QUANTITATIVE ESTIMATES FOR THE RELATIVE ISOPERIMETRIC PROBLEM AND ITS GRADIENT FLOW OUTSIDE CONVEX BODIES IN THE PLANE

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Abstract. We prove three related quantitative results for the relative isoperimetric problem outside a convex body Ω in the plane: (1) Łojasiewicz estimates and quantitative rigidity for critical points, (2) rates of convergence for the gradient flow, and (3) quantitative stability for minimizers. These results come with explicit constants and optimal exponents/rates, and hold whenever a simple two-dimensional auxiliary variational problem for circular arcs outside of Ω is nondegenerate. The proofs are inter-related, and in particular, for the first time in the context of isoperimetric problems, a flow approach is used to prove quantitative stability for minimizers.

1. Introduction

Two central themes in the study of geometric variational problems are stability and dynamics: how does the energy grow near a minimizer or critical point, and how rapidly does the associated gradient flow converge to equilibrium? It is well known that both properties are closely linked to nondegeneracy, and when the second variation at any critical point is nondegenerate modulo symmetries, one expects a quadratic stability estimate for minimizers and exponential convergence to equilibrium of the gradient flow. Verifying nondegeneracy, however, is in practice often delicate or intractable, especially in geometric settings where the space of competitors is constrained by curvature, topology, or boundary behavior.

This paper investigates these themes in the context of the exterior isoperimetric problem in the plane. Given a convex body $\Omega \subset \mathbb{R}^2$, i.e. a compact, convex set with nonempty interior, with C^2 boundary Σ and a prescribed area $\eta > 0$, consider the isoperimetric problem

$$I_\Omega(\eta) = \inf\left\{P(E;\mathbb{R}^2 \setminus \Omega) \;:\; E \subset \mathbb{R}^2 \setminus \Omega,\ \ |E| = \eta\right\}. \tag{1.1}$$

Here $P(E;\mathbb{R}^2 \setminus \Omega)$ is the relative perimeter, which is equal to $\mathcal{H}^1(\partial E \setminus \Sigma)$ when E has C^1 boundary; see Section 2.

An important aspect of this problem is that we can reduce the question of the nondegeneracy of the second variation of critical points for the infinite-dimensional problem (1.1) to the nondegeneracy of the Hessian for critical points of an explicit two-dimensional variational problem over circular arcs; see Assumption 1.1 below. For any convex body Ω and area constraint $\eta > 0$ for which this two-dimensional nondegeneracy condition holds, we establish a suite of sharp results predicted by the second variation theory: (1) quantitative rigidity and Łojasiewicz estimates, (2) exponential convergence of the associated gradient flow, and (3) quadratic stability for minimizers.

We prove these three main results in an interconnected manner. In particular, to address quantitative stability—that is, the question of whether a set almost achieving the infimum in (1.1) must be quantitatively close to a minimizer—we evolve the boundary of a set by the free boundary area-preserving curve shortening flow and then integrate out a Łojasiewicz-type estimate along the trajectory. Flow-based approaches to proving quantitative stability have been developed in recent years in the contexts of maps from $\mathbb{S}^2$ to $\mathbb{S}^2$ [37, 32] and Sobolev-type inequalities [5], but to our knowledge this is the first application of the method in the context of isoperimetric problems. This constructive strategy yields sharp estimates with explicit constants, and puts stability, quantitative rigidity, and gradient flow convergence in a unified analytic framework.

For any $\eta > 0$, the collection of minimizers $\mathcal{M}^\Omega_\eta$ of (1.1) (among sets of finite perimeter, see Section 2) is nonempty by the direct method. The first variation shows that the relative boundary $\partial E_* \setminus \Sigma$ of any minimizer $E_* \in \mathcal{M}^\Omega_\eta$ is a union of equal-radii circles and circular arcs meeting $\Sigma = \partial\Omega$ orthogonally, and a simple competitor argument then ensures that

E_* is connected[1] and intersects Σ nontrivially. Thus, the boundary of E_* is the union of a single circular arc c^* meeting Σ orthogonally and a subarc σ_{c^*} of Σ.

More generally, we consider oriented immersed curves γ that lie outside of Ω° and intersect Σ only at their endpoints $x_1(\gamma)$ and $x_2(\gamma)$; we let $\mathcal{B}$ denote the collection of such curves, see (2.9). We extend the notion of (signed) enclosed area to any $\gamma \in \mathcal{B}$ by letting $A_\Sigma(\gamma) := \text{Area}(\gamma + \sigma_\gamma)$ for the unique subarc σ_γ of Σ for which the concatenation $\gamma + \sigma_\gamma$ is contractable in $\mathbb{R}^2 \setminus \Omega^\circ$. For positively oriented embedded curves $\gamma \in \mathcal{B}$, this of course agrees with the area of the set E_γ whose relative boundary is given by γ. We let

$$\mathcal{B}_\eta := \{\gamma \in \mathcal{B} : |A_\Sigma(\gamma)| = \eta\} \tag{1.2}$$

be the collection of such (oriented) curves with prescribed enclosed area A_Σ. In addition to minimizers of the isoperimetric problem (1.1), or equivalently, minimizers of the length among curves in $\mathcal{B}_\eta$, we consider critical points of the length functional L in this class. As area preserving variations are characterized by $\int_{\gamma_t} \partial_t\gamma_t \cdot \nu_{\gamma_t}\, ds_{\gamma_t} = 0$, c.f. (2.11), the usual first variation formula for the length can be written as

$$dL(\gamma)(X) = -\int (\kappa_\gamma - \bar{\kappa}_\gamma)\langle \nu_\gamma, X\rangle\, ds_\gamma + \langle X(x_2), \tau_\gamma(x_2)\rangle - \langle X(x_1), \tau_\gamma(x_1)\rangle \tag{1.3}$$

for vector fields $X = \partial_t\gamma_t$ induced by such variations. In particular, the set of critical points of the length functional in $\mathcal{B}_\eta$ is given by

$$C^*_\eta := \{\gamma \in \mathcal{B}_\eta : \text{ circular arc intersecting } \Sigma \text{ orthogonally}\}\,. \tag{1.4}$$

Here and in the sequel, we use the convention that τ_γ is the unit tangent of the oriented curve γ, ν_γ is the normal obtained by rotating τ_γ counterclockwise by $\pi/2$, $\bar{\kappa}_\gamma$ is the average of the curvature κ_γ, and s_γ is the arclength parameter of γ.

The quantitative results established in this paper for the aforementioned minimizers and critical points are closely connected to our third main focus: the asymptotic behavior of the associated gradient flow, i.e. the free boundary area-preserving curve shortening flow, or freeAPSCF, introduced by the first author in [23]. This natural area-preserving gradient flow of the length evolves families of curves γ_t by

$$\partial_t\gamma_t = (\kappa_{\gamma_t} - \bar{\kappa}_{\gamma_t})\nu_{\gamma_t}$$

subject to the constraint that the curves intersect Σ orthogonally and from the outside of Ω at their endpoints $x_1(\gamma_t), x_2(\gamma_t) \in \Sigma$, see (1.9) below for the precise definition. In general, this flow can be quite poorly behaved: it neither preserves embeddedness nor remains entirely outside of Ω, and singularities may develop in finite time. However, in [23], the first author established conditions on the initial data guaranteeing that the flow exists and remains outside of Ω for all time.

A fundamental question addressed in this paper is the rate of convergence to equilibrium for such solutions; see Theorem 1.3 below. The key ingredient to prove this is the quantitative control on the behavior of almost critical points of the length functional on the set $\mathcal{B}_\eta$; see Theorem 1.2. This flow, in turn, will be the fundamental tool that we use to establish quantitative stability of minimizers of (1.1) in Theorem 1.4, as it provides a natural way of deforming a curve in a way that preserves the enclosed area A_Σ while decreasing the length, thereby improving the isoperimetric ratio.

We prove results related to all three of the above problems with optimal exponents (respectively rates) in particular in the case where Σ is a circle and more generally whenever $\Sigma = \partial\Omega$ satisfies a simple non-degeneracy condition concerning the behavior of the length functional on the set of circular arcs

$$\text{Circ}_\eta := \{c \in \mathcal{B}_\eta : c \text{ subarc of a circle intersecting } \Sigma \text{ transversally}\}. \tag{1.5}$$

By the implicit function theorem Circ_η is a smooth 2-dimensional manifold, which we may locally parametrize by the centers $z \in \mathbb{R}^2$ of the defining circles of these arcs c. The restriction of the length functional to Circ_η can hence locally be written as a function $z \mapsto \mathcal{L}_\eta(z) = L(c)$ of two variables, compare also Section 3.2.

In our main results below, we assume either that Σ is a circle, or that the pair (Σ, η) satisfies the following nondegeneracy assumption corresponding to the two-dimensional function $\mathcal{L}_\eta$.

Assumption 1.1. *Given $\eta > 0$ and Σ, we ask that the Hessian $d^2\mathcal{L}_\eta(z^*)$ of $\mathcal{L}_\eta$ is non-degenerate at any critical point z^* of $\mathcal{L}_\eta$, i.e. that the eigenvalues of this symmetric 2×2 matrix are non-zero.*

[1] In higher dimensions, it is not known whether isoperimetric sets are connected, while for the interior relative isoperimetric problem, they are known to be connected [36].

Our first main result is the following quantitative rigidity and Łojasiewicz estimate for critical points.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a convex body with C^2 boundary $\Sigma = \partial\Omega$, fix $\eta > 0$ and assume that either Σ is a circle or the pair (Σ, η) satisfies Assumption 1.1. Then for each $\bar{L} > 0$ and $\bar{\phi} > 0$, there exist constants $C_{0,1} = C_{0,1}(\eta, \Sigma, \bar{L}, \bar{\phi})$ such that the following holds. For any curve $\gamma \in \mathcal{B}_\eta$ with length $L(\gamma) \leq \bar{L}$ and turning angle $|\int_\gamma \kappa_\gamma ds_\gamma| \leq \bar{\phi}$, we may find $c^* \in C^*_\eta$ such that*

$$\|\gamma - c^*\|_{C^1(ds_\gamma)} \leq C_0\big[\|\kappa_\gamma - \bar{\kappa}_\gamma\|_{L^2(ds_\gamma)} + |\alpha_1(\gamma) - \tfrac{\pi}{2}| + |\alpha_2(\gamma) - \tfrac{\pi}{2}|\big] \tag{1.6}$$

and

$$|L(\gamma) - L(c^*)| \leq C_1\big[\|\kappa_\gamma - \bar{\kappa}_\gamma\|_{L^2(ds_\gamma)} + |\alpha_1(\gamma) - \tfrac{\pi}{2}| + |\alpha_2(\gamma) - \tfrac{\pi}{2}|\big]^2 \tag{1.7}$$

where $\alpha_{1,2} \in [0, \pi]$ denote the intersection angles at which γ intersects Σ.

We carry out the proof of this theorem using explicit geometric constructions, which in particular avoid any use of compactness arguments. As such, the constants $C_{0,1}$ appearing in the above theorem are computable in terms of basic geometric and analytic quantities associated to problem.[2] Here and in the following we use the convention that the $C^1(ds_\gamma)$ distance $\|\gamma - \tilde{\gamma}\|_{C^1(ds_\gamma)}$ of a curve $\tilde{\gamma}$ to a given curve γ is computed using the reparametrisation of $\tilde{\gamma}$ over the interval $[0, L(\gamma)]$ with constant speed. The expression

$$\varepsilon(\gamma) := \|\kappa_\gamma - \bar{\kappa}_\gamma\|_{L^2(ds_\gamma)} + |\alpha_1(\gamma) - \tfrac{\pi}{2}| + |\alpha_2(\gamma) - \tfrac{\pi}{2}| \tag{1.8}$$

appearing in the above result is equivalent to the norm of the gradient of L on $\mathcal{B}_\eta$, as the first variation of the length along area preserving vector fields is given by (1.3) above and as $|\langle X(x_i), \tau_\Sigma(x_i)\rangle| = |X(x_i)||\cos(\alpha_i)|$ is bounded from above and below by a multiple of $|X(x_i)||\alpha_i - \frac{\pi}{2}|$. Thus (1.6) is a quantitative form of the classification of critical points in (1.4).

Quantitative rigidity (or "quantitative Alexandrov") estimates for isoperimetric problems along the lines of (1.6) have been investigated intensively over the past decade—see, e.g, [10, 20, 21, 19, 18, 31]—motivated in part by applications to flows [20, 21, 19]. For higher dimensional isoperimetric problems, quantitative ridigity theorems must be formulated to account for the possibility of bubbling; this behavior is precluded in the present context, essentially because controlling $\varepsilon(\gamma)$ amounts to controlling the second fundamental form in a super-critical norm. The one-dimensional nature of the problem allows for a hands-on proof of (1.6): we associate to γ an explicit, quantitatively close circular arc $c \in \mathrm{Circ}_\eta$, and then prove (1.6) for curves in Circ_η by using Assumption 1.1 or, in the case that Σ is a circle, direct planar geometric arguments.

The second Łojasiewicz estimate (1.7) will be obtained from (1.6) using a first variation argument and will be the key tool in the proof of our second main result, which establishes exponential convergence to equilibrium of global solutions to the free boundary area-preserving curve shortening flow. We recall that, starting from the seminal work of Simon [35], Łojasiewicz estimates have been used as a powerful tool in the analysis of both asymptotics and singularities for gradient flows in myriad settings.

Theorem 1.3. *Let Ω be convex body with C^2 boundary $\Sigma = \partial\Omega$, fix $\eta, \bar{L}, \bar{\phi} > 0$ and assume that either Σ is a circle or the pair (Σ, η) satisfies Assumption 1.1. Let $\gamma : [a_1, a_2] \times [0, \infty) \to \mathbb{R}^2$ be a global-in-time solution to the flow*

$$\begin{cases} \partial_t \gamma_t = (\kappa_{\gamma_t} - \bar{\kappa}_{\gamma_t})\nu_{\gamma_t} & \text{on } [a_1, a_2] \times [0, \infty), \\ \gamma_t(a_1),\ \gamma_t(a_2) \in \Sigma & \text{on } [0, \infty), \\ \tau_{\gamma_t}(a_1) = -\nu_\Sigma(\gamma_t(a_1)),\ \ \tau_{\gamma_t}(a_2) = \nu_\Sigma(\gamma_t(a_2)) & \text{on } [0, \infty). \end{cases} \tag{1.9}$$

Assume that the turning angle remains bounded by $|\int \kappa_{\gamma_t} ds_{\gamma_t}| \leq \bar{\phi}$ and that γ_t intersects Ω only at the endpoints $x_{1,2}(\gamma_t) \in \Sigma$ for each t. Then there is a unique arc $c^* \in C^*_\eta$ such that γ_t converges smoothly exponentially to c^*. More precisely, for every $k \in \mathbb{N}$, there exists $C_k = C_k(\gamma_0, \Sigma, \eta, \bar{\phi})$ and $c_k = c_k(\gamma_0, \Sigma, \eta, \bar{\phi}) > 0$ so that

$$\|\tilde{\gamma}_t - \tilde{c}^*\|_{C^k([0,1])} \leq C_k \exp(-c_k t) \quad \text{for all } t \in [1, \infty) \tag{1.10}$$

for the constant speed parametrizations $\tilde{\gamma}_t$ and $\tilde{c}^*$ of γ_t and c^* on $[0, 1]$. Furthermore, there is a constant $C = C(\Sigma, \eta, \bar{L}, \bar{\phi})$ such that if $L(\gamma_0) \leq \bar{L}$, then the total L^2-distance traveled by both the original flow and the reparametrized flow is bounded by

$$\int_0^\infty \|\partial_t \gamma_t\|_{L^2(ds_{\gamma_t})} dt + \int_0^\infty \|\partial_t \tilde{\gamma}_t\|_{L^2([0,1])} dt \leq C(L(\gamma_0) - L(c^*))^{1/2}. \tag{1.11}$$

[2] An inspection of the proof shows that the constants can be bounded explicitly in terms of η, the C^2 norm of the arclength parametrization σ of Σ, the modulus of continuity of σ'', and (for Σ not a circle) the spectral gap around 0 of the 2×2 matrix $d^2\mathcal{L}_\eta(z^*)$ at critical points z^*.

We always consider Σ with a positive orientation, so in particular, ν_Σ is the inner unit normal to Ω in (1.9) above. The results of [23] give sufficient conditions on the initial data to ensure the assumptions of Theorem 1.3 hold, which will be essential in its application to the next result.

Our final main result is a sharp quantitative stability estimate for minimizers of the relative isoperimetric problem (1.1). In the statement, $\kappa_{max}(\Sigma) := \max_\Sigma |\kappa_\Sigma|$ and $\bar{\mathfrak{c}} \approx .04$ is an explicit universal constant whose precise value is given in (2.5).

Theorem 1.4. *Let Ω be convex body with C^2 boundary $\Sigma = \partial\Omega$. Fix $\eta \in (0, \bar{\mathfrak{c}}\kappa_{max}(\Sigma)^{-2})$, and assume that either Σ is a circle or the pair (Σ, η) satisfies Assumption 1.1. There is an explicitly computable constant $c = c(\Sigma, \eta)$ such that*

$$P(E; \mathbb{R}^2 \setminus \Omega) - I_\Omega(\eta) \geq c \inf_{E_* \in \mathcal{M}_\eta^\Omega} |E \Delta E_*|^2 \tag{1.12}$$

for any set of finite perimeter E in $\mathbb{R}^2 \setminus \Omega$ with $|E| = \eta$. If $\partial E \setminus \Omega$ is a rectifiable curve, then additionally

$$P(E; \mathbb{R}^2 \setminus \Omega)^2 - I_\Omega(\eta)^2 \geq c \inf_{E_* \in \mathcal{M}_\eta^\Omega} d_H(\partial E, \partial E_*)^2 \,. \tag{1.13}$$

The second statement (1.13) is a free-boundary counterpart of the classical theorem of Bonnesen [6]. Simple examples given by removing or adding a small ball from a minimizer show that (1.13) is false without the additional assumption on E. By [1], any simple set of finite perimeter E in $\mathbb{R}^2$, i.e. one such that E and $\mathbb{R}^2 \setminus E$ are indecomposable, is bounded by a rectifiable Jordan curve up to modification of E on a Lebesgue-null set. The assumption $\eta \in (0, \bar{\mathfrak{c}}\kappa_{max}(\Sigma)^{-2})$ is an artifact of the proof; as described below, this is used to ensure the global existence of a well-behaved solution to the gradient flow.

Let us sketch how the we use the gradient flow to establish (1.12). (The proof of (1.13) is similar.) Let E be a set as in Theorem 1.4, and assume without loss of generality that $P(E; \mathbb{R}^2 \setminus \Omega)$ lies below the energy level of any non-minimizing critical point in (1.4). (Assumption 1.1 guarantees that the critical values of L on $\mathcal{B}_\eta$ are discrete, while if Σ is a circle, minimizers are the only critical points.) In a by-hand reduction procedure in Proposition 5.2, we associate to E a set F with $|F \setminus \Omega| = \eta$ whose relative boundary $\partial F \setminus \Omega$ is a convex $C^{2,\alpha}$ curve γ_F meeting Σ orthogonally such that

$$\delta_\eta(F) + |E \Delta F|^2 \leq C\, \delta_\eta(E)\,.$$

Here we let $\delta_\eta(E) = P(E; \mathbb{R}^2 \setminus \Omega) - I_\Omega(\eta)$ denote the isoperimetric deficit appearing on the right-hand side of (1.12). It thus suffices to show that there is an isoperimetric set $E_* \in \mathcal{M}_\eta^\Omega$ such that

$$L(\gamma_F) - I_\Omega(\eta) = \delta_\eta(F) \geq c|F \Delta E_*|^2. \tag{1.14}$$

To this end, we evolve γ_F by the gradient flow above. The convexity of F and assumed bound $\eta < \bar{\mathfrak{c}}\kappa_{max}(\Sigma)^{-2}$ on the enclosed area allow us to apply results of the first-named author [23], which guarantee that the flow exists, remains embedded, and satisfies the assumptions of Theorems 1.3 for all $t \in [0, \infty)$. By Theorem 1.3 and the monotonicity of length under the gradient flow, γ_t converges exponentially to an arc $c^* \in \mathcal{C}_\eta^*$ that is the relative boundary $\partial E_* \setminus \Omega$ of a minimizer $E_* \in \mathcal{M}_\eta^\Omega$. In particular, $L(c^*) = I_\Omega(\eta)$. Finally, the fundamental theorem of calculus together with some geometric estimates show that the left-hand side of the displacement estimate (1.11) bounds $|F \Delta E_*|$ above. Thus (1.14) follows from (1.11).

