{1/2}(\partial B_\rho) \cap \{\int_{\partial B_\rho} \xi = 0\} \rightarrow L^2(\partial B_\rho)$ denote the Dirichlet-to-Neumann map corresponding to the problem (3.5) translated by the zero order term $\frac{1}{2} \mathcal{H}_{B_\rho} |u'_\rho|^2$. More specifically, given $\xi \in H^{1/2}(\partial B_\rho)$ with $\int_{\partial B_\rho} \xi = 0$, define

$$L_\rho \xi = \partial_\nu w_\xi + \frac{1}{2} \mathcal{H}_{B_\rho} |u'_\rho|^2 \xi, \quad (3.7)$$

where w_ξ is the unique solution to (3.5) and ν is the outer unit normal to B_ρ . In this way, we can express the second variation on the ball as

$$\delta^2 \lambda_1(B_\rho)[\xi, \xi] = 2 \int_{\partial B_\rho} \xi L_\rho \xi.$$

The following theorem shows that the deficit in Sperner's inequality for a nearly spherical set is equal to the second variation, up to an error depending on $\|\xi\|_{C^{2,\alpha}(\partial B_\rho)}$.

Theorem 3.7. *Fix $\gamma \in (0, \pi/2)$. There is a modulus of continuity ω depending only on n and γ such that the following holds. Fix ρ with $|\pi/2 - \rho| < \gamma$ and let $\Omega \subset S^n$ be a set parametrized by ξ over B_ρ with $|\Omega| = |B_\rho|$. Setting $\hat{\xi} = \xi - \int_{\partial B_\rho} \xi$, we have*

$$\lambda_1(\Omega) - \lambda_1(B_\rho) = \frac{1}{2} \delta^2 \lambda_1(B_\rho)[\hat{\xi}, \hat{\xi}] + \omega \left(\|\hat{\xi}\|_{C^{2,\alpha}(\partial B_\rho)} \right) \|\hat{\xi}\|_{H^{1/2}(\partial B_\rho)}^2. \quad (3.8)$$

Theorem 3.7 was established in the Euclidean setting in [23, Theorem 1] (see also [22, Theorem 1.4] for the $C^{2,\alpha}$ norm replaced by the $W^{2,p}$ norm for $p > n$). Adapting the proof to the spherical and hyperbolic settings requires only technical modifications, which we omit.

Remark 3.8. For a function $\xi \in H^{1/2}(\partial B_\rho)$ with nonzero mean, the Fredholm Alternative implies that no solution exists to the problem (3.5) (even without the orthogonality constraint $\int_{B_\rho} w_\xi u_{B_\rho} = 0$), and for this reason the second variation is only defined for mean zero functions. However, for a nearly spherical set Ω , a Taylor expansion of the volume constraint shows that

$$0 = |\Omega| - |B_\rho| = \int_{\partial B_\rho} \xi + o(\|\xi\|_{L^2(\partial B_\rho)});$$

see, for instance, [13]. This implies that $\|\xi - \hat{\xi}\|_{L^2(\partial B_\rho)} = o(\|\xi\|_{L^2(\partial B_\rho)})$. In particular, $\|\xi\|_{H^{1/2}(\partial B_\rho)} = \|\hat{\xi}\|_{H^{1/2}(\partial B_\rho)} + o(\|\xi\|_{L^2(\partial B_\rho)})$ and

$$\frac{1}{2}\|\xi\|_{C^{2,\alpha}(\partial B_\rho)} \leq \|\hat{\xi}\|_{C^{2,\alpha}(\partial B_\rho)} \leq 2\|\xi\|_{C^{2,\alpha}(\partial B_\rho)}.$$

Therefore, the conclusion of Theorem 3.7 above may be equivalently written as

$$\lambda_1(\Omega) - \lambda_1(B_\rho) = \frac{1}{2}\delta^2\lambda_1(B_\rho)[\hat{\xi}, \hat{\xi}] + \omega(\|\xi\|_{C^{2,\alpha}(\partial B_\rho)})\|\xi\|_{H^{1/2}(\partial B_\rho)}^2. \quad (3.9)$$

3.1.2. The spectral gap. We study the spectrum of the shifted Dirichlet-to-Neumann operator L_ρ introduced in (3.7) and establish a spectral gap in Theorem 3.11 below. In contrast to the Euclidean case, the spectrum cannot be computed explicitly and therefore the spectral analysis relies on implicit methods based on the maximum principle.

We let $\{Y_i\}_{i=0}^\infty$ be the orthonormal basis for $L^2(\partial B_\rho)$ of spherical harmonics, i.e. solutions to the eigenvalue equation $\Delta_{S^{n-1}}Y_i = -\mu_i Y_i$, or equivalently $\Delta_{\partial B_\rho}Y_i = -\sin^{-2}(\rho)\mu_i Y_i$, on ∂B_ρ with each Y_i normalized so that $\int_{\partial B_\rho} Y_i^2 = 1$. The eigenfunctions are ordered such that $\mu_i \leq \mu_{i+1}$ for all i . Recall that

$$\begin{aligned} \mu_0 &= 0 \\ \mu_1 &= \dots = \mu_n = (n-1) \\ \mu_{n+1} &= 2n. \end{aligned} \quad (3.10)$$

The unique spherical harmonic corresponding to $\mu_0 = 0$ is constant, and thus for each $i \geq 1$ we have

$$\int_{\partial B_\rho} Y_i = 0. \quad (3.11)$$

We let w_i denote the solution of (3.5) with $\xi = Y_i$. That is, $w_i \in H^1(B_\rho) \cap C(B_\rho)$ is the unique solution to

$$\begin{cases} -\Delta w_i = \lambda_{B_\rho} w_i & \text{in } B_\rho \\ w_i = |u'_\rho| Y_i & \text{on } \partial B_\rho \\ \int_{B_\rho} w_i u_{B_\rho} = 0. \end{cases} \quad (3.12)$$

The following lemma shows that the solutions w_i take a particular form that will be crucial to our spectral analysis.

Lemma 3.9. *For each $i \geq 1$, the unique solution w_i of (3.12) takes the form*

$$w_i(\phi, \theta) = Y_i(\theta)g_i(\phi) \quad (3.13)$$

in spherical coordinates, and g_i satisfies the following three properties:

- (1) $g_i(0) = 0$
- (2) $g_i(\rho) = |u'_\rho|$
- (3) $g_i(\phi) > 0$ for $0 < \phi < \rho$.

Proof. Fix $i \geq 1$ and the corresponding spherical harmonic Y_i and solution w_i of (3.12). We omit the subscript i in the remainder of this proof for notational simplicity. Let us make the ansatz that w takes the form (3.13) in spherical coordinates. Then $w(\theta, \rho) = Y(\theta)g(\rho)$ on ∂B_ρ , so w satisfies the boundary condition in (3.12) provided $g(\rho) = |u'_\rho|$. Furthermore, thanks to the orthogonality (3.11), we have

$$\int_{B_\rho} w u_{B_\rho} = \int_0^\rho \left(\int_{S^{n-1}} Y(\theta) d\mathcal{H}^{n-1}(\theta) \right) u_\rho(\phi) g(\phi) \sin(\phi)^{n-1} d\phi = 0,$$

so w satisfies the orthogonality condition in (3.12). Finally, from the expression (3.2) for the Laplacian of a function f in spherical coordinates (θ, ϕ) , we see that $g(\phi)$ satisfies the ordinary differential equation

$$\frac{d^2 g}{d\phi^2} + (n-1) \frac{\cos \phi}{\sin \phi} \frac{dg}{d\phi} + \left(\lambda_\rho - \frac{\mu}{\sin^2 \phi} \right) g = 0. \quad (3.14)$$

Under the change of variable $t = \cos \phi$, the solution g is a solution to the associated Legendre equation, which are well-known special functions. Thus, up to choosing a constant multiple of g so that the boundary condition is satisfied, this function is the unique $H^1(B_\rho)$ solution of (3.5).