The past two decades have seen tremendous advances in the theory of quantitative stability for isoperimetric inequalities, see e.g. [16, 13, 9, 15, 27, 2], including for the relative (and capillary) isoperimetric problems on half planes and cones [12, 26, 22, 30, 7]. The story for the classical planar isoperimetric problem starts nearly a century earlier with Bonneson's theorem [6] (and the earlier [4] for convex curves); see [29] for a survey of these early developments.

Various proofs of Bonneson's inequality are known, using tools such as Steiner formulas for convex sets [29], Fourier analysis [17, 14], an improved Wirtinger inequality [3], or integral geometry [34]. None of these proofs admit direct generalization to yield alternative proofs of Theorem 1.4. An alternative approach to prove Theorem 1.4 would be to use a selection principle argument [9], where the spectral analysis is carried out in carefully chosen coordinates. While this approach would likely allow for the removal of the assumption $\eta < \bar{\mathfrak{c}}\kappa_{max}(\Sigma)^{-2}$, its use of a compactness argument would prevent one from obtaining explicit constants. In contrast, the constants in all of our main results come from elementary geometric arguments, making them explicitly computable, and our approach highlights the intertwined nature of these three core problems of the quantitative analysis of PDEs.

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2. Preliminaries

2.1. **The isoperimetric profile.** Throughout the paper, $\Omega \subset \mathbb{R}^2$ will denote a convex body, i.e. a compact convex set with nonempty interior, whose boundary $\Sigma = \partial\Omega$ is of class C^2. We let $\kappa_{max}(\Sigma) > 0$ be the maximum curvature of Σ.

For a set of finite perimeter E in $\mathbb{R}^2$, we denote by $P(E)$ its perimeter, $P(E; A)$ its relative perimeter in an open set A, and $\partial^* E$ its reduced boundary, so that $P(E; A) = \mathcal{H}^1(\partial^* E; A)$; see [25, Chapter 12] for basics on sets of finite perimeter. We will tacitly choose a representative of a set of finite perimeter E with $\overline{\partial^* E} = \partial E$; see [25, Prop. 12.19]. In two dimensions, sets of finite perimeter have a simple structure thanks to [1]; this structure will be recalled and utilized in Section 5.2.

The isoperimetric profile is continuous and non-decreasing and satisfies the upper and lower bounds

$$(2\pi)^{\frac{1}{2}}\,\eta^{\frac{1}{2}} \le I_\Omega(\eta) \le 2\pi^{\frac{1}{2}}\,\eta^{\frac{1}{2}}. \tag{2.1}$$

The left-hand side is precisely the isoperimetric profile I_H of the half-plane, and the first inequality follows directly by using the convexity of Ω and the fact that $\eta \mapsto I_H(\eta)$ is increasing.[3] The upper bound in (2.1) comes from choosing as a competitor in (1.1) a ball that is disjoint from Ω.

We recall that the set of circular arcs that intersect Σ orthogonally and enclose a set of area η is denoted by C^*_η. It follows from convexity of Σ that the center z_c of any $c \in C^*_\eta$ is in $\mathbb{R}^2 \setminus \Omega^\circ$. So, for $c \in C^*_\eta$ with positive orientation,

$$\phi_{\text{turn}}(c) \in [\pi, 2\pi), \qquad \pi r_c^2 \ge \eta \ge \tfrac{1}{2}\pi r_c^2, \qquad \text{and} \qquad \kappa_c \in [(\pi/2\eta)^{1/2}, (\pi/\eta)^{1/2}], \tag{2.2}$$

where r_c and κ_c respectively are the radius and curvature of the circle containing c, and $\phi_{\text{turn}}(c) = \int_c \kappa_c ds_c$ is the turning angle of c. Given $r > 0$, we let $\Theta_\Sigma(r)$ be the maximal turning angle of a circular arc of radius r in $\mathbb{R}^2 \setminus \Omega$ which meets Σ orthogonally at the endpoints and note that a trigonometric exercise shows that

$$\Theta_\Sigma(r) \le 2\pi - 2\arctan(1/(r\kappa_{max}(\Sigma)))\,. \tag{2.3}$$

We can hence get a slightly improved upper bound for $I_\Omega(\eta)$ as follows. Let $c^* \in C^*_\eta$ be a circular arc which bounds a minimizer $E_* \in \mathcal{M}^*_\eta$. Then E_* contains a circular sector of $B_{r_{c^*}}(z_{c^*})$ of angle $\phi = \phi_{\text{turn}}(c^*)$. In particular, $\frac{1}{2} r_{c^*}^2 \phi \le \eta$, i.e. $r_{c^*} \le (2\eta/\phi)^{1/2}$. Thus $I_\Omega(\eta) \le (2\theta\eta)^{1/2}$, and so by (2.3),

$$I_\Omega(\eta) \le 2\pi^{\frac{1}{2}}\eta^{\frac{1}{2}}\left(1 - \pi^{-1}\arctan\left(\left(\tfrac{\pi}{2\eta}\right)^{\frac{1}{2}}\kappa_{max}(\Sigma)^{-1}\right)\right)^{\frac{1}{2}}\,. \tag{2.4}$$

We fix the universal constant

$$\bar{\mathfrak{c}} = \frac{4}{25\pi}\arcsin^2\left(\frac{1}{4\pi}\right) \approx .0415. \tag{2.5}$$

In Theorem 1.4, we assume $\eta < \bar{\mathfrak{c}}\kappa_{max}(\Sigma)^{-2}$, which guarantees the following upper bound for the isoperimetric profile.

Lemma 2.1. *Suppose $\eta < \bar{\mathfrak{c}}\kappa_{max}(\Sigma)^{-2}$. There is an explicitly computable constant $\delta_0 = \delta_0(\eta, \kappa_{max}(\Sigma)) > 0$ such that for any $\delta \in [0, \delta_0]$,*

$$I_\Omega(\eta) + \delta < \frac{4}{5\kappa_{max}(\Sigma)}\arcsin\left(\frac{\eta}{(I_\Omega(\eta) + \delta)^2}\right). \tag{2.6}$$

Proof. The estimate (2.6) with $\delta = 0$ follows directly follows from the upper bound in (2.1) and the definition of $\bar{\mathfrak{c}}$. Since the inequality (2.6) with $\delta = 0$ is strict and both sides are continuous functions of δ, it is clear the estimate holds up to some δ_0. This δ_0 can be made explicit with the claimed dependence by using (2.4). ◻

Remark 2.2. The proof of Theorem 1.4 actually goes through for any η small enough such that (2.6) holds with $\delta = 0$.

[3] The analogous comparison theorem holds in higher dimensions but is no longer trivial to prove, see [8].

Remark 2.3. A basic geometric argument shows that $\kappa_{max}(\Sigma)^{-1} \le d_\Sigma/2$, where

$$d_\Sigma := \min\{|x - x'| : x, x' \in \Sigma, \tau_\Sigma(x) = -\tau_\Sigma(x')\} \tag{2.7}$$

is the *width* of Σ, that is, the minimal distance of two parallel lines touching Σ. Hence the right-hand side of (2.6) is bounded by $\frac{2}{5}\arcsin(1/(2\pi))d_\Sigma$, and we can in particular use that $I_\Omega(\eta) + \delta \le d_\Sigma/2$ whenever (2.6) holds for a given (η, δ).

The isoperimetric profile is differentiable outside a countable set and is left- and right-differentiable everywhere. A classical argument shows that the left-and right-derivatives of I_Ω at η are bounded above by the (constant) curvature κ of any minimizer of $I_\Omega(\eta)$, and at differentiability points, $I_\Omega{}'(\eta) = \kappa$. So, I_Ω is absolutely continuous on compact subsets of $(0, \infty)$ and thus by the fundamental theorem of calculus, for any $0 < \eta_1 < \eta_2$,

$$\left(\frac{\pi}{2\eta_2}\right)^{\frac{1}{2}}(\eta_2 - \eta_1) \le I_\Omega(\eta_2) - I_\Omega(\eta_1) \le \left(\frac{\pi}{\eta_1}\right)^{\frac{1}{2}}(\eta_2 - \eta_1). \tag{2.8}$$

2.2. **Notation and basic properties of oriented curves.** For a regular oriented C^2 curve γ from $x_1(\gamma)$ to $x_2(\gamma)$ we let $\tau_\gamma = \frac{\gamma'}{|\gamma'|}$ be the unit tangent to the curve, and let $\nu_\gamma = J\tau_\gamma$, where J denotes a counterclockwise rotation by $+\pi/2$. We let $L(\gamma)$ be the length of γ and s_γ the arclength parameter. We denote by $\kappa_\gamma = \frac{\langle\gamma'', \nu_\gamma\rangle}{|\gamma'|^2}$ the signed curvature with respect to $\nu_\gamma = J\tau_\gamma$. We also set $\bar\kappa_\gamma := \frac{1}{L(\gamma)}\int_\gamma \kappa_\gamma\, ds_\gamma$ and denote by $\phi_{\text{turn}}(\gamma) := \int_\gamma \kappa_\gamma\, ds_\gamma$ the total turning angle of γ. We recall that $\kappa_\gamma, \bar\kappa_\gamma, \phi_{\text{turn}}(\gamma), \tau_\gamma$, and ν_γ are of course independent of the choice of parametrization, but that they flip a sign when the orientation is reversed.

In the following we use the convention that Σ is parametrized with positive orientation (i.e. counterclockwise). With this convention $\nu_\Sigma(p)$ is the inner unit normal and κ_Σ is nonnegative. Throughout the paper we consider curves $\gamma \in \mathcal{B}$ for

$$\mathcal{B} := \Big\{\gamma : \gamma \text{ is an oriented } H^2 \text{ curve for which } \gamma \cap \Omega = \{x_1(\gamma), x_2(\gamma)\}\Big\}. \tag{2.9}$$

Definition 2.4. *Given a curve $\gamma \in \mathcal{B}$, we concatenate γ with the (unique) oriented sub-arc σ_γ of Σ for which the oriented immersed closed curve $\gamma + \sigma_\gamma$ is contractible in $\mathbb{R}^2 \setminus \text{int}(\Omega)$ and define the relative area $A_\Sigma(\gamma)$ enclosed by γ as*

$$A_\Sigma(\gamma) := A(\gamma + \sigma_\gamma) := -\frac{1}{2}\Big(\int \gamma \cdot \nu_\gamma\, ds_\gamma + \int \sigma_\gamma \cdot \nu_{\sigma_\gamma}\, ds_{\sigma_\gamma}\Big). \tag{2.10}$$

An explicit way to construct σ_γ is by taking the projection $\tilde\sigma_\gamma = \pi_\Sigma \circ \gamma$ of γ onto Σ, and defining σ_γ as the (unique) oriented arc of Σ that is homotopic to $-\tilde\sigma_\gamma$ (with fixed endpoints). We note that σ_γ can traverse Σ multiple times, may have the same or opposite orientation as the full curve Σ, and can even just be a point. If γ is embedded and oriented so that $\gamma + \sigma_\gamma$ has positive orientation, then $A_\Sigma(\gamma)$ coincides with the area of the region bounded by γ and Σ.

Remark 2.5. As σ_γ is determined uniquely by the above topological condition, this construction in particular ensures that σ_{γ_t} and $\mathcal{A}_\Sigma(\gamma_t)$ vary continuously along any continuous family of curves $\gamma_t \in \mathcal{B}$, and that $t \mapsto A_\Sigma(\gamma_t)$ is differentiable whenever $t \mapsto \gamma_t \in \mathcal{B}$ is differentiable with

$$\frac{d}{dt}A_\Sigma(\gamma_t) = -\int_\gamma X_t \cdot \nu_{\gamma_t} ds_{\gamma_t} = -\int_{\gamma_t} X_t \cdot J\tau_{\gamma_t}\, ds_{\gamma_t} \qquad \text{where } X_t := \frac{d}{dt}\gamma_t\,. \tag{2.11}$$

We note that the variation of σ_{γ_t} does not appear in the above formula as $\partial_t\sigma_{\gamma_t}$ is tangential.

Since a change in orientation of γ results in a change of sign of $A_\Sigma(\gamma)$, we define for each given $\eta > 0$ the set of admissible curves for the relative isoperimetric problem as

$$\mathcal{B}_\eta := \{\gamma \in \mathcal{B} : |A_\Sigma(\gamma)| = \eta\}.$$

It is immediate from the definition of A_Σ that $L(\gamma) \ge I_\Omega(\eta)$ whenever $\gamma \in \mathcal{B}_\eta$ is embedded and a short argument, which we include in Appendix A, ensures that we can always bound

$$L(\gamma) \ge I_\Omega(\eta) \qquad \text{for every } \gamma \in \mathcal{B}_\eta \text{ and every } \eta > 0 \tag{2.12}$$

and hence in particular $L(\gamma) \ge (2\pi)^{\frac{1}{2}}\eta^{\frac{1}{2}}$ by (2.1).

We denote by $\alpha_i(\gamma) \in [0, \pi]$ the angles at which a given curve $\gamma \in \mathcal{B}$ intersect Σ, characterized by

$$\tau_\Sigma(x_1) = R_{\alpha_1}\tau_\gamma(x_1) \text{ and } \tau_\Sigma(x_2) = R_{-\alpha_2}\tau_\gamma(x_2)\,. \tag{2.13}$$

Here and in the following R_ψ denotes the rotation by angle ψ in positive, i.e. counter-clockwise direction.

In what follows will often consider curves $\gamma \in \mathcal{B}_\eta$ whose length $L(\gamma)$ and turning angle $|\phi_{\text{turn}}(\gamma)|$ are bounded by given numbers $\bar{L}$ and $\bar{\phi}$ and whose intersection angles α_i with Σ defined in (2.13) satisfy $|\alpha_i(\gamma) - \frac{\pi}{2}| \leq \bar{\beta}$ for some $\bar{\beta} < \frac{\pi}{2}$. As Σ is convex, this ensures that $|\phi_{\text{turn}}(\gamma)| \geq \pi - 2\bar{\beta}$ and hence, recalling (2.1), that

$$\frac{\pi - 2\bar{\beta}}{\bar{L}} \leq |\bar{\kappa}| \leq \frac{\bar{\phi}}{(2\pi)^{\frac{1}{2}}\, \eta^{\frac{1}{2}}}. \tag{2.14}$$

In the following we also use that the parametrisation by arclength of any curve $\gamma : [0, L(\gamma)] \to \mathbb{R}^2$ can be expressed as

$$\gamma(p) = \gamma(0) + \int_0^p (\cos\theta_\gamma(q), \sin\theta_\gamma(q))\, dq \qquad \text{for} \qquad \theta_\gamma(q) = \theta_0 + \int_0^p \kappa_\gamma(s)\, ds_\gamma\,, \tag{2.15}$$

where $\theta_0(\gamma)$ denotes the angle formed by the tangent vector $\tau_\gamma(0)$ at the starting point and the e_1 axis.

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Throughout the section, Ω denotes a convex body with boundary Σ of class C^2.

For curves γ as in Theorem 1.2 whose deficit $\varepsilon(\gamma)$ defined in (1.8) is bounded from below by a fixed constant $\varepsilon_0 > 0$, the theorem holds trivially by choosing $\gamma^* \in C^*_\eta$ as a global minimizer of L in $\mathcal{B}_\eta$. Thus, we need only to consider curves $\gamma \in \mathcal{B}_\eta$ for which $\varepsilon(\gamma)$ lies below an explicit threshold. The proof in this case has three main steps: We first show that for any $\eta > 0$ and $\gamma \in \mathcal{B}_\eta$, there exists c in the collection Circ_η of circular arcs belonging to $\mathcal{B}_\eta$ (recall (1.5)) whose distance to γ in C^1 is controlled by $C\varepsilon_\kappa(\gamma)$, where we set

$$\varepsilon_\kappa(\gamma) := \|\kappa_\gamma - \bar{\kappa}_\gamma\|_{L^2(ds_\gamma)}, \tag{3.1}$$

compare Lemma 3.1 and Proposition 3.4. This allows us to reduce the proof of the claimed distance Łojasiewicz estimate (1.6) to the analysis of circular arcs, which is carried out in Section 3.2 and which, unlike the other parts of the proof, exploits the non-degeneracy condition on the curve Σ if Σ is not a circle. Combined, this will allow us to prove the claimed bound (1.6) on the distance of γ to the nearest critical point γ^* with a short argument that is carried out at the beginning of Section 3.3 before we show how this distance Łojasiewicz estimate yields the claimed estimate (1.7) on $|L(\gamma) - L(\gamma^*)|$.

3.1. Reduction to circular arcs.

We first prove that curves for which $\varepsilon(\gamma)$ is small are C^1-close to a circular arc c_γ whose enclosed area will be close, but not necessarily equal, to η. We note that an argument in a similar spirit was used also in [20] for closed curves. In a second step, we modify this initial circular arc in a way that the area constraint is satisfied. In what follows, we will always take $\varepsilon(\gamma)$ small enough such that, in particular,

$$|\alpha_i(\gamma) - \tfrac{\pi}{2}| \leq \tfrac{\pi}{12}, \quad i = 1, 2, \tag{3.2}$$

for the intersection angles α_i defined in (2.13). For the first step we begin with the following lemma.

Lemma 3.1. *For any $\eta, \bar{L}, \bar{\phi} > 0$, there are explicit constants $\varepsilon_1 = \varepsilon_1(\Sigma, \eta, \bar{L}, \bar{\phi}) > 0$ and $C_2 = C_2(\Sigma, \eta, \bar{L}, \bar{\phi}) > 0$ such that the following holds. Let $\gamma \in \mathcal{B}_\eta$ be any curve with $L(\gamma) \leq \bar{L}$, $|\phi_{turn}(\gamma)| \leq \bar{\phi}$, and $\varepsilon(\gamma) \leq \varepsilon_1$. Then γ is embedded. Moreover, letting $c_\gamma \in \mathcal{B}$ be the circular arc with radius $\bar{r} := |\bar{\kappa}_\gamma|^{-1}$ emanating from $\gamma(0)$ which has the same intersection angle $\alpha_1(c_\gamma) = \alpha_1(\gamma)$ at this initial point and turns counterclockwise if $\bar{\kappa}_\gamma > 0$ and clockwise if $\bar{\kappa}_\gamma < 0$, we have*

$$\|\gamma - c_\gamma\|_{C^1(ds_\gamma)} \leq C_2 \|\kappa_\gamma - \bar{\kappa}_\gamma\|_{L^2(ds_\gamma)}. \tag{3.3}$$

Remark 3.2. As mentioned above, we will take ε_1 small enough so that any for any curve γ as in Lemma 3.1, the intersection angles of γ with Σ satisfy (3.2). Then by (2.14), $|\bar{\kappa}_\gamma| \geq \frac{5\pi}{6\bar{L}}$ and in particular, $\bar{\kappa}_\gamma \neq 0$ so $c_\gamma \in \mathcal{B}$ is well defined.

For the proof of Lemma 3.1 we will use the following elementary geometric fact.

Lemma 3.3. *Let $\hat{c}$ be a circle with radius R and let c_1 be an oriented circle with radius r through a point x_1 in the exterior of $\hat{c}$. Suppose that there exists a point $x_0 \in \hat{c}$ so that $|x_1 - x_0| \leq \frac{\pi}{24}\min(r, R)$ and $|\angle(\tau_{c_1}(x_1), \nu_{\hat{c}}(x_0))| \leq \frac{\pi}{12}$, where $\nu_{\hat{c}}$ is the inward unit normal of $\hat{c}$. Then c_1 intersects $\hat{c}$ and the length of the circular arc c from x_1 to the first point x_2 where c_1 (with the given orientation) intersects $\hat{c}$ is bounded by $L(c) \leq 2|x_1 - x_0|$.*

We note that the analogous statement also holds if $\hat{c}$ is replaced by a straight line $\mathcal{T}$ and that the proof of this variation of the lemma can be either obtained in the limit $R \to \infty$ or by a simplified version of the proof below.

Proof of Lemma 3.3. As the claim is invariant under rescaling, translation and rotation we can assume without loss of generality that $|x_1 - x_0| = 1$, that $\hat{c} = \partial B_R(0)$ and that $x_0 = (-R, 0)$ and hence $\nu_{\hat{c}}(x_0) = e_1$. We parametrize $c_1 : \mathbb{R} \to \mathbb{R}^2$ by arclength so that $c_1(0) = x_1$ and denote by $\theta_{c_1}(p)$ the angle between the tangent of τ_{c_1} and e_1, compare (2.15). If c_1 intersects $\hat{c}$ we set $p^* := \inf\{p > 0 : c_1(p) \in \hat{c}\}$, so that p^* is the parameter of the first intersection point, while in the case where $c_1 \cap \hat{c} = \emptyset$ we simply let $p^* = \infty$. We note that this choice of p^* ensures that $|c(p)| \geq R$ for all $p \in [0, p^*]$ and that the claim of the lemma follows provided we show that $p^* < 2$.

To prove this we first note that the assumptions of the lemma (and the above normalisation) ensure that $|\theta_{c_1}(0)| \leq \frac{\pi}{12}$ as well as that $r \geq \frac{24}{\pi}$ and hence that $|\kappa_{c_1}| \leq \frac{\pi}{24}$. This ensures that $|\theta_{c_1}(p)| \leq 2|\kappa_{c_1}| + |\theta_{c_1}(0)| = 2r^{-1} + \frac{\pi}{12} \leq \frac{\pi}{6}$ for all $p \in [0, 2]$.

To obtain a similar estimate for the angle $\beta(p)$ between $-\frac{c_1(p)}{|c_1(p)|}$ and e_1 we recall that the initial point x_1 has distance at least R from the origin and e_2 coordinate no more than $|x_1 - x_0| \leq 1$, hence allowing us to bound $|\beta(0)| \leq |\tan(\beta(0))| \leq R^{-1} \leq \frac{\pi}{24}$. As $|c_1(p)| \geq R$ for all $p \in [0, p^*]$ we can bound the change in β by $|\beta'(p)| = \frac{1}{|c_1(p)|}|\Pi_{c_1(p)}(c_1'(p))| \leq \frac{1}{|c_1(p)|} \leq R^{-1} \leq \frac{\pi}{24}$ for all such p, where we let $\Pi_{c_1(p)}$ denote the projection onto $T_{c_1(p)}\partial B_{|c_1(p)|}$. We hence conclude that $|\beta(p)| \leq |\beta(0)| + \frac{\pi}{24}p \leq \frac{\pi}{24} + \frac{\pi}{12} < \frac{\pi}{6}$ for all $p \in [0, \min(p^*, 2)]$.

Combined this ensures that the angle between $-\frac{c_1(p)}{|c_1(p)|}$ and $\tau_{c_1}(p)$ remains bounded by $|\theta_{c_1}(p)| + |\beta(p)| < \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3}$ on $[0, \min(p^*, 2)]$, which ensures that on this interval

$$-|c_1(p)|' = \langle \tau_{c_1}(p), -\frac{c_1(p)}{|c_1(p)|} \rangle > \cos(\pi/3) = \tfrac{1}{2}.$$

As $-|c_1(p)| \leq -R = -|x_0|$ on $[0, p^*]$ we can integrate this bound to deduce that

$$\tfrac{1}{2}\min(p^*, 2) < -\int_0^{\min\{p^*,2\}} |c_1(p)|' dp \leq |x_1| - R = |x_1| - |x_0| \leq |x_1 - x_0| = 1,$$

and hence indeed that $p^* < 2$ as required. □

Based on this lemma we can now complete the proof of Lemma 3.1.

Proof of Lemma 3.1. Let $d_0 = d_0(\Sigma, \eta, \bar{\phi}) > 0$ be chosen so that any (full) circle with radius $r \geq (2\pi)^{\frac{1}{2}}\eta^{\frac{1}{2}}\bar{\phi}^{-1}$ which intersects Σ at an angle α_1 with $|\alpha_1 - \frac{\pi}{2}| \leq \frac{\pi}{12}$ contains a point in Ω whose distance to Σ is greater than $2d_0$. Then set

$$\varepsilon_1' := 2^{1/2}\bar{L}^{-3/2}\min\Big(\frac{\pi}{24}\kappa_{max}(\Sigma)^{-1}, \frac{\pi}{24}\frac{\eta^{\frac{1}{2}}}{(2\pi)^{1/2}}, \frac{d_0}{4}, \frac{\bar{L}}{4}\Big). \tag{3.4}$$

and $\varepsilon_1 = \min\{\varepsilon_1', \pi/12\}$. As the claim is invariant under change of orientation, it suffices to consider the case where γ has $\bar{\kappa}_\gamma > 0$ and hence $\phi_{\text{turn}}(\gamma) > 0$. We parametrize γ by arclength on the interval $[0, L]$, $L := L(\gamma)$, and recall that this parametrization can be expressed in terms of $\theta_\gamma(p) = \theta_0 + \int_0^p \kappa_\gamma(s)ds_\gamma$ as described in (2.15). Setting $\theta_1(p) = \theta_0 + \bar{\kappa}_\gamma p$ for $p \in [0, \infty)$, we get an analoguous parametrization $\gamma_1 : \mathbb{R} \to \mathbb{R}^2$ by arclength of the circle that contains the circular arc c_γ of radius $\bar{r} := \bar{\kappa}_\gamma^{-1}$.

The fundamental theorem of calculus and the fact that $\theta_\gamma(L) = \theta_1(L)$ ensure that $|\theta_\gamma(p) - \theta_1(p)| \leq (L/2)^{1/2}\varepsilon_\kappa(\gamma)$ for all $p \in [0, L]$, which, when inserted into (2.15), immediately imply that

$$|\gamma'(p) - \gamma_1'(p)| \leq 2^{-1/2}L^{1/2}\varepsilon_\kappa(\gamma) \qquad \text{and} \qquad |\gamma(p) - \gamma_1(p)| \leq 2^{-1/2}L^{3/2}\varepsilon_\kappa(\gamma). \tag{3.5}$$

As the angle between two unit vectors w_1, w_2 is given by $\angle(w_1, w_2) = 2\arcsin(\frac{1}{2}|w_2 - w_1|)$ and as (3.5) ensures that $|\tau_\gamma(p) - \tau_{\gamma_1}(p)| \leq \frac{1}{4} \leq 2\sin(\pi/10)$ we can in particular bound

$$\angle(\tau_\gamma(p), \tau_{\gamma_1}(p)) \leq \frac{\pi}{5} \quad \text{and} \quad |\gamma(p) - \gamma_1(p)| \leq \min(\frac{d_0}{4}, \frac{\pi}{24}\frac{\eta^{\frac{1}{2}}}{(2\pi)^{1/2}}) \quad \text{for all } p \in [0, L]. \tag{3.6}$$

We now want to show that this ensures that γ is embedded. For this we first note that $\phi_{\text{turn}}(\gamma) = \phi_{\text{turn}}(\gamma_1|_{[0,L]}) < 2\pi$, i.e. $L < 2\pi\bar{r}$, since the full circle $\gamma_1|_{[0,2\pi\bar{r}]}$ contains a point $\gamma_1(p_0)$ with $\text{dist}(\gamma_1(p_0), \gamma([0, L])) \geq \text{dist}(\gamma_1(p_0), \mathbb{R}^2 \setminus \Omega) \geq 2d_0$. Combined with (2.1) this also gives an improved upper bound of $\bar{\kappa}_\gamma < (2\pi/\eta)^{\frac{1}{2}}$, and hence an improved lower bound of

$\bar{r} > (\eta/2\pi)^{\frac{1}{2}}$. The second estimate of (3.6) hence in particular ensures that $|\gamma(p) - \gamma_1(p)| < \frac{1}{2}\min(d_0, \bar{r})$ for every p. It is now useful to observe that whenever $0 \le p_1 < p_2 \le L$ are so that $|\gamma_1(p_1) - \gamma_1(p_2)| \le \min(d_0, \bar{r})$ we have

$$\angle(\tau_{\gamma_1}(q_1), \tau_{\gamma_1}(q_2)) \le \angle(\tau_{\gamma_1}(p_1), \tau_{\gamma_1}(p_2)) \quad \text{ for all } q_1, q_2 \in [p_1, p_2].$$

This implication is immediate if $\phi_{\text{turn}}(\gamma_1|_{[0,L]}) \le \pi$. Conversely, if $\phi_{\text{turn}}(\gamma_1|_{[0,L]}) \ge \pi$, the implication holds since in this case we can bound $|\gamma_1(0) - \gamma_1(L)| \ge |\gamma_1(0) - \gamma_1(p_0)| \ge |\gamma(0) - \gamma_1(p_0)| - |\gamma(0) - \gamma_1(0)| > d_0$ which ensures that $\phi_{\text{turn}}(\gamma_1|_{[p_1,p_2]}) < \pi$.

If there were any $0 \le p_1 < p_2 \le L$ for which $\gamma(p_1) = \gamma(p_2)$, then this estimate would be applicable since $|\gamma_1(p_1) - \gamma_1(p_2)| \le |\gamma(p_1) - \gamma_1(p_1)| + |\gamma(p_2) - \gamma_1(p_2)| < \min(d_0, \bar{r})$ by (3.6). At the same time, as γ_1 parametrises a circle with radius $\bar{r}$, this estimate would ensure that $|\tau_{\gamma_1}(p_1) - \tau_{\gamma_1}(p_2)| = |\nu_{\gamma_1}(p_1) - \nu_{\gamma_1}(p_2)| = \bar{r}^{-1}|\gamma_1(p_1) - \gamma_1(p_2)| \le 1$ and hence that $\angle(\tau_{\gamma_1}(p_2), \tau_{\gamma_1}(p_1)) < \pi/2$. We could hence conclude that

$$\angle(\tau_\gamma(q_1), \tau_\gamma(q_2)) < \frac{2\pi}{5} + \frac{\pi}{2} < \pi \text{ for all } q_1, q_2 \in [p_1, p_2]$$

which contradicts the assumption that the curve γ intersects itself at $\gamma(p_1) = \gamma(p_2)$. Hence our choice of ε_1 indeed ensures that γ is embedded.

While the parametrization of c_γ by arclength is $\gamma_1|_{[0,L_1]}$, where $L_1 := L(c_\gamma)$, the constant-speed parametrization of c_γ that is used to compute the C^1 distance in the lemma is given by $\tilde{\gamma}_1(p) := \gamma_1(\frac{L_1}{L}p)$, $p \in [0, L]$. To derive the desired estimate (3.3) from (3.5) it suffices to to prove that

$$|L - L_1| \le 2\,|\gamma(L) - \gamma_1(L)|\,, \tag{3.7}$$

which will allow us to deduce that $|L - L_1| \le 2^{1/2}L^{3/2}\varepsilon_\kappa(\gamma)$.

To prove (3.7) we first note that the tangent to γ_1 at $x_1 := \gamma_1(L)$ agrees with $\gamma'(L)$ since $\theta_\gamma(L) = \theta_1(L)$. Setting $x_0 := \gamma(L)$ we can hence use the assumed bound (3.2) on the intersection angles to conclude that the angle β between $\gamma_1'(L)$ and the inner normal $\nu_\Sigma(x_0)$ to Σ is so that $|\beta| = |\alpha_2(\gamma) - \pi/2| < \frac{\pi}{12}$. We furthermore note that $\bar{r} \ge \eta^{\frac{1}{2}}/(2\pi)^{\frac{1}{2}}$ using (2.14) and $\phi_{\text{turn}}(\gamma) = \phi_{\text{turn}}(\gamma_1) < 2\pi$. We also recall that we have already shown that $|x_0 - x_1| \le 2^{-1/2}L^{3/2}\varepsilon_1$ in (3.5). Our choice of ε_1 hence ensures that $\bar{r}$ and $\hat{r}_\Sigma := (\max_\Sigma \kappa_\Sigma)^{-1}$ are so that $|x_0 - x_1| \le \frac{\pi}{24}\min(\bar{r}, \hat{r}_\Sigma)$. To prove (3.7) we can hence apply Lemma 3.3 for this choice of $x_0 = \gamma(L)$ and $x_1 := \gamma_1(L)$ as follows:

If $L_1 \le L$, we choose $\hat{c}$ as the tangent $\mathcal{T}^\Sigma_{x_0} := \{x_0 + q\tau_\Sigma(x_0), q \in \mathbb{R}\}$ through x_0 to Σ and choose c_1 as the circle which contains γ_1 but is parametrised with opposite orientation. Lemma 3.3 then ensures that c_1 intersects $\mathcal{T}^\Sigma_{x_0}$ and that $p^* := \inf\{p > 0 : \gamma_1(L - p) \in \mathcal{T}^\Sigma_{x_0}\}$ is bounded by $p^* \le 2|x_0 - x_1| = 2|\gamma(L) - \gamma_1(L)|$. We then observe that $\gamma_1|_{(L_1,L]}$ cannot intersect $\mathcal{T}^\Sigma_{x_0}$ as this would force the total turning angle of γ_1 on $[0, L]$ to be at least 2π, which is excluded by our choice of ε_0. Thus we must have $L - p^* \le L_1$, allowing us to deduce the desired bound of $|L_1 - L| \le p^* \le 2|\gamma(L) - \gamma_1(L)|$ in this first case where $L > L_1$.

In the case where $L_1 > L$ we argue analogously, but now choose $\hat{c}$ to be the circle of radius $\hat{r}_\Sigma = (\max \kappa_\Sigma)^{-1}$ which touches Σ in $x_0 = \gamma(L)$ from the inside, and which is hence fully contained in Ω, and let c_1 be the circle that contains γ_1 and that has the same orientation as γ_1. Lemma 3.3 then yields that γ_1 intersects $\hat{c}$ and that $p^* := \inf\{p > 0 : \gamma_1(L + p) \in \hat{c}\}$ is bounded by $p^* \le 2|x_0 - x_1|$. As L_1 is the first (positive) time at which γ_1 intersects Σ, we know that $\gamma_1|_{[L,L_1)}$ is contained in the exterior of Ω and hence disjoint from $\hat{c}$. Thus we must have $L + p^* \ge L_1$ which implies the claimed inequality $|L - L_1| \le p^* \le 2|x_0 - x_1|$ if this second case where $L_1 > L$.

The reparametrised curve $\tilde{\gamma}_1(p) := \gamma_1(\frac{L_1}{L}p)$ hence satisfies

$$|\tilde{\gamma}_1(p) - \gamma_1(p)| \le |L - L_1|, \qquad |\tilde{\gamma}_1'(p) - \gamma_1'(p)| \le \left(\tfrac{1}{L} + \bar{\kappa}\right)|L - L_1| \le \tfrac{(1+2\pi)}{L}|L - L_1| \le \tfrac{\eta^{1/2}(1+2\pi)}{(2\pi)^{1/2}}|L - L_1|, \tag{3.8}$$

where we use in the last step that $\phi_{\text{turn}}(\gamma) < 2\pi$ and (2.1). The proof now follows by combining (3.5), (3.8), and (3.7). □

This allows us to prove the following key ingredient for the proof of Theorem 1.2.

Proposition 3.4. *Let Σ and γ be as in Lemma 3.1. Then there exists a circular arc $\tilde{c}_\gamma \in \mathrm{Circ}_\eta$ so that*

$$\|\gamma - \tilde{c}_\gamma\|_{C^1(ds_\gamma)} \le C_3\varepsilon_\kappa(\gamma) \qquad \textit{for} \qquad \varepsilon_\kappa(\gamma) := \|\kappa_\gamma - \bar{\kappa}_\gamma\|_{L^2(ds_\gamma)} \tag{3.9}$$

and a constant $C_3 = C_3(\Sigma, \eta, \bar{L}, \bar{\phi})$. *In particular, we have*

$$|\alpha_i(\gamma) - \alpha_i(\tilde{c}_\gamma)| \le C_4 \varepsilon_\kappa(\gamma), \qquad i = 1, 2 \tag{3.10}$$

for a constant $C_4 = C_4(\Sigma, \eta, \bar{L}, \bar{\phi})$.

Proof of Proposition 3.4. It suffices to consider curves for which $\varepsilon_\kappa(\gamma)$ is less than a fixed constant $\varepsilon_2 = \varepsilon_2(\Sigma, \eta, \bar{L}, \bar{\phi}) > 0$, as the claims trivially hold for $\varepsilon_\kappa(\gamma) \ge \varepsilon_2$ and any $\tilde{c}_\gamma \in \mathrm{Circ}_\eta$ by choosing the constants depending on ε_2. As before, we write for short $L = L(\gamma)$ and let $\bar{r} = \bar{\kappa}_\gamma^{-1}$ be the radius of the circular arc c_γ of Lemma 3.1. We also recall that by Lemma 3.1, γ is embedded and $|\phi_{\mathrm{turn}}(\gamma)| < 2\pi$, and that combining this latter fact with (2.14) guarantees that $(\frac{\eta}{2\pi})^{1/2} \le \bar{r} \le \frac{6\bar{L}}{5\pi}$. Furthermore, since Σ is embedded, there exists a constant $C = C(\Sigma)$ so that $\angle(\tau_\Sigma(x), \tau_\Sigma(\tilde{x})) \le C|x - \tilde{x}|$ for all $x, \tilde{x} \in \Sigma$.

Lemma 3.1 hence in particular ensures that the angle between the tangent to Σ at the endpoints $x_2(\gamma)$ of γ and $x_2(c_\gamma)$ is bounded by $C\|\gamma - c_\gamma\|_{C^0} \le C\varepsilon_\kappa(\gamma)$. Our construction furthermore ensures that $\tau_\gamma(x_1(\gamma)) = \gamma_1'(L)$ and that the angle between this vector and $\tau_{c_\gamma}(x_2(c_\gamma)) = \gamma_1'(L_1)$ is given by $\frac{1}{\bar{r}}|L(\gamma) - L(c_\gamma)| \le (\frac{\eta}{2\pi})^{-1/2} 2\,|\gamma(L) - \gamma_1(L)| \le C\varepsilon_\kappa(\gamma)$. Combined we hence obtain that $|\alpha_2(\gamma) - \alpha_2(c_\gamma)| \le C\|\gamma - c_\gamma\|_{C^1} \le C\varepsilon_\kappa(\gamma)$, $C = C(\eta, \Sigma)$. For ε_2 small enough this in particular ensures that the second intersection angle $\alpha_2(c_\gamma)$ is bounded away uniformly from 0 and π, say so that $\alpha_i(c_\gamma) \in [\pi/4, 3\pi/4]$ for $i = 2$, while this estimate is trivially true for $i = 1$.

Since γ is embedded, we can bound

$$|A_\Sigma(c_\gamma) - \eta| = |A_\Sigma(c_\gamma) - A_\Sigma(\gamma)| \le 2\pi(\bar{r} + \|c_\gamma - \gamma\|_{C^0(ds_\gamma)})\|c_\gamma - \gamma\|_{C^0(ds_\gamma)} \le C\varepsilon_\kappa(\gamma), \tag{3.11}$$

for an explicitly computable constant $C = C(\Sigma, \eta, \bar{L}, \bar{\phi})$, where in the final inequality we use $\bar{r} \le \frac{6\bar{L}}{5\pi}$. After reducing ε_2 if necessary, we assume $C\varepsilon_2 \le \eta/4$ for this constant, and thus (3.11) guarantees that $|A_\Sigma(c_\gamma) - \eta| \le \eta/4$.

We now let $z \in \mathbb{R}^2$ be the center of the circle that contains c_γ and consider the (continuous) family of circular arcs $c_r(z)$ in $\mathcal{B}$ parametrized on $[0, L]$ with center z for which $c_{\bar{r}}(z) = c_\gamma$. The uniform a priori bounds on the angles $\alpha_i(c_\gamma)$, the radius $\bar{r}$, and the area of c_γ ensure that the map $r \mapsto c_r(z) \in \mathcal{B}$ is a well-defined C^2 map into $C^1([0, L], \mathbb{R}^2)$ at least on an interval of the form $(\bar{r} - c_0, \bar{r} + c_0)$ for a number $c_0 = c_0(\eta, \Sigma) > 0$ (compare Remark 3.5 below). The first variation of the area along families of circular arcs $c_r(z) \in \mathcal{B}$ with fixed center is given by $\partial_r A_\Sigma(c_r(z)) = L(c_r(z))$; see (3.18) below. In particular, $\partial_r A_\Sigma(c_r(z)) \ge I_\Omega(\frac{1}{2}\eta) \ge \pi^{\frac{1}{2}}\eta^{\frac{1}{2}}$ for all r for which $A_\Sigma(c_r(z)) \ge \frac{1}{2}\eta$.