Next we verify properties (1)–(3). Property (2) holds by our choice of constant multiple of g . As a solution of (3.12), w is nonconstant and continuous, so it follows that $g(0) = 0$ and thus (1) holds. Finally, to show (3), we suppose by way of contradiction that $g(\rho_1) = 0$ for some $0 < \rho_1 < \rho$. Then we see that $w \in H_0^1(B_{\rho_1})$ solves

$$\frac{\int_{B_{\rho_1}} |\nabla w|^2}{\int_{B_{\rho_1}} w^2} = \lambda_\rho.$$

Thus, $\lambda_{\rho_1} \leq \lambda_\rho$; however, since $B_{\rho_1} \subset B_\rho$ we have that $\lambda_\rho < \lambda_{\rho_1}$. This gives a contradiction and we see that g is nonvanishing in $(0, \rho)$. Together with (2) this proves (3). \square

Using Lemma 3.9, we show in the following corollary that the spherical harmonics for $i \geq 1$ are eigenfunctions of the Dirichlet-to-Neumann operator (3.7).

Corollary 3.10. *Each spherical harmonic Y_i for $i \geq 1$ is an eigenfunction of the Dirichlet-to-Neumann operator L_ρ defined in (3.7) with eigenvalue η_i given by*

$$\eta_i = g'_i(\rho) + \frac{1}{2} \mathcal{H}_{B_\rho} |u'_\rho|^2, \quad (3.15)$$

where g_i is the function given in Lemma 3.9. If $\mu_i = \mu_{i+1}$, then $\eta_i = \eta_{i+1}$. Furthermore, $\eta_i \geq 0$ for all $i \in \mathbb{N}$ and $\eta_i = 0$ for $i = 1, \dots, n$.

Proof. Making use of Lemma 3.9, we compute that $L_\rho Y_i = g'_i(\rho) Y_i + \frac{1}{2} \mathcal{H}_{B_\rho} |u'_\rho|^2 Y_i$ where g_i is the solution to (3.14) found in Lemma 3.9. In particular, Y_i is an eigenfunction of L_ρ with eigenvalue η_i as in (3.15). If $\mu_i = \mu_{i+1}$, then we see from (3.14) that $g_i = g_{i+1}$ and thus $\eta_i = \eta_{i+1}$ by (3.15).

The next two claims follow from geometric considerations. Suppose by way of contradiction that $\eta_i < 0$ for some $i \geq 1$. For $t > 0$ and small, let Ω_t be the set parametrized by ξ_t over B_ρ where $\xi_t = tY_i + s(t)Y_0$ and $s(t)$ is the function defined such that $|\Omega_t| = |B_\rho|$. By Sperner's inequality and Theorem 3.7,

$$0 \leq \lambda_1(\Omega_t) - \lambda_1(B_\rho) = t^2 \eta_i + t^3 \omega \left(\|Y_i\|_{C^{2,\alpha}(\partial B_\rho)} \right) \|Y_i\|_{H^{1/2}(\partial B_\rho)}^2 \leq \frac{t^2}{2} \eta_i < 0,$$

where the penultimate inequality holds for t sufficiently small depending on Y_i . We reach a contradiction and conclude that $\eta_i \geq 0$ for all $i \geq 1$.

We sketch the proof that $\eta_i = 0$ for $i = 1, \dots, n$. Note that $\eta_1 = \dots = \eta_{n-1}$ by (3.10) so it suffices to show that $\eta_1 = 0$. Let e_1, \dots, e_{n+1} be an orthonormal basis for \mathbb{R}^{n+1} . Consider the standard embedding of S^n in \mathbb{R}^{n+1} such that the north pole is e_{n+1} . The spherical harmonic $Y = Y_1$ is the restriction to ∂B_ρ of the linear function $x \cdot e_1$ on \mathbb{R}^{n+1} .

For $t \in (-\varepsilon, \varepsilon)$, let $x_t = \exp(te_1)$ and let $\Omega_t = B(x_t, \rho)$. For t sufficiently small, there is a smooth and uniquely defined one-parameter family of functions ξ_t such that

$$\partial\Omega_t = \{(\theta, (1 + \xi_t(\theta))\rho) : (\theta, \rho) \in \partial B_\rho\}$$

in spherical coordinates. Moreover, for t sufficiently small, ξ_t takes the form $\xi_t = \sum_{i=0}^{\infty} a_i(t)Y_i$ with

$$a_1(t) = t/\rho, \quad \text{and} \quad \frac{d}{dt}\Big|_{t=0} a_i(t) = \lim_{t \rightarrow 0} a_i(t)/t = 0 \text{ for } i \neq 1.$$

Because $\lambda_1(\Omega_t) = \lambda_1(B_\rho)$ for all t , we have, in particular,

$$\frac{d^2}{dt^2} \lambda_1(\Omega_t) = 0 \tag{3.16}$$

for all t sufficiently small. The expression for second variation at Ω_t changes smoothly with respect to t . So, dividing (3.16) by t^2 and letting $t \rightarrow 0$, we use the expression (3.6) to arrive at

$$0 = \lim_{t \rightarrow 0} \frac{1}{t^2} \left(\frac{d^2}{dt^2} \lambda_1(\Omega_t) \right) = \lim_{t \rightarrow 0} \sum_{i=1}^{\infty} \eta_i \left(\frac{a_i(t)}{t} \right)^2 = \eta_1.$$

This concludes the proof of the corollary. \square

Corollary 3.10 shows that $Y_1, \dots, Y_n \in \ker L_\rho$. The following theorem is the key point in our spectral analysis: we establish a spectral gap to show that the kernel of L_ρ consists *only* of the span of $\{Y_i\}_{i=1}^n$.

Theorem 3.11 (Spectral gap). *Fix $n \geq 2$ and $\rho \in (0, \pi)$. Let $\{\eta_i\}_{i=1}^{\infty}$ be the eigenvalues of L_ρ given in (3.15). Then $\eta_{n+1} > \eta_n = 0$ and so*

$$\eta_i \geq \eta_{n+1} > 0 \quad \text{for all } i \geq n+1.$$

Furthermore, given any $\gamma \in (0, \pi/2)$, there exists $\bar{\eta}$ depending only on n and γ such that $\eta_{n+1} \geq \bar{\eta}$ for $|\rho - \pi/2| \leq \gamma$.