So, after reducing ε_2 if necessary, we deduce that there is a (unique) $\hat{r} \in (\bar{r} - c_0, \bar{r} + c_0)$ for which $A_\Sigma(c_r(z)) = \eta$ and

$$|\hat{r} - \bar{r}| \le \frac{|A_\Sigma(c_\gamma) - \eta|}{\pi^{\frac{1}{2}}\eta^{\frac{1}{2}}} \le C\varepsilon_\kappa(\gamma) \tag{3.12}$$

Thus $\tilde{c}_\gamma = c_{\hat{r}}(z) \in \mathrm{Circ}_\eta$. It is simple to check that $\|\tilde{c}_\gamma - c_\gamma\|_{C^1([0,L])} \le C|r - \hat{r}|$ for an explicit $C = C(\bar{L}, \eta)$. Combining this with (3.12) and the bound (3.3) on $\|c_\gamma - \gamma\|_{C^1(ds_\gamma)}$ obtained in Lemma 3.1, we obtain the first claim (3.9).

The second claim follows from the first. As noted above, $|\angle(\tau_\Sigma(x), \tau_\Sigma(\tilde{x}))| \le C|x - \tilde{x}|$ for all $x, \tilde{x} \in \Sigma$. So, again using that $\angle(w_1, w_2) = 2\arcsin(\frac{1}{2}|w_2 - w_1|)$ for unit vectors w_1, w_2, we have $|\alpha_i(\tilde{c}_\gamma) - \alpha_i(\gamma)| \le C|x_i(\tilde{c}_\gamma) - x_i(\gamma)| + 2\arcsin(\frac{1}{2}|\tau_{\tilde{c}_\gamma}(x_i(\tilde{c}_\gamma)) - \tau_\gamma(x_i(\gamma))|) \le C\|\tilde{c}_\gamma - \gamma\|_{C^1(ds_\gamma)}$, which combined with (3.9) completes the proof. □

3.2. **Analysis of circular arcs.** We let Circ be the subset of $\mathcal{B}$ that is made up of circular arcs which intersect Σ transversally and let $\mathrm{Circ}_\eta := \mathrm{Circ} \cap \mathcal{B}_\eta$, compare (1.5). Given $c \in \mathrm{Circ}$, we denote by z_c and r_c the center and radius of the circle that contains c.

We first note that for any $c \in \mathrm{Circ}$ there exist neighborhoods U_c of z_c, $U_{1,2}$ of the endpoints $x_1(c)$, $x_2(c)$ of c and I of r_c so that for every $z \in U_c$ and any $r \in I$ there is a unique circular arc $c_r(z) \in \mathcal{B}$ with radius r and center z whose endpoints $x_i(c_r(z))$ are in U_i. As Σ is assumed to be C^2, a simple argument using the implicit function theorem, which is applicable when the intersection angles of these circular arcs remain bounded away from 0 and π, furthermore gives the following.

Remark 3.5. For any $r_0 > 0$ and $\beta_0 < \pi/2$ there exist numbers $d_{0,1} = d_{0,1}(r_0, \beta_0, \Sigma) > 0$ so that the following holds true. If the radius and intersection angles of $c \in \mathrm{Circ}$ are so that $r_c \ge r_0$ and $|\alpha_i(c) - \pi/2| \le \beta_0$, $i = 1, 2$, then the above family $(r, z) \mapsto c_r(z)$ is well defined on $(r_c - d_0, r_c + d_0) \times B_{d_1}(z_c)$, and the maps $(r, z) \mapsto x_i(c_r(z))$ and $(r, z) \mapsto \theta_i(c_r(z))$ which assign to each such pair the endpoints and the angles between $x_i - z$ and e_1 are given by C^2 maps whose norms are bounded by a constant that only depends on r_0, β_0 and the C^2 norm of Σ.

In the following it suffices to consider circular arcs with positive orientation and it will be convenient to parametrize these arcs $c_r(z)$ with constant speed over $[0, 1]$, i.e. as

$$p \mapsto z + r(\cos, \sin)(\theta(p)), \quad \theta(p) := \theta_1 + p(\theta_2 - \theta_1), p \in [0, 1], \tag{3.13}$$

which then provides a way of viewing $(r, z) \mapsto c_r(z)$ as a C^2 function from $(r_c - d_0, r_c + d_0) \times B_{d_1}(z_c)$ to $C^2([0, 1], \mathbb{R}^2)$ with uniformly bounded norms. We can furthermore use that the dependence of the intersection angles $\alpha_i(c_r(x))$ on r and x is controlled in C^1 since Σ is C^2.

For families of such circular arcs we now show the following useful lemma.

Lemma 3.6. *Let $c_\varepsilon = c_{r_\varepsilon}(z_\varepsilon)$ be a differentiable family of circular arcs in* Circ *which have positive orientation. Then the variation of the enclosed area is given by*

$$\frac{d}{d\varepsilon} A_\Sigma(c_\varepsilon) = \partial_\varepsilon r \cdot L + \langle \partial_\varepsilon z, -J(x_2 - x_1)) \rangle. \tag{3.14}$$

Furthermore, the variation of the endpoints along general variations c_ε can be expressed as

$$\partial_\varepsilon x_1 = \mu_1 \tau_\Sigma(x_1) \quad \text{and} \quad \partial_\varepsilon x_2 = -\mu_2 \tau_\Sigma(x_2) \quad \text{for} \quad \mu_i = \frac{-\partial_\varepsilon r + \langle \nu_c(x_i), \partial_\varepsilon z \rangle}{\sin(\alpha_i)}. \tag{3.15}$$

For area preserving variations (3.15) *this formula reduces to*

$$\mu_i = \frac{1}{\sin(\alpha_i)} \langle \partial_\varepsilon z, Y_i(c) \rangle \quad \text{for} \quad Y_i(c) := \nu_c(x_i) - L^{-1} J(x_2 - x_1), \quad i = 1, 2 \tag{3.16}$$

where ν_c is the inward unit normal of c, which in turn ensures that the variation of the length along area preserving variations is given by

$$\partial_\varepsilon L(c_\varepsilon) = -\sum_i \cot(\alpha_i) \langle \partial_\varepsilon z, Y_i(c) \rangle. \tag{3.17}$$

In the above lemma and its proof we use the convention that all geometric quantities, such as the length L, the endpoints x_i, the intersection angles α_i are evaluated for the corresponding circular arc $c = c_\varepsilon$.

Remark 3.7. From (3.14), we see that along families of circular arcs with fixed center, the first variation of the area is given by

$$\partial_r A_\Sigma(c_r(z)) = L(c_r(z)), \tag{3.18}$$

and is in particular bounded below by $(2\pi\eta)^{1/2} > 0$ whenever $c_r(z) \in \mathrm{Circ}_\eta$.

Proof of Lemma 3.6. We parametrize the circular arcs c_ε as in (3.13) and use that the orientation of c_ε ensures that ν_{c_ε} is the inner normal to c_ε to write $\nu_\varepsilon(p) = -(\cos, \sin)(\theta_\varepsilon(p))$. As $J\nu_\varepsilon(p) = -\tau_\varepsilon(p) = (\sin, -\cos)(\theta_\varepsilon(p))$, this allows us to express the generating vector field $X = \partial_\varepsilon c_\varepsilon$ as

$$X = \partial_\varepsilon c_\varepsilon = \partial_\varepsilon z_\varepsilon - \partial_\varepsilon r_\varepsilon \nu_c - r \partial_\varepsilon \theta_\varepsilon J \nu_c. \tag{3.19}$$

As $|c'| = L$ and as $\frac{L}{\theta_2 - \theta_1} = r$, the formula (2.11) for the variation of the area hence yields that

$$\frac{d}{d\varepsilon} A_\Sigma(c_{r_\varepsilon}(z_\varepsilon)) = -\int_c X \cdot \nu_c \, ds_c = \partial_\varepsilon r \cdot L + L \cdot \left\langle \partial_\varepsilon z, \int_0^1 (\cos, \sin)(\theta_1 + p(\theta_2 - \theta_1)) dp \right\rangle = I + II. \tag{3.20}$$

For term II, we integrate to find

$$\begin{aligned} II &= \tfrac{L}{\theta_2 - \theta_1} \langle \partial_\varepsilon z, (\sin, -\cos)(\theta_2) - (\sin, -\cos)(\theta_1) \rangle = \langle \partial_\varepsilon z, rJ\nu_c(x_2) - rJ\nu_c(x_1) \rangle = \langle \partial_\varepsilon z_\varepsilon, -J(x_2 - z) + J(x_1 - z) \rangle \\ &= \langle \partial_\varepsilon z_\varepsilon, -J(x_2 - x_1) \rangle \end{aligned} \tag{3.21}$$

which together with (3.20) establishes the first claim (3.14) of the lemma.

Next, we note that since the endpoints x_1, x_2 are contained in Σ, their variation can be written as $\partial_\varepsilon x_i = \pm\mu_i\tau_\Sigma(x_i)$ for some $\mu_i = \mu_i(\varepsilon) \in \mathbb{R}$, where here and in the following $\pm$ is to be understood as $+$ for $i = 1$ and as $-$ for $i = 2$. To determine μ_i we can use that

$$\partial_\varepsilon r_\varepsilon = \tfrac{1}{2}r^{-1}\partial_\varepsilon|x_i - z|^2 = \langle r^{-1}(x_i - z), \pm\mu_i\tau_\Sigma(x_i) - \partial_\varepsilon z\rangle = -\mu_i\langle\nu_c(x_i), \pm\tau_\Sigma(x_i)\rangle + \langle\nu_c(x_i), \partial_\varepsilon z\rangle$$

since $\nu_c(x_i) = -\frac{x_i - z}{r}$ is the inner unit normal. We can now use that $\tau_\Sigma(x_i) = R_{\pm\alpha_i}\tau_c(x_i)$, compare (2.13), to write

$$\langle\nu_c(x_i), \pm\tau_\Sigma(x_i)\rangle = \langle J\tau_c(x_i), \pm R_{\pm\alpha_i}\tau_c(x_i)\rangle = \cos(\pi/2 - \alpha_i) = \sin(\alpha_i).$$

Thus $\partial_\varepsilon r_\varepsilon = -\mu_i\sin(\alpha_i) + \langle\nu_c(x_i), \partial_\varepsilon z\rangle$. This establishes the formula (3.15) for the variation of the endpoints.

Now, by (3.14), area preserving variations are characterized by $\partial_\varepsilon r = L^{-1}\langle\partial_\varepsilon z, J(x_2 - x_1)\rangle$. Making this substitution in (3.15) directly yields the expression (3.16) for μ_i in the case of area preserving variations. Inserting the resulting expression for $\partial_\varepsilon x_i$ into the formula (1.3) for the first variation of the length yields the claimed expression in (3.17) of

$$\partial_\varepsilon L(c_\varepsilon) = \langle\partial_\varepsilon x_2, \tau_c(x_2)\rangle - \langle\partial_\varepsilon x_1, \tau_c(x_1)\rangle = -\sum_i \mu_i\langle\tau_\Sigma(x_i), \tau_c(x_i)\rangle = -\sum_i \cot(\alpha_i)\langle\partial_\varepsilon z, Y_i(c)\rangle. \tag{3.22}$$

□

We note that (3.18) in particular ensures that Circ_η is a C^2 manifold that can be parametrized locally using the center. To be more precise, given a local C^2 parametrization $(r, z) \mapsto c_r(z)$ of a neighborhood of a given $c \in \mathrm{Circ}_\eta$ in Circ as considered above, the implicit function theorem ensures that for z in a sufficiently small neighborhood there exists a unique $r(z)$ so that the corresponding arc $c_{r(z)}(z)$ encloses the required area η and that the function $z \mapsto r(z)$ is C^2.

This allows us to view the restriction of the length functional to this 2-dimensional manifold locally as a C^2 function $\mathcal{L}_\eta(z) := L(c_{r(z)}(z))$ of z whose first variation is described by (3.17). As Circ_η is a submanifold of $\mathcal{B}_\eta$, and since the critical points of the length functional (with prescribed enclosed area) are always circular arcs which intersect Σ transversally (and indeed orthogonally), it is trivially true that any critical point $\gamma \in C^*_\eta$ of our original problem is also a critical point of this restricted functional. Conversely, as the vectors $Y_i = \nu_c(x_i) - L^{-1}J(x_2 - x_1)$, $i = 1, 2$, appearing in the formula (3.17) of $d\mathcal{L}_\eta(z)(\partial_\varepsilon z)$ are trivially linearly independent, we can immediately deduce that $\nabla\mathcal{L}_\eta$ vanishes if and only if $\cot(\alpha_i) = 0$ for both $i = 1, 2$, i.e. if and only if both intersection angles are $\alpha_i = \frac{\pi}{2}$. We also note that $|Y_i(c)| \le 2$ and hence that $|d\mathcal{L}_\eta(\partial_\varepsilon z)| \le 2(|\cot(\alpha_1)| + |\cot(\alpha_2)|)|\partial_\varepsilon z|$.

We can hence use that $\mathcal{L}_\eta$ and L are related by the following.

Lemma 3.8. *For any $\eta > 0$ we have $C^*_\eta = \{c \in \mathrm{Circ}_\eta : d\mathcal{L}_\eta(c) = 0\}$ and, parametrizing Circ_η locally by the center of the circular arcs as described above, we can estimate*

$$|\nabla\mathcal{L}_\eta(z)| \le 2(\sin\delta)^{-1}[|\alpha_1(\gamma) - \tfrac{\pi}{2}| + |\alpha_2(\gamma) - \tfrac{\pi}{2}|] \tag{3.23}$$

for $\delta > 0$ chosen so that $\alpha_i \in [\delta, \pi - \delta]$, $i = 1, 2$.

We now want to use these basic properties of $\mathcal{L}_\eta$ to show

Lemma 3.9. *Let $\eta > 0$ and assume that either Σ is a circle or that (Σ, η) satisfies the non-degeneracy assumption* (1.1). *Then there exists a constant $C_5 = C_5(\eta, \Sigma)$ so that for every $c \in \mathrm{Circ}_\eta$ there exists $c^* \in C^*_\eta$ with*

$$\|c - c^*\|_{C^1([0,1])} \le C_5\varepsilon_\alpha(c) \quad \text{where} \quad \varepsilon_\alpha(c) := |\alpha_1(c) - \tfrac{\pi}{2}| + |\alpha_2(c) - \tfrac{\pi}{2}|. \tag{3.24}$$

Proof. We note that the lemma is trivially true (with c^* chosen as a global minimizer of A_Σ) for curves with $\max|\alpha_i(c) - \pi/2| > \pi/4$. Hence we only need to consider circular arcs with $|\alpha_i(c) - \pi/2| \le \pi/4$ for $i = 1, 2$ which we can furthermore assume to be positively oriented. As this subset of Circ_η is compact and as our assumption on (Σ, η) ensures that the set of critical points is finite (up to symmetries in the case of the circle), the claim of the lemma hence follows provided we show that for any $c^* \in C^*_\eta$ there exist $C > 0$ and a neighborhood $\hat{U}_{c^*}$ of c^* in Circ_η so that (3.24) holds true. As Circ_η can locally be represented by $z \mapsto c_{r(z)}(z)$ as described above and as $\mathcal{L}_\eta$ and L are related as described in Lemma 3.8, this follows provided we prove that for every positively oriented $c^* \in C^*_\eta$ there exist $\varepsilon > 0$ and $C > 0$ so that for any $z \in B_\varepsilon(z^*)$, where $z^* := z_{c^*}$ is the centre of c^*, there exists $\hat{z}^*$ with $\nabla\mathcal{L}_\eta(\hat{z}^*) = 0$ so that

$$|z - \hat{z}^*| \le C|\nabla\mathcal{L}_\eta(z)|. \tag{3.25}$$

We first prove that this holds for $\hat{z}^* = z^*$ in the case where (Σ, η) satisfies the non-degeneracy assumption (1.1), i.e. where the eigenvalues $\lambda_{1,2}$ of the Hessian $D^2\mathcal{L}_\eta(z^*)$ are non-zero. In this case we set $\lambda_0 := \min(|\lambda_1|, |\lambda_2|) > 0$ and use that $z \mapsto \mathcal{L}_\eta(z)$ is C^2 and that $z \mapsto \alpha_i(z) := \alpha_i(c_{r(z)}(z))$ is C^1 with $\alpha_i(z^*) = \pi/2$ to choose $\varepsilon > 0$ (depending on the modulus of continuity of $D^2\mathcal{L}_\eta$ and thus on η and the modulus of continuity of the arc length parametrization of Σ) so that

$$\|D^2\mathcal{L}_\eta(z^*) - D^2\mathcal{L}_\eta(z)\| \le \tfrac{1}{2}\lambda_0 \quad \text{and} \quad |\alpha_i(c_{r(z)}(z)) - \tfrac{\pi}{2}| \le \tfrac{\pi}{4} \quad \text{for all } z \in B_\varepsilon(z^*). \tag{3.26}$$

We then let $E_{1,2}$ be orthonormal eigenvectors of $D^2\mathcal{L}_\eta(z^*)$ to the eigenvalues $\lambda_{1,2}$, and given any unit vector $w \in \mathbb{R}^2$ consider the unit vector $\tilde{w} := \sum_i \operatorname{sign}(\lambda_i)\langle w, E_i\rangle E_i$ which is chosen so that

$$D^2\mathcal{L}_\eta(\tilde{z})(w, \tilde{w}) \ge D^2\mathcal{L}_\eta(z^*)(w, \tilde{w}) - \tfrac{1}{2}\lambda_0 = |\lambda_1|\langle w, E_1\rangle^2 + |\lambda_2|\langle w, E_2\rangle^2 - \tfrac{1}{2}\lambda_0 \ge \tfrac{1}{2}\lambda_0$$

for all $\tilde{z} \in B_\varepsilon(z^*)$. Given $z \in B_\varepsilon(z^*)$ we apply this for $\tilde{z}_t := z^* + t(z - z^*)$, $t \in [0, 1]$ and $w = \frac{z-z^*}{|z-z^*|} = \frac{\partial_t \tilde{z}_t}{|z-z^*|}$. As $\nabla\mathcal{L}_\eta(z^*) = 0$ this yields

$$\begin{aligned} |\nabla\mathcal{L}_\eta(z)| \ge \nabla\mathcal{L}_\eta(z) \cdot \tilde{w} = (\nabla\mathcal{L}_\eta(z) - \nabla\mathcal{L}_\eta(z^*)) \cdot \tilde{w} = \int_0^1 \frac{d}{dt}\nabla\mathcal{L}_\eta(\tilde{z}_t) \cdot \tilde{w}\,dt \\ = |z - z^*| \int_0^1 D^2\mathcal{L}_\eta(\tilde{z}_t)(w, \tilde{w})dt \ge \tfrac{1}{2}\lambda_0|z - z^*| \end{aligned} \tag{3.27}$$

hence establishing that

$$|z - z^*| \le 2\lambda_0^{-1}|\nabla\mathcal{L}_\eta(z)| \qquad \text{for all } z \in B_\varepsilon(z^*).$$

It hence remains to prove the analogous claim in the case where Σ is circle, without loss of generality given by $\Sigma = \partial B_{\rho_0}(0)$ for some $\rho_0 > 0$. As the symmetries of this setting ensure that $\mathcal{L}_\eta(z) = \mathcal{L}_\eta(|z|)$, we can assume without loss of generality that $\hat{z}^* = z_1^* e_1$ where $z_1^* > 0$ is given by a critical point of $x \mapsto \mathcal{L}_\eta(xe_1)$ and it suffices to prove that there exists $\varepsilon > 0$ and $C > 0$ so that

$$|z_1 - z_1^*| \le C|\partial_{z_1}\mathcal{L}_\eta(z_1e_1)| \qquad \text{for all } |z_1 - z_1^*| < \varepsilon \tag{3.28}$$

The main step in the proof of this is to establish that

$$\partial_{z_1}^2\mathcal{L}_\eta(z^*) > 0. \tag{3.29}$$

To see this, we can differentiate the expression $\partial_{z_1}\mathcal{L}_\eta(z_1e_1) = -\sum \cot(\alpha_i(z_1))\langle Y_i(z_1e_1), e_1\rangle$ which results from (3.17) and use that $\alpha_1(z_1e_1) = \alpha_2(z_1e_1)$ for all z_1 by symmetry. Evaluating the resulting expression at z_1^* where $\alpha_i = \pi/2$ we hence obtain that at $z^* = z_1^*e_1$,

$$\partial_{z_1}^2\mathcal{L}_\eta(z^*) = -\cot'(\pi/2)(\partial_{z_1}\alpha_1)(z^*)\langle e_1, Y_1(z^*) + Y_2(z^*)\rangle = (\partial_{z_1}\alpha_1)(z^*)\langle e_1, \nu_{c^*}(x_1^*) + \nu_{c^*}(x_2^*) - 2L^{-1}J(x_2^* - x_1^*)\rangle \tag{3.30}$$

Letting $\beta_{1,2}^*$ be so that the endpoints $x_i^* = x_i(c^*)$ of $c^* = c_{r(z^*)}(z^*)$ are given by $x_i* = \rho_0(\cos, \sin)(\beta_i^*)$, we now note that the symmetries ensure that $\beta_1^* = -\beta_2^*$. Furthermore, as the intersection of c^* and $\Sigma = \partial B_{\rho_0}(0)$ is perpendicular, we can use that $\beta_2^* \in (0, \pi/2)$ and that $\nu_{c^*}(x_1^*) = \tau_\Sigma(x_1^*) = (-\sin, \cos)(\beta_1^*) = (\sin, \cos)(\beta_2^*)$ while $\nu_{c^*}(x_2^*) = -\tau_\Sigma(x_2^*) = (\sin, -\cos)(\beta_2^*)$.