Proof of Theorem 3.11. As noted in Corollary 3.10, if $\mu_i = \mu_{i+1}$, then $\eta_i = \eta_{i+1}$. Let $w_i = Y_i(\theta)g_i(\phi)$ be the solution to (3.5). We consider $h = g_n - g_{n+1}$. From properties (1) and (2) in Lemma 3.9, we have that $h(\rho) = h(0) = 0$. Furthermore, from (3.14) and property (3) of Lemma 3.9, we see that h is a strict supersolution to the equation that g_n satisfies. That is,

$$\frac{d^2 h}{d\phi^2} + (n-1) \frac{\cos \phi}{\sin \phi} \frac{dh}{d\phi} + \left(\lambda_\rho - \frac{\mu_n}{\sin^2 \phi} \right) h = \frac{\mu_n - \mu_{n+1}}{\sin^2 \phi} g_{n-1}(\phi) < 0. \tag{3.17}$$

We will show that $h > 0$ for $\phi \in (0, \rho)$ and $h'(\rho) > 0$. Notice that in the case of the hemisphere $\rho = \pi/2$, we have $\lambda_\rho = n$ and $\mu_{n+1} = 2n$ (recall (3.10)), hence the zero-order term $(\lambda_{\pi/2} - \mu_{n+1} \sin^{-2}(\phi)) \leq 0$ for all ϕ . Moreover, there exists ε depending on dimension n , such that for $\rho \in (\pi/2 - \varepsilon, \pi)$, the zero-order term $(\lambda_\rho - \mu_{n+1} \sin^{-2}(\phi)) \leq 0$ for all ϕ . From (3.17) we have h_ρ is a strict supersolution, and from the comparison principle we conclude that $h > 0$, and so $g_n < g_{n+1}$ on $(0, \rho)$. Furthermore, from the Hopf principle we conclude that $h'(\rho) < 0$.

We will now show the same two properties for any $\rho \in (0, \pi)$. If h_ρ is the function associated with ρ , then define the set

$$A := \{\rho \in (0, \pi) \mid h_\rho > 0 \text{ in } (0, \rho)\}.$$

We will show A is both relatively open and closed in $(0, \pi)$. We have shown already that $A \supset (\pi/2 - \epsilon, \pi)$, and therefore A is nonempty. Since h_ρ is the difference of two Legendre functions defined on the whole interval $(0, \pi)$, then h_ρ varies continuously with ρ in C^2 on $[0, \phi_0]$ for any $\phi_0 < \pi$. Suppose $\rho_k \in A$ and $\rho_k \rightarrow \rho \in (0, \pi)$. Then by uniform convergence $h_\rho \geq 0$. If $h_\rho(\phi) = 0$, then ϕ is a local minimum, and so $h'_\rho(\phi) = 0$ and $h''_\rho(\phi) \geq 0$. But from (3.17) we have $h''_\rho(\phi) < 0$, which is a contradiction. Therefore, $h_\rho(\phi) > 0$, and so $h_\rho \in A$. This proves that A is closed.

To show A is also open, we will use the fact (shown below) that if $h_\rho \geq 0$, then $h'(\rho) < 0$. Suppose by way of contradiction that $\rho \in A$ and there exists a sequence $\rho_k \rightarrow \rho$ and $\phi_k \in (0, \rho_k)$ such that $h_{\rho_k}(\phi_k) \leq 0$. Since h_{ρ_k} converges uniformly to h_ρ , it follows that for a subsequence, either $\phi_k \rightarrow \rho$ or $\phi_k \rightarrow 0$. If $\phi_k \rightarrow \rho$, then from the uniform convergence, there also exists a second sequence $\phi_{1,k} \rightarrow \rho$, such that $h_{\rho_k}(\phi_{1,k}) = 0$. From the mean value theorem, there exists a third sequence $\phi_{2,k} \rightarrow \rho$, such that $h'_{\rho_k}(\phi_{2,k}) = 0$. From the C^1 convergence up to the boundary point ρ , we would then have that $h'_\rho(\rho) = 0$ which is a contradiction. If instead $\phi_k \rightarrow 0$, then from the uniform convergence, there exists a second sequence of local minima $\phi_{1,k} \rightarrow 0$. But for ϕ close to zero, the zero-order term $(\lambda_{\rho_k} - \mu_{n+1} \sin^{-2}(\phi)) \leq 0$, so from the comparison principle, we have a contradiction. We thus conclude that $A = (0, \pi)$.

We now show the claim that $h'(\rho) < 0$. This claim is not only necessary for the proof above, but we will also use it to show the eigenvalue gap. We only need to consider the situation when $(\lambda_\rho - \mu_{n+1} \sin^{-2}(\rho)) > 0$ and necessarily $\rho \in (0, \pi/2 - \epsilon)$. Under appropriate conditions one may still apply the Hopf principle; however, in our one-dimensional case, we give the following simpler argument.

Since $h \geq 0$ and the zero-order term is positive in $(\rho - \delta, \rho)$ we have that

$$h'' + (n-1) \cot \phi h' = \zeta(\phi) \leq 0 \quad \text{for } \phi \in (\rho - \delta, \rho). \quad (3.18)$$

Assume by way of contradiction that $h'(\rho) = 0$. Then applying the explicit representation formula for first order linear equations to (3.18) together with the initial condition $h'(\rho) = 0$, we have that

$$h'(\phi) = \frac{1}{p(\phi)} \int_\rho^\phi p(s) \zeta(s) ds \geq 0 \quad (\text{since } \zeta \leq 0 \text{ and } \phi < \rho),$$

and where $p(s) > 0$ is the integrating factor given by

$$p(s) = \text{Exp} \left\{ \int_\rho^s (n-1) \cot t dt \right\}.$$

Then $h' \geq 0$ in $(\rho - \delta, \rho)$. Since $h \geq 0$, we must then have that $h' \equiv 0$, so that $h \equiv 0$ on $(\rho - \delta, \rho)$. But this is a contradiction since as already noted h is a strict supersolution to the equation that g_{n+1} satisfies. We must then conclude that $h'(\rho) < 0$.

Consequently, we have $g'_{n+1}(\rho) > g'_n(\rho)$ and thus $\eta_{n+1} > \eta_n$. In particular, $\eta_i \geq \eta_{n+1} > 0$ for all $i \geq n+1$. Finally, from (3.14), we see that η_{n+1} depends continuously on ρ , and so $\eta_{n+1} = \eta_{n+1}(\rho) \geq \bar{\eta}$ for ρ in any compact subset $[\pi/2 - \gamma, \pi/2 + \gamma]$ of $(0, \pi)$. \square

Theorem 3.11 will allow us to easily show that the second variation controls the L^2 norm squared of any function $\xi \in H^{1/2}(\partial B_\rho)$ that is orthogonal to Y_0, \dots, Y_{n-1} ; see the proof of Corollary 3.13 below. In order to improve this L^2 control to $H^{1/2}$ control, we need the following lemma.

Lemma 3.12. *Fix $n \geq 2$ and $\gamma \in (0, \pi/2)$. There exists a constant C depending only on n and γ such that the following holds. Let $|\pi/2 - \rho| \leq \gamma$ and let $\xi \in H^{1/2}(\partial B_\rho)$ with $\int_{\partial B_\rho} \xi = 0$. For the solution w_ξ of (3.5), we have $\|w_\xi\|_{L^2(B_\rho)} \leq \|\xi\|_{L^2(\partial B_\rho)}$.*

Proof. Let $w = w_\xi$ for notational simplicity. Let φ be the unique solution to the following auxiliary problem:

$$\begin{cases} (-\Delta - \lambda_\rho)\varphi = w & \text{on } B_\rho \\ \varphi = 0 & \text{on } \partial B_\rho \\ \int_{B_\rho} \varphi u_{B_\rho} = 0. \end{cases}$$

Such a solution exists and is unique by the Fredholm alternative. We multiply the equation by w and integrate by parts twice to find

$$\int_{B_\rho} w^2 = \int_{B_\rho} w(-\Delta - \lambda_\rho)\varphi = - \int_{\partial B_\rho} \partial_\nu \varphi w + \int_{\partial B_\rho} \partial_\nu w \varphi + \int_{B_\rho} \varphi(-\Delta - \lambda_\rho)w = - \int_{\partial B_\rho} \partial_\nu \varphi w.$$

Then by Hölder's inequality and the trace embedding (see [2, Theorem 5.22]) respectively, we see that

$$\int_{B_\rho} w^2 \leq \|\partial_\nu \varphi\|_{L^2(\partial B_\rho)} |u'_\rho| \|\xi\|_{L^2(\partial B_\rho)} \leq C \|\nabla \varphi\|_{H^1(B_\rho)} |u'_\rho| \|\xi\|_{L^2(\partial B_\rho)} \leq C \|\varphi\|_{H^2(B_\rho)} |u'_\rho| \|\xi\|_{L^2(\partial B_\rho)},$$

where the constant $C > 0$ depends on n and ρ , but is uniformly bounded above for $|\pi/2 - \rho| \leq \gamma$. By standard elliptic estimates (see [29, Theorem 9.13]), we have $\|\varphi\|_{H^2(B_\rho)} \leq C \|w\|_{L^2(B_\rho)}$, where again $C > 0$ is uniformly bounded above depending on n and γ . Since additionally $|u'_\rho|$ is bounded by a constant $C = C(n, \gamma)$, we find that

$$\int_{B_\rho} w^2 \leq C \|w\|_{L^2(B_\rho)} \|\xi\|_{L^2(\partial B_\rho)}.$$

Dividing through by $\|w\|_{L^2(B_\rho)}$ establishes the claim. \square

From Theorem 3.11 and Lemma 3.12, we can conclude that the second variation controls $\|\xi\|_{H^{1/2}(\partial B)}^2$ for any function $\xi \in H^{1/2}(\partial B_\rho)$ that is orthogonal to Y_0, \dots, Y_{n-1} :

Corollary 3.13. *Fix $n \geq 2$ and $\gamma \in (0, \pi/2)$. There is a positive constant $\hat{\eta}$ depending only on n and γ such that the following holds. Let $\rho \in (0, \pi)$ satisfy $|\pi/2 - \rho| \leq \gamma$. For any $\xi \in H^{1/2}(\partial B_\rho)$ with $\int_{\partial B_\rho} \xi Y_i = 0$ for $i = 0, \dots, n$, we have*

$$\delta^2 \lambda_1(B_\rho)[\xi, \xi] \geq \hat{\eta} \|\xi\|_{H^{1/2}(\partial B_\rho)}^2. \quad (3.19)$$

Proof. We express $\xi = \sum_{i=n+1}^\infty a_i Y_i$ in the basis of spherical harmonics. By the linearity of L_ρ , we see that $L_\rho \xi = \sum_{i=n+1}^\infty \eta_i a_i Y_i$. Thus, Theorem 3.11 shows that

$$\frac{1}{2} \delta^2 \lambda_1(B_\rho)[\xi, \xi] = \sum_{i=n+1}^\infty \eta_i a_i^2 \geq \eta_n \sum_{i=n+1}^\infty a_i^2 = \bar{\eta} \|\xi\|_{L^2(\partial B_\rho)}^2, \quad (3.20)$$

where $\bar{\eta}$ depends only on n and γ . In order to improve this estimate to replace $\|\xi\|_{L^2(\partial B_\rho)}$ with $\|\xi\|_{H^{1/2}(\partial B_\rho)}$, let h_ξ denote the harmonic extension of $|u'_\rho| \xi$ as defined in (3.4) and note that

$$|u'_\rho|^2 \|\xi\|_{H^{1/2}(B_\rho)}^2 = \int_{B_\rho} |\nabla h_\xi|^2 \leq \int_{B_\rho} |\nabla w_\xi|^2.$$

Therefore, using the expression (3.6) for the second variation and then applying Lemma 3.12 to w_ξ , we have

$$\begin{aligned} 2|u'_\rho|^2 \|\xi\|_{H^{1/2}(B_\rho)}^2 &\leq 2 \int_{B_\rho} |\nabla w_\xi|^2 \\ &\leq \delta^2 \lambda_1(B_\rho)[\xi, \xi] + |\mathcal{H}_{B_\rho}| |u'_\rho|^2 \int_{\partial B_\rho} \xi^2 + 2\lambda_\rho \int_{B_\rho} w_\xi^2 \\ &\leq \delta^2 \lambda_1(B_\rho)[\xi, \xi] + C \int_{\partial B_\rho} \xi^2, \end{aligned}$$

where $C = C(n, \gamma)$. Noting that $|u'_\rho| > 0$ depends continuously on ρ and thus $|u'_\rho| \geq c(n, \gamma) > 0$, we apply the estimate (3.20) to the second term on the right-hand side to conclude the proof. \square

Finally, we can combine Corollary 3.13 with Theorem 3.7 to obtain the following theorem.

Theorem 3.14. *Fix $n \geq 2$ and $\gamma \in (0, \pi/2)$. There are positive constants ε and $\hat{\eta}$ depending only on n and γ such that the following holds. Let $|\pi/2 - \rho| \leq \gamma$ and let Ω be a nearly spherical set parametrized by ξ over B_ρ with $\|\xi\|_{C^{2,\alpha}(\partial B_\rho)} \leq \varepsilon$. Then*

$$\lambda_1(\Omega) - \lambda_1(B_\rho) \geq \hat{\eta} \|\xi\|_{H^{1/2}(\partial B_\rho)}^2.$$

Proof. We express ξ in the basis of spherical harmonics as $\xi = \sum_{i=0}^{\infty} a_i Y_i$ where $a_i = \int_{\partial B_\rho} Y_i \xi$. The volume constraint and the set center constraint respectively imply that

$$a_0^2 = o\left(\|\xi\|_{L^2(\partial B_\rho)}^2\right), \quad a_1^2, \dots, a_n^2 = o\left(\|\xi\|_{L^2(\partial B_\rho)}^2\right) \quad (3.21)$$

with the $o(\cdot)$ term depending on n and γ . Indeed, as we noted in Remark 3.8, the first fact is a standard computation that can be found in [13, Theorem 3.1]. To see the second, consider the standard embedding of $S^n \subset \mathbb{R}^{n+1}$ with the north pole $o = e_{n+1}$, and fix a basis vector e_i for $i = 1, \dots, n$. In spherical coordinates we may write $e_i = (1, \bar{\theta}, \pi/2)$ for some $\bar{\theta} \in S^{n-1}$. So, from the barycenter constraint $x_\Omega = e_{n+1}$, we have

$$\begin{aligned} 0 &= e_i \cdot \int_{\Omega} x \, d\text{vol} = \int_{S^{n-1}} \int_0^{(1+\xi(\theta)\rho)} ((\theta \cdot \bar{\theta}) \sin \phi) \sin^{n-1} \phi \, d\phi \, d\theta \\ &= \int_{S^{n-1}} (\theta \cdot \bar{\theta}) \int_0^{(1+\xi(\theta)\rho)} \sin^n \phi \, d\phi \, d\theta \\ &= \int_{S^{n-1}} (\theta \cdot \bar{\theta}) F((1 + \xi(\theta))\rho) \, d\theta, \end{aligned}$$

where $F(t) = \int_0^t \sin(s)^n \, ds$ is a smooth hypergeometric function and $\theta \cdot \bar{\theta}$ indicates the inner product with respect to the standard metric on S^{n-1} . A Taylor expansion of F and Hölder's inequality thus show that

$$\begin{aligned} 0 &= \int_{S^{n-1}} (\theta \cdot \bar{\theta}) \left(F(\rho) + C_\rho \xi(\theta) + o(\|\xi\|_{L^2(\partial B_\rho)}) \right) \, d\theta \\ &= C_\rho \int_{S^{n-1}} (\theta \cdot \bar{\theta}) \xi(\theta) \, d\theta + o(\|\xi\|_{L^2(\partial B_\rho)}). \end{aligned}$$