Thus the vector

$$\nu_{c^*}(x_1^*) + \nu_{c^*}(x_2^*) - 2L^{-1}J(x_2^* - x_1^*) = 2\sin(\beta_2^*)e_1 - 2L^{-1}J(2\rho_0\sin(\beta_2^*)e_2) = [2 + 4\rho_0L^{-1}]\sin(\beta_2^*)e_1.$$

appearing in the above formula is given by a positive multiple of e_1 since $\beta_2^* \in (0, \pi/2)$. Inserted into (3.30), this yields $\partial_{z_1}^2\mathcal{L}_\eta(z^*) = [2 + 4\rho_0L^{-1}]\sin(\beta_2^*)\partial_{z_1}\alpha_1(z^*)$, and thus to show (3.29) it remains to show $\partial_{z_1}\alpha_1(z^*) > 0$.

To see this we differentiate the relation

$$\cos(\alpha_1) = \langle\tau_\Sigma(x_1), \tau_c(x_1)\rangle = \langle\tau_\Sigma(x_1), -J\nu_c(x_1)\rangle = \langle\tau_\Sigma(x_1), -Jr^{-1}(z - x_1)\rangle \tag{3.31}$$

along $c_\varepsilon = c_{r(z_\varepsilon)}(z_\varepsilon)$ for $z_\varepsilon = (z_1^* + \varepsilon)e_1$ and evaluate all resulting expressions at $\varepsilon = 0$ where we can use that $\alpha_1(z^*) = \pi/2$ and hence $\nu_{c^*}(x_1^*) = \tau_\Sigma(x_1^*)$. This yields

$$\begin{aligned} -\partial_{z_1}\alpha_1(z^*) &= \langle\partial_\varepsilon|_{\varepsilon=0}\tau_\Sigma(x_1), -J\nu_{c^*}(x_1^*)\rangle + \langle\tau_\Sigma(x_1^*), -J\partial_\varepsilon|_{\varepsilon=0}(r^{-1}(z - x_1))\rangle] \\ &= -\langle\mu_1\kappa_\Sigma\nu_\Sigma(x_1^*), J\nu_{c^*}(x_1^*)\rangle - r^{-1}\langle\tau_\Sigma(x_1^*), Je_1\rangle + r^{-1}\mu_1\langle\tau_\Sigma(x_1), J\tau_\Sigma(x_1)\rangle + r^{-2}\partial_\varepsilon r\langle\tau_\Sigma(x_1^*), J(z^* - x_1^*)\rangle \\ &= -[\mu_1\kappa_\Sigma + \langle\tau_\Sigma(x_1), e_2\rangle] \end{aligned} \tag{3.32}$$

where we use in the last step that $J(z^* - x_1) = -rJ\nu_{c^*}(x_1^*) = -rJ\tau_\Sigma(x_1^*)$ is normal to $\tau_\Sigma(x_1^*)$. Combined with the formula for μ_1 obtained in (3.16) this yields that indeed

$$\partial_{z_1}\alpha_1(z^*) = \kappa_\Sigma[\langle\nu_{c^*}(x_1^*), e_1\rangle + 2\rho_0L^{-1}\sin(\beta_2^*)] + \langle\tau_\Sigma(x_1^*), e_2\rangle = \kappa_\Sigma[1 + 2\rho_0L^{-1}]\sin(\beta_2^*) + \cos(\beta_2^*) > 0. \tag{3.33}$$

Having hence established that $\partial^2_{z_1}\mathcal{L}_\eta(z^*) > 0$ we can now choose $\varepsilon > 0$ small enough so that $\partial^2_{z_1}\mathcal{L}_\eta(z_1e_1) \geq \frac{1}{2}\partial^2_{z_1}\mathcal{L}_\eta(z^*) =: C^{-1} > 0$ on $(z_1^* - \varepsilon, z_1^* + \varepsilon)$ to deduce that $|\partial_{z_1}\mathcal{L}_\eta(z_1e_1)| = |\partial_{z_1}\mathcal{L}_\eta(z_1e_1) - \partial_{z_1}\mathcal{L}_\eta(z_1^*e_1)| \geq C^{-1}|z_1 - z_1^*|$, which immediately implies (3.28) for this choice of C. This completes the proof of the lemma in this second case where Σ is a circle. □

3.3. **Proof of Theorem 1.2.** In this section we explain how the results from the previous two subsections allow us to complete the proof of the Łojasiewicz estimates (1.6) and (1.7) claimed in Theorem 1.2. So let $\bar{L} > 0$ and $\bar{\phi}$ be any given numbers and let $\gamma \in \mathcal{B}_\eta$ be so that $L(\gamma) \leq \bar{L}$ and $|\phi_{\text{turn}}(\gamma)| \leq \bar{\phi}$. As discussed at the beginning of the section, it suffices to consider curves $\gamma \in \mathcal{B}_\eta$ with $\varepsilon(\gamma) \leq \varepsilon_0$ for a fixed constant $\varepsilon_0 = \varepsilon_0(\Sigma, \eta, \bar{L}, \bar{\phi}) > 0$, which we can in particular choose so that $\varepsilon_0 \leq \varepsilon_1$ for the constant ε_1 obtained in Lemma 3.1.

For such curves, Proposition 3.4 applies and ensures that there exists a circular arc $c \in \text{Circ}_\eta$ so that

$$\|\gamma - c\|_{C^1(ds_\gamma)} \leq C_3\varepsilon_\kappa(\gamma) \quad \text{and} \quad |\alpha_i(c) - \alpha_i(\gamma)| \leq C_4\varepsilon_\kappa(\gamma), \quad i = 1, 2. \tag{3.34}$$

Lemma 3.9 then yields the existence of $c^* \in C^*_\eta$ for which we can bound

$$\|c - c^*\|_{C^1(ds_\gamma)} \leq \max(1, L(\gamma)^{-1})\|c - c^*\|_{C^1([0,1])} \leq C_5 \max(1, I_\Omega(\eta)^{-1})\varepsilon_\alpha(c) \leq C_5 \max(1, I_\Omega(\eta)^{-1})[\varepsilon_\alpha(\gamma) + 2C_4\varepsilon_\kappa(\gamma)]. \tag{3.35}$$

Combined, (3.34) and (3.35) provide the claimed distance Łojasiewicz estimate (1.6).

It remains to show that this estimate on the C^1 distance of γ to c^* ensures that the difference of their lengths is controlled by (1.7). To this end we first note that the length of the segments σ_i of Σ between $x_i(c^*)$ and $x_i(\gamma)$ is bounded by

$$L(\sigma_i) \leq C\|\gamma - c^*\|_{C^0} \leq C\varepsilon(\gamma) \leq \min\Big(\frac{\pi}{4\kappa_{max}(\Sigma)}, (\frac{\eta}{2\pi})^{\frac{1}{2}}\Big), \tag{3.36}$$

where the last estimate holds after reducing ε_0 if necessary.

We now note that since c^* meets Σ orthogonally, the turning angles of c^* (when parametrized with positive orientation) and of the subarc σ_{c^*} by which we close up c^* are related by $\phi_{\text{turn}}(c^*) = \pi + \phi_{\text{turn}}(-\sigma_{c^*})$. This immediately implies that $\phi_{\text{turn}}(-\sigma_{c^*}) \in [0, \pi)$ and hence that $L(\Sigma \setminus \sigma_{c^*}) \geq \frac{\pi}{\kappa_{max}(\Sigma)}$. If $\phi_{\text{turn}}(-\sigma_{c^*}) \in [\pi/2, \pi]$ we can furthermore bound $L(\sigma_{c^*}) \geq \frac{\pi}{2\kappa_{max}(\Sigma)}$ while otherwise we can use that $\phi_{\text{turn}}(c^*) \in [\pi, \frac{3\pi}{2}]$ and hence $L(\sigma_{c^*}) \geq |x_1(c^*) - x_2(c^*)| = (2 - 2\cos(\phi_{\text{turn}}(c^*))^{1/2} r_{c^*} \geq 2^{\frac{1}{2}} r_{c^*} \geq 2^{\frac{1}{2}}(\eta\pi^{-1})^{\frac{1}{2}}$. The above estimate (3.36) hence in particular implies that $\min(L(\sigma_{c^*}), L(\Sigma \setminus \sigma_{c^*})) \geq L(\sigma_1) + L(\sigma_2)$ which ensures that σ_1 and σ_2 are disjoint.

We now consider the modified support curve $\hat{\Sigma}$ which we obtain from Σ by replacing these short segments σ_i of Σ with the line segments $\hat{\sigma}_i$ between $x_i(\gamma)$ and $x_i(c^*)$ and denote by $\hat{\mathcal{B}}$ and $A_{\hat{\Sigma}} : \hat{\mathcal{B}} \to \mathbb{R}$ the set of admissible curves and enclosed area for this modified support curve.

The convexity of Σ ensures that the sets E_i, $i = 1, 2$, which are enclosed by σ_i and $\hat{\sigma}_i$ are contained in the triangle formed by the line segment $\hat{\sigma}_i$, whose length is $|x_1(c^*) - x_1(\gamma)| \leq C\varepsilon(\gamma)$, and the two tangents $T_{x_i(c^*)}\Sigma$ and $T_{x_i(\gamma)}\Sigma$, whose intersection angles with $\hat{\sigma}_i$ can be no larger than $L(\sigma_i)\kappa_{max}(\Sigma) \leq C\varepsilon(\gamma) \leq \pi/4$. This implies that

$$|\hat{E}_i| \leq \frac{1}{2}|x_i(c^*) - x_i(\gamma)|^2 \tan(\kappa_{max}(\Sigma)L(\sigma_i)) \leq C\varepsilon(\gamma)^3, \tag{3.37}$$

from which we deduce that $|A_\Sigma(\tilde{\gamma}) - A_{\hat{\Sigma}}(\tilde{\gamma})| \leq C\varepsilon(\gamma)^3$ for all $\tilde{\gamma} \in \hat{\mathcal{B}} \cap \mathcal{B}$. Applied for c^* and γ and combined with the fact that $A_\Sigma(\gamma) = \eta = A_\Sigma(c^*)$ this in particular ensures that

$$|A_{\hat{\Sigma}}(c^*) - A_{\hat{\Sigma}}(\gamma)| \leq C\varepsilon(\gamma)^3. \tag{3.38}$$

We now want to argue that this implies that the first variation of both the modified area functional and of the length functional at c^* in direction of the vector field $X(p) = (\gamma(p) - c^*(p))$, are of order $O(\varepsilon(\gamma))$. To this end, we note that since the modified support curve is flat between the respective endpoints of c^* and γ, the curves $\gamma_t \in C^1([0, L(\gamma)], \mathbb{R}^2)$, $t \in [0, 1]$, obtained by interpolating linearly

$$\gamma_t(p) = c^*(p) + t(\gamma(p) - c^*(p)), \quad p \in [0, L], \quad L := L(\gamma)$$

between the constant speed parameterizations of c^* and γ, are all in $\hat{\mathcal{B}}$ and so that $\partial_t\gamma_t = X$ for every t. While these parameterizations of γ_t, $t \in (0, 1)$ will in general not be by constant speed, we can use that $\||\gamma_t'| - 1\|_{C^0} \leq \|\gamma_t - \gamma\|_{C^1([0,L])} \leq \|c^* - \gamma\|_{C^1([0,L])} \leq C\varepsilon(\gamma) \leq \frac{1}{2}$, where the last step holds after reducing ε_0 if necessary, and hence that $\||\gamma_t'|^{-1} - 1\|_{C^0} \leq 2\|c^* - \gamma\|_{C^1([0,L])} \leq C\varepsilon(\gamma)$.

As the first variation of the length along these curves can be computed as $dL(\gamma_t)(X) = \int_0^L |\gamma_t'(p)|^{-1}\gamma_t'(p)\cdot X'(p)dp$ while $dA_{\hat{\Sigma}}(\gamma_t)(X) = -\int_0^L X(p)\cdot J\gamma_t'(p)dp$, we can bound

$$|dL(\gamma_t)(X) - dL(c^*)(X)| + |dA_{\hat{\Sigma}}(\gamma_t)(X) - dA_{\hat{\Sigma}}(c^*)(X)| \le C\|\gamma_t - c^*\|_{C^1}\cdot\|X\|_{C^1} \le C\varepsilon(\gamma)^2. \tag{3.39}$$

Writing $A_{\hat{\Sigma}}(\gamma) - A_{\hat{\Sigma}}(c^*) = \int_0^1 \frac{d}{dt}A_{\hat{\Sigma}}(\gamma_t)dt = \int_0^1 dA_{\hat{\Sigma}}(\gamma_t)(X)dt$ and using (3.38) and (3.39) we can thus conclude that

$$\Big|\int_{c^*} X\cdot\nu_{c^*}ds_{c^*}\Big| = |dA_{\hat{\Sigma}}(c^*)(X)| \le |A_{\hat{\Sigma}}(\gamma) - A_{\hat{\Sigma}}(c^*)| + C\varepsilon(\gamma)^2 \le C\varepsilon(\gamma)^3 + C\varepsilon(\gamma)^2 \le C\varepsilon(\gamma)^2. \tag{3.40}$$

Since the curvature κ_{c^*} of the circular arc c^* is constant, the first variation of the length along the (in general not area preserving) vector field X can be written as

$$dL(c^*)(X) = -\kappa_{c^*}\int X\cdot\nu_{c^*}ds_{c^*} + \langle X(L), \tau_{c^*}(L)\rangle - \langle X(0), \tau_{c^*}(0)\rangle.$$

Since X is parallel to the line segment $\hat{\sigma}_i$ at the endpoints and since the intersection angles $\hat{\alpha}_i(c^*)$ between c^* and $\hat{\sigma}_i$ are so that $|\hat{\alpha}_i(c^*) - \pi/2| = |\hat{\alpha}_i(c^*) - \alpha_i(c^*)| = \angle(\hat{\sigma}_i, T_{x_i(\gamma)}\Sigma) \le C\varepsilon(\gamma)$, we can hence bound

$$|dL(c^*)(X)| \le C\varepsilon(\gamma)^2 + C\varepsilon(\gamma)\left(\cos(\hat{\alpha}_1(c^*)) + \cos(\hat{\alpha}_2(c^*))\right) \le C\varepsilon(\gamma)^2. \tag{3.41}$$

Combined with (3.39) we thus conclude that

$$|L(c^*) - L(\gamma)| = \Big|\int_0^1 \frac{d}{dt}L(\gamma_t)dt\Big| = \Big|\int_0^1 dL(\gamma_t)(X)dt\Big| \le |dL(c^*)(X)| + C\varepsilon(\gamma)^2 \le C\varepsilon(\gamma)^2 \tag{3.42}$$

which establishes the second claim (1.7) of Theorem 1.2.

Remark 3.10. If Σ is analytic, then the expected analogues of the Łojasiewicz estimates (1.6) and (1.7) hold (without imposing any form of non-degeneracy). Namely, there exist $\beta_{1,2} = \beta_{1,2}(\eta, \Sigma) \in (0, 1]$ such that (1.6) holds with exponent β_1 replacing 1 on the right-hand side, and (1.7) holds with exponent $1 + \beta_2$ replacing 2 on the right-hand side. Indeed, as the map $z \mapsto \mathcal{L}_\eta(z)$ is analytic whenever Σ is analytic, the classical Łojasiewicz inequality for analytic functions on finite dimensional spaces guarantees that $|z - \hat{z}^*| \le C|\nabla\mathcal{L}_\eta(z)|^{\beta_1}$ for some $\beta_1 \in (0, 1]$. The first estimate can be shown in the same way as (1.6) but with this Łojasiewicz inequality in place of the estimate (3.25).
To prove the second estimate, observe that, up to minor modifications, one could replace the arc $c^* \in C^*_\eta$ with the arc $\tilde{c}_\gamma \in \mathrm{Circ}_\eta$ obtained in Proposition 3.4 in the estimates starting from (3.36) and ending with (3.42), to obtain $|L(\tilde{c}_\gamma) - L(\gamma)| \le C\varepsilon(\gamma)^2$ in place of (3.42). Next, the classical finite dimensional (gradient) Łojasiewicz estimate for analytic functions guarantees that $|\mathcal{L}_\eta(c_\gamma) - \mathcal{L}_\eta(c^*)| \le C|\nabla\mathcal{L}_\eta(c_\gamma)|^{1+\beta_2}$ for some $\beta_2 \in (0, 1]$. Pairing Lemma 3.8, with the estimate (3.34), we know $|\nabla\mathcal{L}_\eta(\tilde{c}_\gamma)| \le C\varepsilon(\gamma)$, thus allowing us to conclude the second estimate.

4. Exponential convergence of the gradient flow

In this section, we recall some background on the area preserving curve shortening flow with Neumann free boundary conditions, including results of the first author that will be crucial to apply the flow to prove Theorem 1.4 in the next section. Then, we prove Theorem 1.3.

4.1. Background on the flow.

Parabolic regularity theory implies that a solution of (1.9) with initial regularity $C^{2+\alpha}$ satisfies

$$\gamma \in C^{2+\alpha, 1+\frac{\alpha}{2}}\left([a_1, a_2]\times[0, T_{max}), \mathbb{R}^2\right) \cap C^\infty\left([a_1, a_2]\times(0, T_{max}), \mathbb{R}^2\right) \quad \text{for } 0 < \alpha < 1$$

where $C^{2+\alpha,1+\frac{\alpha}{2}}$ denotes the usual parabolic Hölder spaces and $T_{max} > 0$ is the maximal time of existence. We will use the notation $\gamma_t := \gamma(\cdot, t)$. This flow was constructed as (formal) L^2-gradient flow of the length with the constraint that the enclosed area is constant. It satisfies in particular

$$\frac{d}{dt}L(\gamma_t) = -\|\kappa_{\gamma_t} - \bar{\kappa}_{\gamma_t}\|^2_{L^2(ds_{\gamma_t})}. \tag{4.1}$$

Due to the flow's non-local nature and the free boundary condition, preserved properties are rare to find. In [23] it was shown that convexity of γ_t is preserved under the flow. However, embeddedness and the property of being contained in $\mathbb{R}^2 \setminus \Omega$ is not preserved in general. Thus, for general initial data, understanding how to close up the curve to define the enclosed area is *subtle*, as doing this naively might produces a family of closed curves whose enclosed area jumps by $\pm|\Omega|$. An example to illustrate this behavior can be found in Figure 1.

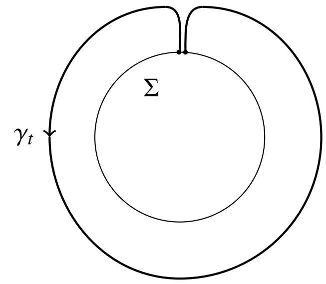

Figure 1. A solution of the flow γ_t can cross at the endpoints $\gamma_t(a_1)$, $\gamma_t(a_2)$ at some time $t > 0$. If we naively close the curves γ_t for each t by connecting $\gamma_t(a_2)$ with $\gamma_t(a_2)$ via the positive orientation of Σ, then the enclosed area jumps by $|\Omega|$ or $-|\Omega|$. Our choice of boundary curves closing up γ_t, Definition 2.4, ensures that the algebraic enclosed area is continuous with respect to t along the flow.