So, we see that for $i = 1, \dots, n$, we have $a_i = \int_{S^{n-1}} (\theta \cdot \bar{\theta}) \xi(\theta) \, d\theta = o(\|\xi\|_{L^2(S^{n-1})})$ and conclude (3.21). Define the functions

$$\hat{\xi} = \xi - a_0 Y_0, \quad \bar{\xi} = \hat{\xi} - \sum_{i=1}^n a_i Y_i.$$

As a consequence of (3.21), we see that

$$\begin{aligned} \|\xi\|_{H^{1/2}(\partial B_\rho)} &= \|\hat{\xi}\|_{H^{1/2}(\partial B_\rho)} + o\left(\|\xi\|_{L^2(\partial B_\rho)}\right) \\ &= \|\bar{\xi}\|_{H^{1/2}(\partial B_\rho)} + o\left(\|\xi\|_{L^2(\partial B_\rho)}\right). \end{aligned}$$

By Theorem 3.7 and applying Corollary 3.13 to $\bar{\xi}$, we have

$$\begin{aligned} \lambda_1(\Omega) - \lambda_1(B_\rho) &\geq \frac{1}{2} \delta^2 \lambda(B_\rho) [\hat{\xi}, \hat{\xi}] + o\left(\|\xi\|_{H^{1/2}(\partial B_\rho)}^2\right) \\ &= \frac{1}{2} \delta^2 \lambda(B_\rho) [\bar{\xi}, \bar{\xi}] + o\left(\|\xi\|_{H^{1/2}(\partial B_\rho)}^2\right) \\ &= \frac{\hat{\eta}}{2} \|\bar{\xi}\|_{H^{1/2}(\partial B_\rho)}^2 + o\left(\|\xi\|_{H^{1/2}(\partial B_\rho)}\right) \\ &= \frac{\hat{\eta}}{2} \|\xi\|_{H^{1/2}(\partial B_\rho)}^2 + o\left(\|\xi\|_{H^{1/2}(\partial B_\rho)}\right). \end{aligned}$$

Choosing ε sufficiently small depending on $\bar{\eta}$ (and thus on γ and n), we may absorb the second term on the right-hand side to conclude. \square

3.1.3. The $H^{1/2}$ norm controls the distance. Thanks to Theorem 3.14, we will conclude the proof of Proposition 3.6 as soon as we can show that the terms $|\Omega \Delta B_\rho|$ and $\int_{S^n} |u_\Omega - u_{B_\rho}|$ are controlled linearly by $\|\xi\|_{H^{1/2}(B)}$. This is the content of the following proposition.

In the case of Euclidean space and hyperbolic space, the proof of this proposition will be identical up to replacing the coarea factor $\sin^{n-1} \phi$ with ϕ^{n-1} and $\sinh^{n-1} \phi$ respectively in various integrals throughout the proof.

Proposition 3.15. *Fix $n \geq 2$ and $\gamma \in (0, \pi/2)$. There are positive constants ε and C depending on n and γ such that the following holds. Let $|\pi/2 - \rho| < \gamma$. Let $\Omega \subset S^n$ be a nearly spherical set parameterized by ξ with $\|\xi\|_{C^{2,\alpha}(\partial B_\rho)} \leq \varepsilon$ and $\lambda_1(\Omega) - \lambda_1(B_\rho) \leq \varepsilon \lambda_1(B_\rho)$. Then*

$$|\Omega \Delta B_\rho|^2 + \int_{S^n} |u_\Omega - u_{B_\rho}|^2 \leq C \|\xi\|_{H^{1/2}(\partial B_\rho)}^2. \quad (3.22)$$

Proof. The fact that $|\Omega \Delta B_\rho| \leq C \|\xi\|_{L^2(\partial B)} \leq C \|\xi\|_{H^{1/2}(\partial B_\rho)}$ is a standard computation; see [21, Proof of Theorem 4.3] or [16, Lemma 4.2]. Establishing $\int_{S^n} |u_\Omega - u_{B_\rho}|^2 \leq C \|\xi\|_{H^{1/2}(\partial B_\rho)}^2$ is more involved and will be carried out in several steps.

Step 1. First, we define a diffeomorphism between B_ρ and Ω and express the equation solved by u_Ω pulled back by this diffeomorphism. To this end, let $h : B_\rho \rightarrow \mathbb{R}$ be the harmonic extension of ξ in B_ρ defined in (3.4). Let $\Psi : B_\rho \rightarrow S^n$ be the smooth map defined by

$$\Psi(x) = (1 + h(x)) \exp_o(x), \quad (3.23)$$

where $\exp_o(x)$ is the exponential map at o . In normal coordinates $\{x^\alpha\}$, the differential $d\Psi$ has coefficients

$$\frac{\partial \Psi^\alpha}{\partial x^i} = (1 + h(x)) \delta_i^\alpha + \frac{\partial}{\partial x^i} h(x) x^\alpha. \quad (3.24)$$

Since h is harmonic and $|\nabla h|^2$ is subharmonic, the maximum principle ensures that $|h(x)|, |\nabla h(x)| \leq \|\xi\|_{C^1(\partial B_\rho)} \leq C \|\xi\|_{C^{2,\alpha}(\partial B_\rho)} \leq C\varepsilon$ for all $x \in B_\rho$, where $C = C(n, p)$. The metric coefficients g_{ij} are uniformly bounded above, so $|\frac{\partial}{\partial x^i} h(x)| \leq C\varepsilon$ as well. We thus see that

$$\left| \frac{\partial \Psi^\alpha}{\partial x^i} - \delta_i^\alpha \right| \leq C\varepsilon, \quad (3.25)$$

where C is a constant depending only on n and α . In particular, the differential $d\Psi$ is non-vanishing on B_ρ , so Ψ is injective and hence a diffeomorphism onto its image. Since Ψ maps ∂B_ρ to $\partial\Omega$, it follows that Ω is the image of B_ρ under Ψ and so Ψ defines a smooth diffeomorphism from B_ρ to Ω . We let $\Phi : \Omega \rightarrow B_\rho$

denote the inverse of Ψ .

Recall that u_Ω denotes the first Dirichlet eigenfunction of Ω , normalized so that $\int u_\Omega^2 dx = 1$ and extended by 0 to be defined on all of S^n . We define $\hat{u}_\Omega : B_\rho \rightarrow \mathbb{R}$ to be the pullback $\hat{u}_\Omega(x) = u_\Omega(\Psi(x))$ of u_Ω by Ψ . Then \hat{u}_Ω satisfies the equation

$$\begin{cases} -\mathcal{L}\hat{u}_\Omega = m\lambda_\Omega\hat{u}_\Omega & \text{in } B_\rho \\ \hat{u}_\Omega = 0 & \text{on } \partial B_\rho. \end{cases} \quad (3.26)$$

Here and in the remainder of the proof we use the short-hand $\lambda_\Omega = \lambda_1(\Omega)$, and \mathcal{L} is the a linear divergence form operator given by

$$\mathcal{L}f = \operatorname{div}(A\nabla f), \quad (3.27)$$

where $A = m\mathcal{A}$ for $\mathcal{A}(x) = d\Phi(\Psi(x))(d\Phi(\Psi(x)))^*$ and $m = 1/\sqrt{\det \mathcal{A}}$. From the definition of \mathcal{L} and the coordinate expression (3.24), we see that the coefficients satisfy the pointwise estimates

$$|A - \operatorname{Id}| \leq C(h + |\nabla h|), \quad |m - 1| \leq C(h + |\nabla h|) \quad (3.28)$$

for all x in B_ρ , where C is a dimensional constant. Furthermore, we have

$$\|A - \operatorname{Id}\|_{C^{0,\alpha}(B_\rho)} \leq C\|h\|_{C^{1,\alpha}(B_\rho)}, \quad \|m - 1\|_{C^{0,\alpha}(B_\rho)} \leq C\|h\|_{C^{1,\alpha}(B_\rho)}. \quad (3.29)$$

In particular, by (3.25) and by [29, Corollary 8.35], we have

$$\|u_\Omega\|_{C^{1,\alpha}(\bar{\Omega})} \leq C, \quad \|\hat{u}_\Omega\|_{C^{1,\alpha}(\bar{B}_\rho)} \leq C. \quad (3.30)$$

Additionally, we have

$$\|A - \operatorname{Id}\|_{L^2(B_\rho)} \leq C\|h\|_{W^{1,2}(B_\rho)} \leq C\|\xi\|_{H^{1/2}(B_\rho)}. \quad (3.31)$$

Step 2: We now prove the following integral estimate for $u_\Omega - \hat{u}_\Omega$:

$$\int_{S^n} |u_\Omega - \hat{u}_\Omega|^2 dx \leq C \|\xi\|_{H^{1/2}(\partial B)}^2. \quad (3.32)$$

We argue in three separate regions: $B_\rho \cap \Omega$, $\Omega \setminus B_\rho$, and $B_\rho \setminus \Omega$. First, given any $x \in B_\rho \cap \Omega$, we have

$$|\hat{u}_\Omega(x) - u_\Omega(x)| = |u_\Omega(\Psi(x)) - u_\Omega(x)| \leq \|u_\Omega\|_{C^1(\Omega)} d(x, \Psi(x)) \leq Ch(x).$$

The final two inequalities follow from (3.30) and the definition of Ψ respectively. So, squaring and integrating this pointwise estimate and applying Lemma 3.12 (or more specifically, the analogous statement with λ_ρ replaced by 0), we have

$$\int_{B_\rho \cap \Omega} |u_\Omega - \hat{u}_\Omega|^2 dx \leq C\|\xi\|_{L^2(\partial B_\rho)}^2. \quad (3.33)$$

Next, for any $x \in \Omega \setminus B_\rho$, note that $\hat{u}_\Omega = 0$ and so $|u_\Omega(x) - \hat{u}_\Omega(x)| = u_\Omega(x)$. Thanks to the estimate (3.30) and the fact that u_Ω vanishes on $\partial\Omega$, we see that $u_\Omega(x) \leq \frac{C}{\rho} d(x, \partial\Omega)$. Thus, in spherical coordinates (θ, ϕ) , we have $u_\Omega(\theta, \phi) \leq C(1 + \xi(\theta) - \phi/\rho)$. Hence, using the coarea formula, we have

$$\begin{aligned} \int_{\Omega \setminus B_\rho} u_\Omega^2 dx &= \int_{\{\xi > 0\}} \int_\rho^{(1+\xi)\rho} u(\theta, \phi)^2 \sin(\phi)^{n-1} d\phi d\theta \\ &\leq C \sin(\rho)^{n-1} \int_{\{\xi > 0\}} \int_\rho^{(1+\xi)\rho} (1 + \xi(\theta) - \phi/\rho)^2 d\phi d\theta \\ &= C \sin(\rho)^{n-1} \rho \int_{\{\xi > 0\}} \xi^2 d\theta \leq C\rho \|\xi\|_{L^2(\partial B_\rho)}^2. \end{aligned}$$

where the penultimate equality comes from the change of variable $1 + \xi(\theta) - \phi/\rho = s$. The analogous argument on $B_\rho \setminus \Omega$ using \hat{u}_Ω in place of u_Ω . Together, these estimates along with (3.33) show (3.32).

Step 3: Finally, we show that

$$\int_{B_\rho} |\hat{u}_\Omega - u_{B_\rho}|^2 dx \leq C \|\xi\|_{H^{1/2}(\partial B_\rho)}^2. \quad (3.34)$$

Let us begin with the following simplification. Let $\alpha_\Omega = \int_{B_\rho} \hat{u}_\Omega u_{B_\rho}$. Then

$$\int_{B_\rho} |\hat{u}_\Omega - u_{B_\rho}|^2 dx \leq 4 \int_{B_\rho} |\hat{u}_\Omega - \alpha_\Omega u_{B_\rho}|^2 dx,$$

since the left-hand side is equal to $2(1 - \alpha_\Omega^2)$ while the right-hand side is equal to $1 - \alpha_\Omega^2$. It therefore suffices to show that

$$\int_{B_\rho} |\hat{u}_\Omega - \alpha_\Omega u_{B_\rho}|^2 dx \leq C \|\xi\|_{H^{1/2}(\partial B_\rho)}^2. \quad (3.35)$$

To this end, set $v = \hat{u}_\Omega - \alpha_\Omega u_{B_\rho}$. The idea behind showing (3.35) is that v solves an equation with right-hand side controlled in terms of ξ . More specifically, we see that v satisfies the equation

$$\begin{aligned} -\mathcal{L}v &= -\mathcal{L}\hat{u}_\Omega + \alpha_\Omega \mathcal{L}u_{B_\rho} \\ &= m\lambda_\Omega \hat{u}_\Omega - \alpha_\Omega \lambda_\rho u_{B_\rho} + \alpha_\Omega (\mathcal{L} - \Delta)u_{B_\rho} \\ &= \lambda_\Omega v + \alpha_\Omega (\lambda_\Omega - \lambda_\rho)u_{B_\rho} + \alpha_\Omega (\mathcal{L} - \Delta)u_{B_\rho} + (m-1)\lambda_\Omega \hat{u}_\Omega. \end{aligned}$$

We multiply this equation by v and integrate over B_ρ . On one hand, note that (3.28) ensures that $A \geq (1 - C\varepsilon)\text{Id}$. So integrating by parts, we find that

$$- \int_{B_\rho} v \mathcal{L}v \geq (1 - C\varepsilon) \int_{B_\rho} |\nabla v|^2. \quad (3.36)$$