In this paper, we only deal with flows that stay outside of Ω, and therefore we avoid this subtlety. Instead, Definition 2.4 gives a simple canonical way to close up the curve and hence to define the enclosed area, which by Remark 2.5 is continuous along the flow.

The following result due to the first author in [23] shows that, if the initial curve γ_0 is short enough in relation to the maximal curvature of Σ and if its shape is not too bad regarding its isoperimetric quotient, then the above mentioned pathological behavior does not appear. In particular, the curves stay embedded and stay outside of Ω. This theorem will be essential in the proof of Theorem 1.4 together with Theorem 1.3.

Theorem 4.1 ([23])**.** *Let Σ be a convex C^2-Jordan curve and suppose $\eta \in (0, \bar{c}\kappa_{max}(\Sigma)^{-2})$. Let $\gamma_0 \in \mathcal{B}_\eta$ be an embedded, convex curve of class $C^{2,\alpha}$ satisfying*

$$L(\gamma_0) < \tfrac{4}{5\kappa_{max}(\Sigma)} \arcsin(\eta/L(\gamma_0)^2). \tag{4.2}$$

Then the solution $\{\gamma_t\}_{t>0}$ of the free boundary area preserving curve shortening flow emanating from γ_0 exists for all $t > 0$. Moreover, $|\int \kappa ds_{\gamma_t}| \leq 2\pi$ and for each $t \geq 0$, γ_t is embedded and intersects Ω only at the endpoints. In particular, $\gamma_t \in \mathcal{B}_\eta$ for all $t \in [0, \infty)$.

Proof. This theorem is a summary of the several main results of [23]. We note that condition (4.2) implies $L(\gamma_0) < \frac{1}{2\kappa_{max}(\Sigma)} < d_\Sigma$, where d_Σ is the width defined in (2.7). By [24, Proposition 2.7], if $\{\gamma_t\}_{t\in[0,T]}$ is a solution to the flow (1.9) with initial curve γ_0 satisfying $L(\gamma_0) < d_\Sigma$, then for every $t \in [0, T_{max})$, the closeup curve σ_{γ_t} from Definition 2.4 has turning angle at most π, and moreover there exists some $l \in \mathbb{Z}$ such that $(2l-2)\pi < \int \kappa_{\gamma_t} ds_{\gamma_t} < 2l\pi$ for all $t \in [0, T_{max})$. The assumption that γ_0 is convex and embedded guarantees that $l = 1$ if γ_0 is positively oriented and $l = 0$ if γ_0 is negatively oriented.
Furthermore, by [23, Theorem 5.6], we know that γ_t stays outside of $\bar{\Omega}$ and, together with the line segment from $x_2(\gamma_t)$ to $x_1(\gamma_t)$, γ_t traces out a convex domain for $t \in [0, \infty)$. In particular, $\gamma_t \in \mathcal{B}_\eta$. □

4.2. **Proof of Theorem 1.3.** In this section, we prove Theorem 1.3. In the proof, we will need to compare the L^2 norm of the velocities of γ_t and of its constant speed reparametrization on a fixed interval:

Lemma 4.2. *Let $T > 0$ and $\gamma : [a_1, a_2] \times [0, T) \to \mathbb{R}^2$ be a $C^{2,1}$-family of curves moving in normal direction, i.e. $(\partial_t\gamma)^T = 0$. We denote by $\tilde{\gamma}$ its orientation preserving reparametrization by constant speed on* $[0, 1]$. *Then we have*

$$\Big| L(\gamma_t) \cdot \|\partial_t\tilde{\gamma}_t\|^2_{L^2([0,1])} - \|\partial_t\gamma_t\|^2_{L^2(ds_{\gamma_t})} \Big| \leq 4L(\gamma_t) \Big(\int |\kappa||\partial_t\gamma| \, ds_{\gamma_t} \Big)^2. \tag{4.3}$$

Proof. We closely follow [33]. We write $\gamma_t(p) = \gamma(p, t)$ and use the notation $\gamma'(p, t)$ to mean $\partial_p\gamma(p, t)$. For $t \in [0, T)$, consider the strictly increasing function $p \mapsto \varphi(p, t) := L(\gamma_t)^{-1} \int_{a_1}^p |\gamma'(q, t)| \, dq$ and let $\psi(\cdot, t) : [0, 1] \to [a_1, a_2]$ denote its inverse. This way, the unit speed reparametrzation of γ_t on $[0, 1]$ is given by $\tilde{\gamma}(p, t) := \gamma(\psi(p, t), t)$. By the chain rule,

$$\partial_t\tilde{\gamma}(p, t) = \partial_t\gamma(\psi, t) + \partial_t\psi(p, t)\, \gamma'(\psi, t) \qquad \text{where } \psi = \psi(p, t).$$

For the normal speed, the change of variable $p = \varphi(q,t)$ shows $\int_0^1 |\partial_t\gamma(\psi(p,t),t)|^2\,dp = L(\gamma_t)^{-1}\|\partial_t\gamma_t\|^2_{L^2(ds_{\gamma_t})}$. So, to prove the lemma, it remains to estimate the squared L^2 norm of the tangential speed, which by the same change of variable is

$$I := \int_0^1 |\partial_t\psi(p,t)|^2\,|\gamma'(\psi(p,t),t)|^2\,dp = L(\gamma_t)^{-1}\int_{a_1}^{a_2}\left|\partial_t\psi\big|_{(\varphi(q,t),t)}\right|^2|\gamma'(q,t)|^3\,dq\,. \tag{4.4}$$

Toward this aim, we differentiate the identity $q = \psi(\varphi(q,t),t)$ with respect to t and q to find

$$\partial_t\psi\big|_{(\varphi(p,t),t)} = -\partial_p\psi\big|_{(\varphi(p,t),t)}\partial_t\varphi(p,t) = \tfrac{L(\gamma_t)}{|\gamma'(p,t)|}\partial_t\varphi(p,t).$$

Now, from the Frenet equation $\partial_p\tau = \kappa\nu|\partial_p\gamma|$ and the fact that $\partial_t\gamma$ is orthogonal to τ, we have

$$\partial_t\int_{a_1}^p|\gamma'| = \int_{a_1}^p\langle\partial_t\gamma',\tau\rangle = \int_{a_1}^p\partial_p(\langle\partial_t\gamma,\tau\rangle) - \langle\partial_t\gamma,\partial_p\tau\rangle = -\int_{a_1}^p\langle\partial_t\gamma,\kappa\nu\rangle|\gamma'|.$$

Therefore, $\partial_t\varphi(p,t) = -L(\gamma_t)^{-1}(\partial_t L(\gamma_t)\,\varphi(p,t) + \int_{a_1}^p\langle\partial_t\gamma,\kappa\nu\rangle|\gamma_t'|)$. Substituting this into the expression for I in (4.4) and keeping in mind that $|\varphi|\le 1$, we have

$$I = L(\gamma_t)^{-1}\int_{a_1}^{a_2}\left(\partial_t L(\gamma_t)\,\varphi(p,t) + \int_{a_1}^q\langle\partial_t\gamma,\kappa\nu\rangle|\gamma'|\right)^2 ds_{\gamma_t}(q) \le 2\,(\partial_t L(\gamma_t))^2 + 2\left(\int|\partial_t\gamma||\kappa|ds_{\gamma_t}\right)^2.$$

Finally, noting that $\frac{d}{dt}L(\gamma_t) = -\int\langle\partial_t\gamma,\kappa\nu\rangle\,ds_{\gamma_t} \le \int|\kappa||\partial_t\gamma|\,ds_{\gamma_t}$ completes the proof. □

Proof of Theorem 1.3. We proceed in several steps. Let γ_t be a global-in-time solution to the flow as in the statement of the theorem. We recall that $L(t) := L(\gamma_t)$ is monotone along the flow and bounded below by $L(t)\ge I_\Omega(\eta)$. In particular, and thus $L(\infty) := \lim_{t\to\infty}L(t)$ exists and is contained in $[I_\Omega(\eta), L(0)]$. Let $\bar{L}$ be an upper bound on $L(\gamma)$.

We recall that by the Łojasiewicz estimate (1.7) of Theorem 1.2, there is a constant $C_1 = C_1(\eta,\Sigma,\bar{L},\bar{\phi}) > 0$ such that

$$|L(t) - \ell_*(t)| \le C_1\|\kappa_\gamma - \bar{\kappa}_{\gamma_t}\|^2_{L^2(ds_\gamma)} \tag{4.5}$$

for $\ell_*(t)$ chosen so that $|L(t)-\ell_*(t)| = \min\{|L(t) - L(c^*)| : c^*\in C^*_\eta\}$. Note that this minimum is achieved: in the case of the circle, $\ell_0(\eta) = I_\Omega(\eta)$ is the only critical value, while otherwise our assumption on (Σ,η) implies that the set of critical values of the length L is discrete for the given η. In the following, we denote by

$$I_\Omega(\eta) =: \ell_0(\eta) < \ell_1(\eta) < \dots < \ell_m(\eta) < \dots \tag{4.6}$$

the critical values of the length, and let N be so that $\ell_{N-1}(\eta)\le\bar{L}<\ell_N(\eta)$.

Step 1: As a first step, we show there is a constant $C = C(\eta,\Sigma,\bar{L},\bar{\phi})$ such that for any $0\le t_1<t_2\le\infty$,

$$\int_{t_1}^{t_2}\|\partial_t\gamma_t\|_{L^2(ds_{\gamma_t})}dt \le C(L(t_1)-L(t_2))^{1/2}\,. \tag{4.7}$$

We divide the time interval $[0,T)$ into subintervals defined by $T_0 = 0 < T_1 < \dots < T_K$, where T_i are the times where $L(\gamma_t)$ either reaches a critical value or one of the midpoints of the intervals $[\ell_m(\eta),\ell_{m+1}(\eta)]$, i.e. $L(\gamma_t)\in\{\ell_m(\eta) : m\le N\}\cup\{\frac12(\ell_{m+1}(\eta)-\ell_m(\eta)) : m\le N-1\}$. Notice that $K\le 2N$ is bounded by a constant depending only on $L(\gamma_0)$ since the length is monotone along the flow.

On intervals $[T_i,T_{i+1}]$ on which $L(\gamma_t)\in[\ell_m(\eta),\frac12(\ell_{m+1}(\eta)-\ell_m(\eta))]$ for some m, we can apply (4.5) with $\ell_*(t)\equiv\ell_m(\eta)$, and use that length decays according to (4.1) to bound

$$\begin{aligned}-\frac{d}{dt}\,(L(\gamma_t)-\ell_m(\eta))^{\frac12} &= \frac12\,(L(\gamma_t)-\ell_m(\eta))^{-\frac12}\left(-\frac{d}{dt}L(\gamma_t)\right)\\ &= \frac12\,(L(\gamma_t)-\ell_m(\eta))^{-\frac12}\int(\kappa_{\gamma_t}-\bar{\kappa}_{\gamma_t})^2ds_{\gamma_t} \ge \frac{1}{2C_1}\|\partial_t\gamma\|_{L^2(ds_{\gamma_t})}\,.\end{aligned}$$

Integrating over subintervals $[t',t'']\subset[T_i,T_{i+1}]$ and using the fact that $(a-b)^2\le a^2-b^2$ for $a\ge b\ge 0$ yields

$$\int_{t'}^{t''}\|\partial_t\gamma_t\|_{L^2(ds_{\gamma_t})}dt \le 2C_1\left((L(\gamma_{t'})-\ell_m(\eta))^{\frac12} - (L(\gamma_{t''})-\ell_m(\eta))^{\frac12}\right) \le 2C_1\,(L(\gamma_{t'})-L(\gamma_{t''}))^{\frac12}\;. \tag{4.8}$$

The same argument is applicable also on intervals $[T_i,T_{i+1}]$ on which instead $L(\gamma_t)\in[\frac12(\ell_{m-1}(\eta)-\ell_m(\eta)),\ell_m(\eta)]$, except that we have to consider the evolution of the square root of $-(L(\gamma_t)-\ell_m(\eta))$ instead of $(L(\gamma_t)-\ell_m(\eta))$ since $L(\gamma_t)\le\ell_m(\eta)$ on such intervals. Thus (4.8) holds in this case as well.

Given arbitrary $0 \le t_1 \le t_2 \le \infty$, we can divide the interval $[t_1, t_2]$ into finitely many subintervals I_k such that each I_k is fully contained in one of the intervals $[T_i, T_{i+1}]$ and add the inequalities from the two cases above. Applying Cauchy-Schwarz inequality to bound the resulting sum on the right-hand side and using that the resulting series is telescoping, this yields (4.7) with $C = 2C_1K^{1/2}$.

Step 2: Next, we prove analogue for the constant speed reparametrization $\tilde{\gamma}_t$ of γ_t on $[0, 1]$, i.e. show that for any $0 \le t_1 < t_2 \le \infty$,

$$\int_{t_1}^{t_2} \|\partial_t\tilde{\gamma}_t\|_{L^2([0,1])}\, dt \le C(L(t_1) - L(t_2))^{1/2} . \tag{4.9}$$

To this end, we first recall from Lemma Lemma 4.2 that

$$\|\partial_t\tilde{\gamma}_t\|_{L^2([0,1])} \le \frac{1}{L(\gamma_t)^{1/2}}\|\partial_t\gamma_t\|_{L^2(ds_{\gamma_t})} + 2\|\kappa_{\gamma_t}\|_{L^2(ds_{\gamma_t})}\, \|\partial_t\gamma_t\|_{L^2(ds_{\gamma_t})} .$$

Since $|\partial_t\gamma_t| = |\kappa_{\gamma_t} - \bar{\kappa}_{\gamma_t}|$, we have $\|\kappa_{\gamma_t}\|_{L^2(ds_{\gamma_t})} = (\|\partial_t\gamma_t\|^2_{L^2(ds_{\gamma_t})} + \bar{\kappa}^2_{\gamma_t} L(\gamma_t))^{1/2} \le \|\partial_t\gamma_t\|_{L^2(ds_{\gamma_t})} + \bar{\kappa}_{\gamma_t} L(\gamma_t)^{1/2}$. Moreover, $\gamma_t \in \mathcal{B}_\eta$ and thus $L(\gamma_t) \ge I_\Omega(\eta)$. So, keeping in mind that $|\bar{\kappa}_{\gamma_t}|L(\gamma_t)^{1/2} = |\phi_{\text{turn}}(\gamma_t)|\, L(\gamma_t)^{-1/2}$, we find

$$\|\partial_t\tilde{\gamma}_t\|_{L^2([0,1])} \le \frac{2(1+\bar{\phi})}{I_\Omega(\eta)^{1/2}}\, \|\partial_t\gamma_t\|_{L^2(ds_{\gamma_t})} + 2\|\partial_t\gamma_t\|^2_{L^2(ds_{\gamma_t})} .$$

Using the fact that $\frac{d}{dt}L(\gamma_t) = -\|\partial_t\gamma_t\|^2_{L^2(ds_{\gamma_t})}$ and integrating, we find

$$\int_{t_1}^{t_2} \|\partial_t\tilde{\gamma}_t\|_{L^2([0,1])}\, dt \le \frac{2(1+\bar{\phi})}{I_\Omega(\eta)^{1/2}} \int_{t_1}^{t_2} \|\partial_t\gamma_t\|_{L^2(ds_{\gamma_t})}\, dt + 2(L(t_1) - L(t_2)) . \tag{4.10}$$

Combining (4.10) with (4.7) and using that $L(t_1) - L(t_2) \le \bar{L}^{1/2}(L(t_1) - L(t_2))^{1/2}$ completes the proof of (4.9).

As (4.7) and (4.9) are in particular applicable for $t_1 = 0$ and $t_2 = \infty$, and in this case yield the claimed estimate (1.11), it remains to prove the asymptotic convergence (1.10).

Step 3. We recall that it is one of the main results of [23] that for global-in-time solutions with bounded turning angle $|\bar{\kappa}(t)| \le c_0$ and $L(\gamma_t) \ge c_1 > 0$ for all $t \in [0, \infty)$, all derivatives of the curves are bounded $|\partial_t^k \partial_p^l \tilde{\gamma}| \le C(k, l, \Sigma, c_0, c_1, L(\gamma_0))$ on $[0, 1] \times [1, \infty)$, see also Theorem 2.11 in [24].

At the same time, from the Sobolev inequality and Gagliardo-Nirenberg interpolation inequalities (see, e.g. [28, p.125]), for any $k \in \mathbb{N}$, there is a constant $C = C(k)$ such that

$$\|u\|_{C^k[0,1]} \le C\|u\|_{H^{k+1}([0,1])} \le C(\|u\|^{1/2}_{H^{2k+2}([0,1])}\, \|u\|^{1/2}_{L^2([0,1])} + \|u\|_{L^2([0,1])}) \tag{4.11}$$

for any $u \in H^{2k+2}([0, 1])$. As the estimate (4.9) obtained in step 2 allows us to bound

$$\|\tilde{\gamma}_{t_1} - \tilde{\gamma}_{t_2}\|_{L^2([0,1])} \le \int_{t_1}^{t_2} \|\partial_t\tilde{\gamma}_t\|_{L^2([0,1])}\, dt \le C(L(t_1) - L(t_2))^{1/2} \tag{4.12}$$

we can thus apply (4.11) and the aforementioned derivative estimates to γ_t for $t \ge 1$ to find that

$$\|\tilde{\gamma}_{t_1} - \tilde{\gamma}_{t_2}\|_{C^k([0,1])} \le C_k(L(t_1) - L(t_2))^{1/4} \text{ for any } k \text{ and for all } 1 \le t_1 < t_2 < \infty . \tag{4.13}$$

In particular, the curves γ_t are Cauchy with respect to any C^k norm, and so converge smoothly to a unique limit γ_∞, which must be an element of C^*_η since $\varepsilon(\gamma_t) \to 0$, compare (1.6). Thanks to (4.13), it remains to show that $L(t) - L(\infty)$ decays exponentially as $t \to \infty$. Since this trivially holds for t less than a fixed constant t_1 and suitably chosen C, it suffices to consider times $t \ge t_1$ with t_1 chosen below.

If $L(t) = L(\infty)$ for some $t < \infty$, then this trivially holds by monotonicity (4.1) of the length. Otherwise, we let m be so that $L(\infty) = \ell_m(\eta)$ (recall (4.6)) and choose $t_1 > 1$ sufficiently large so that $L(t_1) \le \frac{1}{2}(\ell_m(\eta) + \ell_{m+1}(\eta))$. This allows us to apply the the Łojasiewicz inequality (4.5) with $\ell_*(t) = \ell_m(\eta) = L(\infty)$ on all of $[t_1, \infty)$.

This ensures that for any $t \ge t_1$

$$-\frac{d}{dt} \log (L(t) - L(\infty)) = \frac{-\frac{d}{dt}L(t)}{L(t) - L(\infty)} = \frac{\int (\kappa_{\gamma_t} - \bar{\kappa}_{\gamma_t})^2 ds_{\gamma_t}}{L(t) - L(\infty)} \ge C_1^{-1} .$$

Integrating this over $[t_1, t]$ yields $-\log(L(t) - L(\infty)) \geq C_1^{-1}(t-t_1) - \log(L(t_1) - L(\infty)) = C_1^{-1}t - C$ which yields the desired exponential decay $(L(t) - L(\infty)) \leq Ce^{C_1^{-1}t}$ that is required to complete the proof of the theorem. □

5. Quantitative stability

This section is dedicated to the proof of Theorem 1.4. We fix, for the entirety of the section, a convex body Ω whose boundary Σ is a C^2 curve with positively oriented parametrization $\sigma : \mathbb{S}^1 \to \mathbb{R}^2$. Recall the universal constant $\bar{\mathfrak{c}}$ defined in (2.5). Given a set of finite perimeter E in $\mathbb{R}^2 \setminus \Omega$ with $|E| = \eta$, define the isoperimetric deficit

$$\delta_\eta(E) := P(E; \mathbb{R}^2 \setminus \Omega) - I_\Omega(\eta). \tag{5.1}$$

Theorem 1.4 is shown in two steps. First, in Proposition 5.1, we prove Theorem 1.4 for relatively convex sets with acute contact angle and small isoperimetric deficit. More precisely, we call an open set $F \subset \mathbb{R}^2 \setminus \Omega$ *relatively convex* (with respect to $\mathbb{R}^2 \setminus \Omega$) if F is connected and F is the intersection of $\mathbb{R}^2 \setminus \Omega$ and an open convex set in $\mathbb{R}^2$. If F is a relatively convex set whose boundary coincides with Σ on a (connected) set of positive length, then $\partial F \setminus \Sigma$ is a (geometrically convex) rectifiable curve in $\mathbb{R}^2 \setminus \Omega$ with distinct endpoints $x_0, x_1 \in \Sigma$. Let H_* be the half-plane containing $F \setminus \Omega$ with x_0 and x_1 in ∂H_*. The interior contact angle $\alpha_0 \in (0, \pi)$ of $\partial F \setminus \Sigma$ at x_0 is defined as the smallest interior angle of a wedge containing $F \setminus \Omega$ formed by H_* and a half plane H with $x_0 \in \partial H$. The interior contact angle α_1 at x_1 is defined analogously.