On the other hand, note that $\int_{B_\rho} v u_{B_\rho} = 0$ and v vanishes on ∂B_ρ , and thus v satisfies the improved Poincaré inequality

$$\int_{B_\rho} |\nabla v|^2 dx \geq \lambda_2 \int_{B_\rho} v^2 dx \quad (3.37)$$

where $\lambda_2 = \lambda_2(B_\rho)$ is the second Dirichlet eigenvalue of B_ρ . So, again using the orthogonality of u_{B_ρ} and v , we find that

$$\begin{aligned} - \int_{B_\rho} v \mathcal{L}v &= \int_{B_\rho} v \{ \lambda_\Omega v + \alpha_\Omega (\lambda_\Omega - \lambda_\rho)u_{B_\rho} + \alpha_\Omega (\mathcal{L} - \Delta)u_{B_\rho} + (m-1)\lambda_\Omega \hat{u}_\Omega \} \\ &= \int_{B_\rho} \{ \lambda_\Omega v^2 + \alpha_\Omega (\mathcal{L} - \Delta)u_{B_\rho} v + (m-1)\lambda_\Omega \hat{u}_\Omega v \} dx \\ &\leq \frac{\lambda_\Omega}{\lambda_2} \int_{B_\rho} |\nabla v|^2 + \alpha_\Omega \int_{B_\rho} (\mathcal{L} - \Delta)u_{B_\rho} v + \lambda_\Omega \int_{B_\rho} (m-1)\hat{u}_\Omega v. \end{aligned} \quad (3.38)$$

Provided ε is sufficiently small with respect to the spectral gap λ_1/λ_2 , we may combine (3.36) and (3.38) and absorb the first term on the right-hand side of (3.38) to find

$$c \int_{B_\rho} |\nabla v|^2 \leq \alpha_\Omega \int_{B_\rho} (\mathcal{L} - \Delta)u_{B_\rho} v + \lambda_\Omega \int_{B_\rho} (m-1)\hat{u}_\Omega v. \quad (3.39)$$

Now, we bound the two terms on the right-hand side of (3.39) separately. For the first term, we integrate by parts and then apply the Cauchy-Schwarz inequality and the bound (3.31) on the coefficients to find

$$\begin{aligned} \alpha_\Omega \int_{B_\rho} (\mathcal{L} - \Delta)u_{B_\rho}v &= \alpha_\Omega \int_{B_\rho} \langle (A - \text{Id})\nabla u_{B_\rho}, \nabla v \rangle \\ &\leq \frac{\eta \alpha_\Omega}{2} \int_{B_\rho} |\nabla v|^2 + \frac{\alpha_\Omega}{2\eta} \|\nabla u_{B_\rho}\|_{L^\infty(B_\rho)} \int_{B_\rho} |A - \text{Id}|^2 dx \\ &\leq \frac{\eta \alpha_\Omega}{2} \int_{B_\rho} |\nabla v|^2 + \frac{\alpha_\Omega}{2\eta} \|\nabla u_{B_\rho}\|_{L^\infty(B_\rho)} \|\xi\|_{H^{1/2}(\partial B_\rho)}^2. \end{aligned}$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner product on S^n with respect to the round metric. Provided that η is chosen to be sufficiently small, we can absorb the first term to the left-hand side of (3.39). In a similar way, for the second term on the right-hand side of (3.39), we have

$$\begin{aligned} \lambda_\Omega \int_{B_\rho} (m-1)\hat{u}_\Omega v &\leq \frac{\lambda_\Omega}{\eta} \int_{B_\rho} |m-1|^2 + \lambda_\Omega \eta \|\hat{u}_\Omega\|_{L^\infty(B_\rho)}^2 \int_{B_\rho} v^2 \\ &\leq \frac{\lambda_\Omega}{\eta} \|\xi\|_{H^{1/2}(\partial B_\rho)}^2 + \lambda_\Omega \eta C \int_{B_\rho} v^2 \\ &\leq \frac{\lambda_\Omega}{\eta} \|\xi\|_{H^{1/2}(\partial B_\rho)}^2 + \frac{(1+\varepsilon)\lambda_1}{\lambda_2} \eta C \int_{B_\rho} |\nabla v|^2. \end{aligned}$$

Choosing η to be sufficiently small, we then find that

$$\frac{1}{4} \int_{B_\rho} |\nabla v|^2 \leq \frac{\alpha_\Omega}{2\eta} \|\nabla u_{B_\rho}\|_{L^\infty(B_\rho)} \|\xi\|_{H^{1/2}(\partial B_\rho)}^2. \quad (3.40)$$

This, together with the Poincaré inequality on B_ρ , establishes (3.35) and thus (3.34). We combine (3.34) and (3.32) to conclude the proof of the proposition. \square

Finally we can prove Theorem 3.6.

Proof of Theorem 3.6. Together Theorem 3.14 and Proposition 3.15 directly imply the theorem when $\lambda(\Omega) - \lambda(B_\rho) \leq \varepsilon \lambda(B_\rho)$, while the result holds trivially when $\lambda(\Omega) \geq (1+\varepsilon)\lambda(B_\rho)$ by choosing the constant c to be sufficiently small. \square

3.2. Torsional rigidity, the Kohler Jobin inequality, and Quantitative Stability for the Faber-Krahn inequality. Thus far, we have proven Theorem 3.1 in the special case of nearly spherical sets. The remainder of the proof of Theorem 3.1 relies on a selection principle argument, as described in the introduction. The proof of the selection principle is carried out in our companion paper [4]. We restate the main result there, stated only in the generality needed here, in Theorem 3.16 below. For regularity reasons discussed in [4], the functional involved in the selection principle involves an additional, possibly unexpected term: the torsional rigidity.

Given an open set $\Omega \subset S^n$, the *torsional rigidity* of Ω is defined by

$$\text{tor}(\Omega) = \inf \left\{ \int_\Omega \frac{1}{2} |\nabla u|^2 - \int_\Omega u : u \in W_0^{1,2}(\Omega) \right\}. \quad (3.41)$$

Naturally, the same quantity can be defined for an open subset on any Riemannian manifold. We note that our sign convention for the torsional rigidity differs by a sign from the definition stated in other contexts such

as [33, 14]. With our sign convention, $\text{tor}(\Omega) \leq 0$. Note also that $\text{tor}(\Omega)$, like the first Dirichlet eigenvalue, is decreasing under set inclusion. The infimum in $\text{tor}(\Omega)$ is achieved by the unique solution to the equation

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.42)$$

Applying the Polya-Szegö inequality to any function in the minimization problem (3.41), one sees that balls minimize the torsional rigidity among all quasi-open sets of a fixed volume:

$$\text{tor}(\Omega) \geq \text{tor}(B) \quad \text{if } \text{vol}(\Omega) = \text{vol}(B). \quad (3.43)$$

Since the Polya-Szegö principle holds on Euclidean space and hyperbolic space, balls also minimize the torsional rigidity among sets of a fixed volume in these spaces.

Combined with the Faber-Krahn inequality, we see that balls uniquely minimize the functional $\lambda_1(\Omega) + \mathfrak{T}\text{tor}(\Omega)$ for any $\mathfrak{T} \geq 0$ among sets of a fixed volume. In our companion paper, we establish the following global-to-local stability theorem; see [4, Corollary 1.2 and Remark 1.3].