Proposition 5.1. *Fix $\eta \in (0, \bar{\mathfrak{c}}\,\kappa_{max}(\Sigma)^{-2}]$ and assume that either Σ is a circle or that the pair (Σ, η) satisfies Assumption 1.1. There are explicit constants $\delta_0 = \delta_0(\Sigma, \eta) > 0$ and $C_0 = C_0(\Sigma, \eta)$ such that the following holds. Let $F \subset \mathbb{R}^2 \setminus \Omega$ be an open, relatively convex set with $|F| = \eta$ whose boundary coincides with Σ on a set of positive length and whose interior contact angles are at most $\pi/2$. If $\delta_\eta(F) \leq \delta_0$, then*

$$\inf_{E_* \in \mathcal{M}_\eta^\Omega} d_H(\partial F, \partial E_*)^2 + \inf_{E_* \in \mathcal{M}_\eta^\Omega} |F \Delta E_*|^2 \leq C_0\, \delta_\eta(F). \tag{5.2}$$

Recall that $\mathcal{M}_\eta^\Omega$ denotes the collection minimizers of (1.1). Proposition 5.1 is shown in Section 5.1. Next, through a by-hand reduction procedure, we show that it is always possible to reduce to the setting of Proposition 5.1.

Proposition 5.2. *Fix $\eta \in (0, \bar{\mathfrak{c}}\,\kappa_{max}(\Sigma)^{-2}]$. There are explicit constants $\delta_1 = \delta_1(\Sigma, \eta) > 0$ and $C_1 = C_1(\Sigma, \eta) > 0$ such that, for any set of finite perimeter E in $\mathbb{R}^2 \setminus \Omega$ with $|E| = \eta$ and $\delta_\eta(E) \leq \delta_1$, there is an open, relatively convex set $F \subset \mathbb{R}^2 \setminus \Omega$ with $|F| = \eta$ whose boundary coincides with Σ on a connected subset of Σ of positive length and whose interior contact angles are at most $\pi/2$ such that*

$$\delta_\eta(F) + |E \Delta F|^2 \leq C_1\, \delta_\eta(E)\,. \tag{5.3}$$

Moreover, if $\partial E \setminus \Omega$ is a rectifiable curve, then

$$d_H(\partial E, \partial F)^2 \leq C_1\, \delta_\eta(E)\,. \tag{5.4}$$

Proposition 5.2 is shown in Section 5.4. Once we have Proposition 5.1 and Proposition 5.2, Theorem 1.4 follows in a straightforward manner:

Proof of Theorem 1.4. Let $\bar{\delta} = \min\{\delta_0/C_1, \delta_1\}$ where δ_0 is from Proposition 5.1 and δ_1 and C_1 are from Proposition 5.2. Since $|E \Delta E_*| \leq 2\eta$ for any $E_* \in \mathcal{M}_\eta^\Omega$, the first statement (1.12) in Theorem 1.4 holds trivially when $\delta_\eta(E) > \bar{\delta}$ by choosing $c \leq \bar{\delta}/(4\eta^2)$.

The second statement (1.13) also holds trivially when $\delta_\eta(E) > \bar{\delta}$, after the following small argument. Up to a translation, assume $0 \in \Omega^\circ$. Choose ρ_0 such that $\Omega \subset B_{\rho_0}$ and let $\rho = \max\{\rho_0 + 2(2\eta/\pi)^{1/2}, 2I_\Omega(\eta)\}$. From (2.2), we know any minimizer $E_* \in \mathcal{M}_\eta^\Omega$ is contained in B_ρ. Let E be a set as in the second part of Theorem 1.4, and first consider the case that $E \setminus B_{2\rho}$ is nonempty. Let $R_E > 2\rho$ be the smallest radius such that $E \subset B_{R_E}$. Then $d_H(\partial E, \partial E_*)^2 \leq 4R_E^2$ for any $E_* \in \mathcal{M}_\eta^\Omega$, while by the connectedness of ∂E, we have $P(E; \mathbb{R}^2 \setminus \Omega)^2 - I_\Omega(\eta)^2 \geq 4(R_E - \rho)^2 - (\rho/2)^2 \geq \frac{15}{16}R_E^2 \geq \frac{1}{2}R_E^2$. Thus (1.13) holds for such a set with $c = 1/8$. Next consider the case when $E \subset B_{2\rho}$. In this case $d_H(\partial E, \partial E_*) \leq 4\rho$, and thus if $\delta_\eta(E) > \bar{\delta}$, the estimate holds by taking $c = \bar{\delta}/(16\rho^2)$.

We henceforth assume $\delta_\eta(E) \leq \bar{\delta}$. Let F be the set obtained by applying Proposition 5.2 to E. Observe that $\delta_\eta(F) \leq C_1\delta_\eta(E) \leq \delta_0$ thanks to (5.3) and the choice of $\bar{\delta}$, and thus F satisfies the assumptions of Proposition 5.1. So, letting

$F_* \in \mathcal{M}_\eta^\Omega$ achieve the infimum in $\inf_{E_*\in\mathcal{M}_\eta^\Omega} |F\Delta E_*|^2$ and combining (5.2) and (5.3) yields

$$\inf_{E_*\in\mathcal{M}_\eta^\Omega} |E\Delta E_*|^2 \le |E\Delta F_*|^2 \le 2|F\Delta F_*|^2 + 2|E\Delta F|^2 \le 2C_0\delta_\eta(F) + 2C_1\delta_\eta(E) \le (2C_0C_1 + 2C_1)\delta_\eta(E).$$

Thus the first statement (1.12) of Theorem 1.4 holds with $c = \min\{\bar\delta/(4\eta^2), 1/(2C_0C_1 + 2C_1)\}$. When ∂E is a rectifiable Jordan curve, the second statement (1.13) of Theorem 1.4 follows analogously using (5.4). □

5.1. **Proof of Proposition 5.1.** To prove Proposition 5.1, we need three preparatory lemmas. The first lets us approximate a set F as in the statement of Proposition 5.1 by a set bounded by Σ and a convex $C^{2,\alpha}$ curve meeting Σ orthogonally.

Lemma 5.3. *Fix $\eta \in (0, \bar{c}\,\kappa_{max}(\Sigma)^{-2}]$ and $\alpha \in (0,1)$ and let $\delta_0 = \delta_0(\eta, \kappa_{max}(\Sigma))$ be chosen according to Lemma 2.1. Let $F \subset \mathbb{R}^2 \setminus \Omega$ be an open, relatively convex set with $|F| = \eta$ and $\delta_\eta(F) \le \delta_0$ such that $\partial F \setminus \Sigma$ meets Σ with contact angles at most $\pi/2$. Then for any $\varepsilon > 0$, there is an open, relatively convex set $F_\varepsilon \subset \mathbb{R}^2 \setminus \Omega$ with $|F_\varepsilon| = \eta$ for which $\partial F_\varepsilon \setminus \Sigma$ is given by a convex curve of class $C^{2,\alpha}$ which meets Σ orthogonally at the endpoints and which is so that*

$$d_H(\partial F, \partial F_\varepsilon) + |F\Delta F_\varepsilon| + |\delta_\eta(F) - \delta_\eta(F_\varepsilon)| \le \varepsilon.$$

The proof of Lemma 5.3 is postponed to Appendix B. The next lemma will let us estimate the symmetric difference between the regions enclosed by curves γ_t at different times along the flow.

Lemma 5.4. *Fix $\bar L > 0$ and let $\gamma_t \in \mathcal{B}$, $t \in [t_1, t_2]$, be a smooth family of embedded curves of class C^2 with $L(\gamma_t) \le \bar L$ such that γ_t meets Σ orthogonally at its endpoints. Let E_t be the open bounded set bounded by γ_t and the sub-arc σ_{γ_t} of Σ described in Definition 2.4 and let $V_t = \partial_t\gamma_t \cdot \nu_{\gamma_t}$ denote the normal velocity. Then*

$$|E_{t_2}\Delta E_{t_1}| \le \bar L^{\frac12} \int_{t_1}^{t_2} \|V_t\|_{L^2(ds_{\gamma_t})}\, dt.$$

Proof. Fix $t_0 \in (t_1, t_2)$. Since the curves $\gamma_{t_0+\varepsilon}$ and $\sigma_{\gamma_{t_0+\varepsilon}}$ vary smoothly in ε, so does the area of the region $R_\varepsilon = E_{t_0+\varepsilon}\Delta E_{t_0}$. We claim that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} |R_\varepsilon| \le \int_{\gamma_{t_0}} |V_{t_0}|\, ds_{\gamma_{t_0}}. \tag{5.5}$$

Indeed, let $L = L(\gamma_{t_0})$ and reparametrize each γ_t by constant speed on $[0, L]$. Since

$$\gamma_{t_0+\varepsilon}(p) = \gamma_{t_0}(p) + \varepsilon\, \partial_t\gamma_t(p)\Big|_{t=t_0} + o(\varepsilon)$$

for $p \in [0, L]$, and since tangential motion does not affect the region enclosed by the curve, we have $|R_\varepsilon\Delta S_\varepsilon| = o(\varepsilon)$ where

$$S_\varepsilon := \{\gamma_{t_0}(p) + q\nu_{t_0}(p) : p \in [0, L],\ q \in (-\varepsilon V^-(p), \varepsilon V^+(p))\}.$$

Here $V^\pm(p) := \max(\pm V_{t_0}(p), 0)$, so the interval takes the form $[0, \varepsilon V_{t_0}(p)]$ or $[\varepsilon V_{t_0}(p), 0]$. Since γ_{t_0} is smooth and embedded, there exists $\varepsilon_0 > 0$ such that the map $\Psi(p, q) = \gamma_{t_0}(p) + q\nu_{t_0}(p)$ is a diffeomorphism from $[0, L] \times (-\varepsilon_0, \varepsilon_0)$ onto its image. By the area formula,

$$|S_\varepsilon| = \int_0^L \int_{-\varepsilon V^-(p)}^{\varepsilon V^+(p)} \det D\Psi(p, q)\, dq\, dp = \varepsilon \int_0^L |V_{t_0}(p)|\, dp + o(\varepsilon).$$

In the second identity we use the fact that the Jacobian $\det D\Psi(p, q) = 1 - q\kappa_{\gamma_{t_0}}(p)$ satisfies $\det D\Psi(p, q) = 1 + O(q)$. Dividing by ε and passing $\varepsilon \to 0$ establishes (5.5).

Since (5.5) holds for every $t_0 \in [t_1, t_2]$, integrating with respect to t_0 shows that

$$|E_{t_2}\Delta E_{t_1}| \le \int_{t_1}^{t_2} \|V_t\|_{L^1(ds_{\gamma_t})}\, dt.$$

The lemma then follows from Hölder's inequality and the assumed bound $L(\gamma_t) \le \bar L$. □

The final preparatory lemma lets us upgrade from L^2 control to H^1 control of the displacement along the flow. The lemma holds in greater generality than stated, but we only state and prove it in the setting where it is applied.

Lemma 5.5. *Fix $\eta \in (0, \bar{c}\kappa_{max}(\Sigma)^{-2}]$ and let $\delta_0 = \delta_0(\eta, \kappa_{max}(\Sigma))$ be chosen according to Lemma 2.1. There exists a constant $C = C(\eta, \Sigma)$ such that the following holds. Let $\gamma \in \mathcal{B}_\eta$ be an embedded curve with $L(\gamma) \leq I_\Omega(\eta) + \delta_0$ meeting Σ orthogonally and bounding an open, relatively convex set in $\mathbb{R}^2 \setminus \Omega$. Let $\gamma_* \in C^*_\eta$ be the relative boundary $\partial E_* \setminus \Omega$ for some $E_* \in \mathcal{M}^\Omega_\eta$, and suppose both curves are parametrized by constant speed on $[0,1]$ and oriented so that the normal to the curve coincides with the inward unit normal of the bounded set. Then*

$$\int_0^1 |\gamma'(p) - \gamma_*'(p)|^2 \, dp \leq C\,(L(\gamma) - L(\gamma_*)) + C \int_0^1 |\gamma(p) - \gamma_*(p)|^2 \, dp. \tag{5.6}$$

Proof. Let $w(p) := \gamma(p) - \gamma_*(p)$. First, let us see the information we get from the fact that γ and γ_* both enclose area η. Recall from (2.10) that $A_\Sigma(\gamma) = -\frac{1}{2}\int_0^1 \gamma \cdot J\gamma' \, dp - \frac{1}{2}\int_{\sigma_\gamma} \sigma_\gamma \cdot J\sigma_\gamma' \, ds_{\sigma_\gamma}$ where σ_γ is as in Definition 2.4. Since E is relatively convex by assumption, σ_γ coincides with the projection of $-\gamma$ onto Σ. The analogue holds for the subarc $\sigma_* = \sigma_{\gamma_*}$.

If the traces of σ_γ and σ_* do not intersect, then γ and γ_* lie in two separate half spaces. Denote by l_0 a line dividing them and $\Pi_{l_0} = \Pi_0$ the projection onto the line l_0. Then $\int_0^1 |\gamma - \gamma_*|^2 \geq \int_0^1 |\gamma_* - \Pi_0(\gamma_*)|^2 = \frac{1}{I_\Omega(\eta)} \int |\gamma_* - \Pi_0(\gamma_*)|^2 \, ds_{\gamma_*}$, which is bounded below by the corresponding quantity for a semi-circle of the same radius and its diameter. Keeping in mind (2.1) and $\pi r^2 \geq \eta$, compare (2.2), we hence get $\int_0^1 |\gamma - \gamma_*|^2 \geq \frac{1}{I_\Omega(\eta)} r^2 \int_0^\pi \sin^2 \geq \frac{1}{2\pi^{1/2}\eta^{1/2}} \frac{\eta}{\pi} \frac{\pi}{2} = \eta^{1/2}/(4\pi^{1/2})$. Since the left-hand side of (5.6) is bounded above by $4(I_\Omega(\eta) + \delta_0)^2$, the estimate (5.6) holds trivially in this case.

We can thus assume the traces of σ_γ and σ_* intersect nontrivially. To prove the claim in this main case, it is now convenient to fix the coordinate system so that the center z_{γ_*} of the circular arc γ_* is at the origin. This ensures that $\gamma_*(i)$ is normal to $\gamma_*'(i)$, $i = 0, 1$. As γ_* intersects Σ orthogonally, this hence allows us to use that $\gamma_*(i)$ is normal to $\nu_\Sigma(\gamma_*(i))$ in the following proof. As a first step we show that

$$I := \Big| \int_{\sigma_\gamma} \sigma_g \cdot J\sigma_g' \, ds_{\sigma_\gamma} - \int_{\sigma_*} \sigma_* \cdot J\sigma_*' \, ds_{\sigma_*} \Big| \leq C(|w(0)|^2 + |w(1)|^2) \tag{5.7}$$

To see this we first note that the integrands $\sigma_\gamma \cdot J\sigma_\gamma'$ and $\sigma_* \cdot J\sigma_*'$ coincide on the intersection of their traces. The symmetric difference between their traces is parametrized by two sub-arcs σ_0 and σ_1 of Σ such that the endpoints of σ_0 are $\gamma(0)$ and $\gamma_*(0)$, and the endpoints of σ_1 are $\gamma(1)$ and $\gamma_*(1)$. Moreover, since $\sigma_\gamma \cap \sigma_* \neq \emptyset$, we can bound since $L(\sigma_0) \leq I_\Omega(\eta) + \delta_0 < d_\Sigma/2$ by Remark 2.3. Hence there is an explicit constant C depending on Σ such that $L(\sigma_0) \leq C|\gamma(0) - \gamma_*(0)| = C|w(0)|$. Similarly, $L(\sigma_1) \leq C|w(1)|$. The above quantity I can hence be bounded by

$$I \leq \Big| \int_{\sigma_0} \sigma_0 \cdot J\sigma_0' \, ds_{\sigma_0} \Big| + \Big| \int_{\sigma_1} \sigma_1 \cdot J\sigma_1' \, ds_{\sigma_1} \Big| \leq L(\sigma_0) \sup_{\sigma_0} |\sigma_0 \cdot \nu_\Sigma(\sigma_0)| + L(\sigma_1)) \sup_{\sigma_1} |\sigma_1 \cdot \nu_\Sigma(\sigma_1)| \tag{5.8}$$

As remarked above, our choice of coordinate system ensures that $\gamma_*(0)$ is orthogonal to $\nu_\Sigma(\gamma_0^*(0))$. The inner product $\sigma_0 \cdot \nu_\Sigma(\sigma_0)$ hence vanishes at one of the endpoints of σ_0, namely at $\gamma_*(0)$. Since Σ is a fixed C^2 curve, this ensures that $\sup_{\sigma_0} |\sigma_0 \cdot \nu_\Sigma(\sigma_0)| \leq \operatorname{osc}_{\sigma_0} |\sigma_0 \cdot \nu_\Sigma(\sigma_0)| \leq CL(\sigma_0)$ for a constant $C = C(\Sigma)$ that only depends on an upper bound on the C^2 norm of (the arclength parametrisation) of Σ. Inserting this, and the analogue bound on σ_1, into (5.8), immediately yields the claimed estimate (5.7).