Theorem 3.16 (Selection Principle). *Let M denote the round sphere, Euclidean space, or hyperbolic space. Fix $n \geq 2$ and an interval $[v_0, v_1] \in (0, |M|)$. There exist $\mathfrak{T}_0 > 0$ and $\varepsilon > 0$ depending on n, v_0 , and v_1 such that the following holds.*

Suppose there exists a constant $c > 0$ such that for all $v \in [v_0, v_1]$, $\mathfrak{T} < \mathfrak{T}_0$ and for every ε -nearly spherical set parametrized over B where $|B| = v$, we have

$$c \left(|\Omega \Delta B|^2 + \int_M |u_\Omega - u_B|^2 \right) \leq \lambda_1(\Omega) - \lambda_1(B) + \mathfrak{T}(\text{tor}(\Omega) - \text{tor}(B)). \quad (3.44)$$

Then for all open sets Ω with $|\Omega| = |B|$,

$$c \inf_{x \in M} \left(|\Omega \Delta B(x)|^2 + \int_M |u_\Omega - u_B(x)|^2 \right) \leq \lambda_1(\Omega) - \lambda_1(B) + \mathfrak{T}(\text{tor}(\Omega) - \text{tor}(B)), \quad (3.45)$$

where $B(x)$ is a ball of the same radius as B .

Although the form of Theorem 3.16 does not appear to be immediately compatible with proving Theorem 3.1, an essential point is that the quantities

$$\delta(\Omega) = \lambda_1(\Omega) - \lambda_1(B), \quad \hat{\delta}(\Omega) = \lambda_1(\Omega) - \lambda_1(B) + \mathfrak{T}(\text{tor}(\Omega) - \text{tor}(B)) \quad (3.46)$$

are equivalent. While it is immediate that $\delta(\Omega) \leq \hat{\delta}(\Omega)$, we must call upon a result of Kohler-Jobin, Theorem 3.17 below, to show that $\hat{\delta}(\Omega) \leq C\delta(\Omega)$.

Polya and Szegö [42] conjectured that among sets $\Omega \subset \mathbb{R}^n$ of a fixed torsional rigidity $\text{tor}(\Omega)$, balls minimize the first Dirichlet eigenvalue $\lambda_1(\Omega)$. This conjecture was proven by Kohler-Jobin through the introduction of a new symmetrization technique in [33, 32] (see also [34], as well as [14] for extensions to the p -Laplace energy). Fundamentally, the proof is underpinned only by the coarea formula and the isoperimetry of balls. Basic modifications of Kohler-Jobin's argument show that this fact, now known as the Kohler-Jobin inequality, holds on all simply connected space forms.

Theorem 3.17 (Kohler-Jobin inequality on simply connected space forms). *Let M denote the round sphere, Euclidean space, or hyperbolic space. Fix an open set $\Omega \subset M$ and let $B_\Omega^* \subset M$ denote a geodesic ball with the unique radius such that $\text{tor}(\Omega) = \text{tor}(B_\Omega^*)$. Then*

$$\lambda_1(\Omega) \geq \lambda_1(B_\Omega^*). \quad (3.47)$$

Equality is achieved in (3.47) if and only if Ω is a ball of this radius.

Theorem 3.17 allows us to deduce the following comparison between the Faber-Krahn deficit and the torsional rigidity deficit and in particular show that $\delta(\Omega)$ and $\hat{\delta}(\Omega)$ are comparable.

Corollary 3.18. *Let M denote the round sphere, Euclidean space, or hyperbolic space and fix $0 < \bar{v} < \bar{V} < \text{vol}(M)$. There exists a constant $C = C(\bar{v}, \bar{V}) > 0$ so that for any $\Omega \subset M$ with $\text{vol}(\Omega) = v_0 \in [\bar{v}, \bar{V}]$,*

$$C(\lambda_1(\Omega) - \lambda_1(B)) \geq \text{tor}(\Omega) - \text{tor}(B) \quad (3.48)$$

where B is a geodesic ball with $\text{vol}(B) = v_0$. In particular, $\hat{\delta}(\Omega) \leq \tilde{C}\delta(\Omega)$, where $\tilde{C} = 1 + C\mathfrak{T}$.

Remark 3.19. One may use the scaling invariance of Euclidean space to more directly conclude Corollary 3.18 from the Kohler-Jobin inequality; see [16, Proposition 2.1]. For the sake of a unified approach, we simply include Euclidean space in Corollary 3.18.

Proof of Corollary 3.18. As in Theorem 3.17, let B_Ω^* denote a ball with the uniquely chosen radius such that $\text{tor}(\Omega) = \text{tor}(B_\Omega^*)$. Theorem 3.17 implies that

$$\lambda_1(\Omega) - \lambda_1(B) = [\lambda_1(B_\Omega^*) - \lambda_1(B)] + [\lambda_1(\Omega) - \lambda_1(B_\Omega^*)] \geq \lambda_1(B_\Omega^*) - \lambda_1(B),$$

and so to establish (3.48), it suffices to prove that $C(\lambda_1(B_\Omega^*) - \lambda_1(B)) \geq \text{tor}(B_\Omega^*) - \text{tor}(B)$. Noting that the radius of B_Ω^* is smaller than that of B , the corollary will be proven once we establish the following claim.

Claim: Fix $0 < \bar{r} < \bar{R}$, with $\bar{R} < \pi$ if $M = S^n$, and for each $R \in (\bar{r}, \bar{R})$, define the functions

$$\begin{aligned} f_R(r) &= \lambda_1(B_r) - \lambda_1(B_R), \\ g_R(r) &= \text{tor}(B_r) - \text{tor}(B_R). \end{aligned}$$

There is a constant $C = C(\bar{r}, \bar{R})$ such that for all $r \in (0, R)$,

$$g_R(r) \leq C f_R(r). \quad (3.49)$$

We first establish (3.49) with a constant depending on R . Notice that $\lim_{r \rightarrow 0} f_R(r) = +\infty$ and $g_R(0) = -\text{tor}(B_R)$. So, we may find $\varepsilon = \varepsilon(R)$ such that $g_R(r) \leq f_R(r)$ for all $r \in (0, \varepsilon]$. Next, observe that the functions f_R and g_R are strictly decreasing and differentiable. Define the constants

$$\begin{aligned} m &:= \min\{-f'_R(r) : r \in [\varepsilon, R]\} > 0, \\ M &:= \max\{-g'_R(r) : r \in [\varepsilon, R]\} < \infty. \end{aligned}$$

By the fundamental theorem of calculus and $f(R) = g(R) = 0$, we see that

$$\begin{aligned} f_R(r) &= \int_r^R -f'_R(r) dr \geq m(R-r), \\ g_R(r) &= \int_r^R -g'_R(r) dr \leq M(R-r), \end{aligned}$$

and so $g_R(r) \leq f_R(r) \times M/m$ for $r \in [\varepsilon, R]$. Letting $C = \max\{M/m, 1\} > 0$, we see that (3.49) holds for $C = C(R)$. Finally, by compactness, it is clear that we may choose C uniformly for all $R \in [r_0, R_0]$. This completes the proof of the corollary. \square

Together Theorem 3.6, Theorem 3.16, and Corollary 3.18 lead to the proof of Theorem 3.1:

Proof of Theorem 3.1. Combining Theorem 3.6, we see that (3.44) holds for nearly spherical sets, and so by Theorem 3.16, we see that (3.45) holds for all open sets Ω with $|\Omega| = |B|$. Corollary 3.18 applied to the right-hand side of (3.45) concludes the proof. \square

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