To address the other term coming from the difference of areas, we add and subtract terms, the integrate by parts, and use the identity $Ja \cdot b = -a \cdot Jb$ to find

$$\begin{aligned} \int_0^1 \gamma \cdot J\gamma' - \int_0^1 \gamma_* \cdot J\gamma_*' &= \int_0^1 w \cdot Jw' + \int_0^1 w \cdot J\gamma_*' + \int_0^1 \gamma_* \cdot Jw' \\ &= \int_0^1 w \cdot Jw' + \int_0^1 w \cdot J\gamma_*' - \int_0^1 \gamma_*' \cdot Jw + (\gamma_* \cdot Jw)\Big|_0^1 \\ &= \int_0^1 w \cdot Jw' + 2\int_0^1 w \cdot J\gamma_*' + (\gamma_* \cdot Jw)\Big|_0^1. \end{aligned} \tag{5.9}$$

While we could of course estimate the boundary terms $\gamma_*(i) \cdot Jw(i) = \gamma_*(i) \cdot J(\gamma_*(i) - \gamma(i))$ by a multiple of $|w(i)|$, $i = 0, 1$, this would not suffice to prove our result. Instead, we can exploit that

$$w(i) = \gamma(i) - \gamma_*(i) = \pm L(\sigma_i)\tau_\Sigma(\gamma_*(i)) + err_i \tag{5.10}$$

(with $\pm$ chosen according to the orientation of the subarc σ_i from $\gamma_*(i)$ to $\gamma(i)$), for an error term that is bounded by $|err_i| \leq L(\sigma_i)\text{osc}_{\sigma_i}\tau_\Sigma \leq CL(\sigma_i)^2 \leq C|w(i)|^2$. As our choice of coordinate system ensures that $\gamma_*(i)$ is orthogonal to $\nu_\Sigma(\gamma_*(i)) = J\tau_\Sigma(\gamma_*(i))$, we can hence indeed bound $|\gamma_*(i) \cdot Jw(i)| \leq C|w(i)|^2$

Inserting this into (5.9), rearranging this identity and applying the bounds above and the fact that $2(A_\Sigma(\gamma) - A_\Sigma(\gamma_*)) = 0$ by assumption, we hence find

$$2\Big|\int_0^1 w \cdot J\gamma_*'\Big| \leq \|w\|_{L^2}\|w'\|_{L^2} + C(\,|w(0)|^2 + |w(1)|^2). \tag{5.11}$$

Now we turn to the main estimate. Letting $\ell = L(\gamma)$ and $\ell_* = L(\gamma_*)$ and using the fact that both curves are parametrized with constant speed, we have $\ell^2 = \ell_*^2 + 2\,\gamma_*'(p) \cdot w'(p) + |w'(p)|^2$. Rearranging and integrating over $[0, 1]$ gives

$$\int_0^1 |w'|^2 dp = (\ell^2 - \ell_*^2) - 2\int_0^1 \gamma_*' \cdot w'\,dp. \tag{5.12}$$

We integrate the second term by parts, using that $\gamma_*' = \ell_*\tau_{\gamma_*}$ and $\gamma_*''(p) = \ell_*\kappa_* J\gamma_*'(p)$ for the (constant) curvature κ_* of γ_*:

$$-2\int_0^1 \gamma_*' \cdot w'\,dp = 2\int_0^1 \gamma_*'' \cdot w\,dp - 2\gamma_*' \cdot w\Big|_0^1 = 2\ell_*\kappa_* \int_0^1 J\gamma_*' \cdot w\,dp - 2\ell_*\tau_{\gamma_*} \cdot w\Big|_0^1. \tag{5.13}$$

As above, the boundary terms can be controlled by $C(|w(0)|^2 + |w(1)|^2)$ since τ_{γ_*} is normal to τ_Σ at the endpoints and since w can be written as in (5.10) So, applying (5.11) gives us

$$2\Big|\int_0^1 \gamma_*' \cdot w'\,dp\Big| \leq \ell_*\kappa_*\|w\|_{L^2}\|w'\|_{L^2} + C(|w^2(0)| + |w^2(1)|) \tag{5.14}$$

where C depends only on Σ and η (from (2.1) we can bound κ_* above in terms of η). We estimate the boundary terms using the Sobolev embedding (which simply follows from the fundamental theorem of calculus and Hölder's inequality):

$$\|w\|_{C^0}^2 \leq 2\|w\|_{L^2}\|w'\|_{L^2} + \|w\|_{L^2}^2.$$

Substituting these estimates in (5.12) and using $\ell^2 - \ell_*^2 \leq (2I_\Omega(\eta) + \delta_0)(\ell - \ell_*)$, we find

$$\int_0^1 |w'|^2 dp \leq (2I_\Omega(\eta) + \delta_0)(\ell - \ell_*) + C'(\|w\|_{L^2}\|w'\|_{L^2} + \|w\|_{L^2}^2) \tag{5.15}$$

for $C' = C'(\Sigma, \eta)$. Applying Young's inequality and absorbing the resulting term $\frac{1}{2}\int |w'|^2$ completes the proof. □

We are now ready to prove Proposition 5.1.

Proof of Proposition 5.1. Thanks to Lemma 5.3, it suffices to prove the proposition when $\partial F \setminus \Sigma$ is parametrized by an embedded convex $C^{2,\alpha}$ curve $\gamma \in \mathcal{B}_\eta$ meeting Σ orthogonally at the endpoints.

Taking $\delta_0 = \delta_0(\eta, \kappa_{max}(\Sigma)) > 0$ as in Lemma 2.1, the assumption $\delta_\eta(F) \leq \delta_0$ and Lemma 2.1 together ensure that γ satisfies the hypotheses of Theorem 4.1. Theorem 4.1 guarantees the existence of a global-in-time solution $\{\gamma_t\}$ to the free boundary area-preserving curve shortening flow with initial data $\gamma = \gamma_0$ such that $|\int \kappa ds_{\gamma_t}| \leq 2\pi$ and $\gamma_t \in \mathcal{B}_\eta$ for all $t \geq 0$.

By Theorem 1.3, there is a unique arc $c^* \in C_\eta^*$ such that γ_t converges (smoothly, exponentially) to c^*. Since $L(c^*) \leq L(\gamma)$, c^* is a minimizer of L in $\mathcal{B}_\eta$ provided we choose $\delta_0 < \ell_1(\eta, \Sigma) - I_\Omega(\eta)$, where $\ell_1(\eta, \Sigma) > I_\Omega(\eta)$ is the lowest energy level of a non-minimizing critical point. Let $F_* \in \mathcal{M}_\eta^\Omega$ denote the set bounded by c^* and Σ. By (1.11) and Lemma 5.4 (passing $t_1 \to 0$ and $t_2 \to \infty$), we find

$$|F\Delta F_*| \leq \int_0^\infty \|\partial_t\gamma_t\|_{L^2(ds_{\gamma_t})}\,dt \leq C\delta_\eta(F)^{1/2}.$$

Next, to bound $d_H(\partial F, \partial F_*)$, let $\tilde{\gamma}, \tilde{\gamma}_t$, and $\tilde{c}^*$ be the constant speed parametrizations of γ, γ_t, and c^* on $[0, 1]$. We apply the Sobolev inequality and Lemma 5.5 to find

$$\|\tilde{c}^* - \tilde{\gamma}\|_{C^0([0,1])}^2 \leq C\|(\tilde{c}^*)' - \tilde{\gamma}'\|_{L^2([0,1])}^2 + C\|\tilde{c}^* - \tilde{\gamma}\|_{L^2([0,1])}^2 \leq C\delta_\eta(F) + C\|\tilde{c}^* - \tilde{\gamma}\|_{L^2([0,1])}^2. \tag{5.16}$$

Then, by (1.11) we find

$$\|\tilde{c}^* - \tilde{\gamma}\|_{L^2([0,1])} = \Big\|\int_0^\infty \partial_t\tilde{\gamma}_t\,dt\Big\|_{L^2([0,1])} \leq \int_0^\infty \|\partial_t\tilde{\gamma}_t\|_{L^2([0,1])}dt \leq C\delta_\eta(F)^{1/2}.$$

This completes the proof of the proposition. □

5.2. **Reduction to a set bounded by a rectifiable curve.** The remainder of the paper is dedicated to proving Proposition 5.2. The first step is to replace E with a simply connected set. To do so, we need the following quantitative sub-additivity estimate for the isoperimetric profile.

Lemma 5.6. *Fix $\eta > 0$. There is a positive constant $c_0 = c_0(\kappa_{max}(\Sigma), \eta)$ such that the following holds. Let $\{\eta_i\}_{i\in I}$ be a non-increasing finite or countable sequence of positive numbers with $\sum_{i\in I} \eta_i = \eta$. Then*

$$\sum_{i\geq 1} I_\Omega(\eta_i) - I_\Omega(\eta) \geq c_0 \sum_{i\geq 2} \eta_i^{1/2}. \tag{5.17}$$

Proof. If the index set I has cardinality 1, there is nothing to show, so we assume it is at least 2. For each $i \in I$, let r_i denote the radius of a minimizer of $I_\Omega(\eta_i)$.[4] Up to reindexing, we may assume that $r_1 \geq r_2 \geq \dots$. It suffices to show (5.17) for this reindexed sequence, since for the two sequences, the left-hand sides of (5.17) are equal and the right-hand sides are ordered.

Set $\bar\varepsilon = \pi^{-1} \arctan(1/(r_1\kappa_{max}(\Sigma)))$. The bound (2.3) guarantees that $I_\Omega(\eta_i) \leq 2\pi(1-\bar\varepsilon)r_i$ for each $i \in I$. Combining this with the lower bound (2.1) on the isoperimetric profile shows that

$$\eta_i \leq \frac{I_\Omega(\eta_i)^2}{2\pi} \leq (1-\bar\varepsilon)I_\Omega(\eta_i)r_i \leq (1-\bar\varepsilon)I_\Omega(\eta_i)r_1. \tag{5.18}$$

We construct a competitor for the area-η isoperimetric problem as follows. Up to a rotation and translation, we may assume that a minimizer E_1 of $I_\Omega(\eta_1)$ is bounded by Σ and a circular arc of radius r_1 centered at the origin with one endpoint at $(r_1, 0)$ and the other endpoint in the third quadrant. For each $i \geq 2$, let $\ell_i = \eta_i/r_1$, so that $\ell_i \leq (1-\bar\varepsilon)\, I_\Omega(\eta_i)$. Define the rectangle $R = (-r_1, r_1)\times(0, \sum_{i\geq 2} \ell_i/2)$. Then, letting $H^\pm = \{(x, y) : \pm y > 0\}$ and $E_1^\pm = E_1 \cap H^\pm$, consider the set

$$F = E_1^- \cup R \cup (E_1^+ + (0, \textstyle\sum_{i\geq 2} \ell_i/2)).$$

The convexity of Ω guarantees that $F \subset \mathbb{R}^2 \setminus \Omega$, and by construction $|F| = \sum_{i\in I} \eta_i = \eta$ and

$$P(F;\mathbb{R}^2 \setminus \overline{\Omega}) = I_\Omega(\eta_1) + \sum_{i\geq 2} \ell_i \leq I_\Omega(\eta_1) + (1-\bar\varepsilon) \sum_{i\geq 2} I_\Omega(\eta_i).$$

Using $P(F;\mathbb{R}^2 \setminus \overline{\Omega}) \geq I_\Omega(\eta)$ and applying the lower bound $I_\Omega(\eta_i) \geq (2\pi\eta_i)^{1/2}$ from (2.1) once again, the desired estimate (5.17) follows with the constant

$$(2\pi)^{\frac12}\bar\varepsilon \geq \Big(\frac{2}{\pi}\Big)^{\frac12} \arctan\Big(\Big(\frac{\pi}{2\eta}\Big)^{\frac12} \kappa_{max}(\Sigma)^{-1}\Big) =: c_0.$$

This completes the proof. □

Lemma 5.7. *Fix $\eta > 0$. There exist positive constants $C_2 = C_2(\kappa_{max}(\Sigma), \eta)$ and $\delta_2 = \delta_2(\kappa_{max}(\Sigma), \eta)$ such that the following holds. Let $E \subset \mathbb{R}^2 \setminus \Omega$ be a set of finite perimeter with $|E| = \eta$ and $\delta_\eta(E) \leq \delta_2$. Then there is a connected open set F in $\mathbb{R}^2 \setminus \Omega$ with $|F| \geq \eta$ whose boundary is a rectifiable Jordan curve, coinciding with Σ on a connected, positive $\mathcal{H}^1$-measure set, such that*

$$|E\Delta F|^{\frac12} \leq C_2\, \delta_\eta(E) \quad \text{and} \quad P(F;\mathbb{R}^2 \setminus \Omega) \leq P(E;\mathbb{R}^2 \setminus \Omega) + C_2\, \delta_\eta(E)\,. \tag{5.19}$$

Proof. Step 1: Recall that a set of finite perimeter G is said to be indecomposable if $|G_1|\,|G_2| = 0$ for any disjoint sets G_1, G_2 such that $G = G_1 \cup G_2$ and $P(G) = P(G_1) + P(G_2)$. By [1, Theorem 1], E admits a unique decomposition as the union of at most countably many pairwise disjoint indecomposable sets $\{E_i\}_{i\in I}$ such that $|E_i| > 0$ and $P(E) = \sum_{i\in I} P(E_i)$. It follows from [1, Proposition 3] and Federer's and De Giorgi's theorems [25, Theorem 16.2, Theorem 15.9] that $P(E;\mathbb{R}^2 \setminus \Omega) = \sum_{i\in I} P(E_i;\mathbb{R}^2 \setminus \Omega)$ as well. Therefore, letting $\eta_i = |E_i|$ and reindexing so the η_i are non-increasing, Lemma 5.6 implies that

$$\delta_\eta(E) \geq \sum_{i\in I} I_\Omega(\eta_i) - I_\Omega(\eta) \geq c_0 \sum_{i\geq 2} \eta_i^{\frac12} \geq c_0\Big(\sum_{i\geq 2} \eta_i\Big)^{\frac12} = c_0(\eta-\eta_1)^{\frac12}. \tag{5.20}$$

Here $c_0 = c_0(\kappa_{max}(\Sigma), \eta)$ is the constant from Lemma 5.6 and the final inequality follows from concavity. Therefore, the indecomposable set E_1 satisfies

$$\delta_\eta(E) \geq c_0(\eta-\eta_1)^{\frac12} = c_0|E\Delta E_1|^{\frac12} \quad \text{and} \quad P(E_1;\mathbb{R}^2 \setminus \overline{\Omega}) \leq P(E;\mathbb{R}^2 \setminus \overline{\Omega})\,. \tag{5.21}$$

We choose $\delta_2 \leq c_0\eta^{1/2}/2$, so that the first estimate in (5.21) guarantees that $\eta_1 \geq \eta/2$.

[4] While not needed for the proof, we recall from Section 2 that for almost every η, there is a unique such r.

We claim that $P(E_1;\mathbb{R}^2\setminus\Omega) < P(E_1)$, which in turn guarantees that ∂E_1 intersects Σ on a set of positive $\mathcal{H}^1$ measure and that $E_1\cup\Omega$ is indecomposable. To see this, take a ball of area $\eta - |E_1| = |E\Delta E_1|$ at positive distance from E_1 and Ω. By the first bound in (5.21), this ball has perimeter at most $C\delta_\eta(E)$ for $C = C(\kappa_{max}(\Sigma),\eta)$. So, using the second bound in (5.21), the union $\tilde{E}$ of E_1 and this ball has perimeter

$$P(\tilde{E};\mathbb{R}^2\setminus\Omega) \le I_\Omega(\eta) + C\delta_2 < 2\pi^{\frac{1}{2}}\eta^{\frac{1}{2}}.$$

Here, the second inequality holds thanks to the upper bound (2.4) for the isoperimetric profile, provided we choose $\delta_2 \le \frac{1}{C}\,2\pi^{\frac{1}{2}}\eta^{\frac{1}{2}}\{1-(1-\pi^{-1}\arctan((\frac{\pi}{2\eta})^{\frac{1}{2}}\kappa_{max}(\Sigma)^{-1})^{\frac{1}{2}}\}$. On the other hand, $P(\tilde{E}) \ge 2\pi^{1/2}\eta^{1/2}$ by the isoperimetric inequality. Thus $P(\tilde{E};\mathbb{R}^2\setminus\Omega) < P(\tilde{E})$, which by definition of $\tilde{E}$ proves the claim.

Step 2: Next we "fill in the holes" of $E_1\cup\Omega$. More specifically, since $E_1\cup\Omega$ is indecomposable, [1, Corollary 1] says that the essential boundary $\partial^M(E_1\cup\Omega)$ admits a unique decomposition into at most countably many rectifiable Jordan curves C^+ and $\{C_j^-\}_{j\in J}$ with $\mathrm{int}(C_j^-)\subset\mathrm{int}(C^+)$ such that

$$E_1\cup\Omega = \mathrm{int}(C^+)\setminus\bigcup_{j\in J}\mathrm{int}(C_j^-), \qquad P(E_1\cup\Omega) = \mathcal{H}^1(C^+) + \sum_{j\in J}\mathcal{H}^1(C_j^-).$$

Let $G = \mathrm{int}(C^+)$ and $F_1 = G\setminus\Omega$. Notice that by construction, $F_1\supset E_1$, $\partial F_1\setminus\Sigma = C^+\setminus\Sigma$ is a single rectifiable curve in $\mathbb{R}^2\setminus\Omega$ with endpoints on Σ, and $\partial F_1\cap\Sigma = \Sigma\setminus C^+$ is a connected set with positive $\mathcal{H}^1$ measure. Letting $\hat{C}^+ := C^+\setminus\Sigma$ and $\hat{C}_j^- = C_j^-\setminus\Sigma$ for each $j\in J$, it follows from the decomposition above together with Federer's and De Giorgi's theorems that $P(E_1;\mathbb{R}^2\setminus\Omega) = \mathcal{H}^1(\hat{C}^+) + \sum_{j\in J}\mathcal{H}^1(\hat{C}_j^-)$ and hence

$$P(F_1;\mathbb{R}^2\setminus\Omega) \le P(E_1;\mathbb{R}^2\setminus\Omega)\,. \tag{5.22}$$

To bound $|F_1\Delta E_1|$, let $a_j = |\mathrm{int}(C_j^-)|$, so that $|F_1\Delta E_1|^{1/2} = (\sum_{j\in J}a_j)^{1/2} \le \sum_{j\in J}a_j^{1/2}$. From the lower bound (2.1) on the isoperimetric profile, $(2\pi a_j)^{1/2} \le \mathcal{H}^1(\hat{C}_j^-)$. So, as $|F_1|\ge|E_1| = \eta_1$ and hence $I_\Omega(|F_1|)\ge I_\Omega(\eta_1)$, we have

$$\begin{aligned}(2\pi)^{\frac{1}{2}}|E_1\Delta F_1|^{\frac{1}{2}} &\le P(E_1;\mathbb{R}^2\setminus\Omega) - P(F_1;\mathbb{R}^2\setminus\Omega)\\ &\le P(E_1;\mathbb{R}^2\setminus\Omega) - I_\Omega(\eta_1) \le \delta_\eta(E) + (I_\Omega(\eta) - I_\Omega(\eta_1)).\end{aligned} \tag{5.23}$$

So, recalling from above that $\eta_1\ge\eta/2$, the local Lipschitz estimate (2.8) and (5.20) show that $I_\Omega(\eta) - I_\Omega(\eta_1) \le (\pi/\eta_1)^{1/2}(\eta - \eta_1) \le C\,\delta_\eta(E)$ where $C = \pi^{1/2}/c_0$. Combining this with (5.23), (5.22) and (5.21) shows the existence of $C = C(\kappa_{max}(\Sigma),\eta)$ such that

$$|E\Delta F_1|^{\frac{1}{2}} \le C\delta_\eta(E) \qquad\text{and}\qquad P(F_1;\mathbb{R}^2\setminus\Omega) \le P(E;\mathbb{R}^2\setminus\Omega). \tag{5.24}$$

Step 3: If $|F_1|\ge\eta$, we complete the proof by taking $F = F_1$. Otherwise, as in step 1, take a ball of area $\eta - |F_1|$, and thus of perimeter at most $C\delta_\eta(E)$. Since F_1 is bounded, we may translate this ball from infinity along some ray so that it is disjoint from $\Omega\cup F_1$ and its boundary intersects $\partial F_1\setminus\Sigma$. Then, by a slight deformation of $F_1 = F\cup B$ gives a set satisfying the conclusions of the lemma. □

5.3. **Reduction to a set bounded by a convex curve.** The next step toward Proposition 5.2 is to replace a set of the type obtained in Lemma 5.7 by a relatively convex set with acute contact angle. First we prove the following elementary geometric lemma.

Lemma 5.8. *Fix $\bar{L} > 0$. There exists $C = C(\bar{L}) > 0$ such that the following holds. Let $\gamma : [0,1]\to\mathbb{R}^2$ be a rectifiable curve with $L(\gamma)\le\bar{L}$ and let $\tilde{\gamma} : [0,1]\to\mathbb{R}^2$ be a parametrization of the linear segment joining $\gamma(0)$ and $\gamma(1)$.*

$$d_H(\gamma,\tilde{\gamma})^2 \le C(L(\gamma) - L(\tilde{\gamma})). \tag{5.25}$$

Proof. It suffices to bound the distance between any point $z\in\gamma$ to its projection $\hat{z}$ onto the line through the endpoints x_0 and x_1 of γ. Pythagoras, applied to the triangles $\Delta(x_0,z,\hat{z})$ and $\Delta(x_1,z,\hat{z})$, immediately gives the required bound of

$$\begin{aligned}2|z-\hat{z}|^2 &= |z-x_0|^2 - |\hat{z}-x_0|^2 + |z-x_1|^2 - |\hat{z}-x_1|^2\\ &\le 2\bar{L}\cdot(|z-x_0| + |z-x_1| - [|\hat{z}-x_0| + |\hat{z}-x_1|]) \le 2\bar{L}\cdot(L(\gamma) - L(\tilde{\gamma})).\end{aligned} \tag{5.26}$$

□

Lemma 5.9. *Fix $\eta \in (0, \bar{c}\kappa_{max}(\Sigma)^{-2}]$. There are positive constants $\delta_3 = \delta_3(\eta, \kappa_{max}(\Sigma))$ and $C_3 = C_3(\eta, \kappa_{max}(\Sigma))$ such that the following holds. Let $E \subset \mathbb{R}^2 \setminus \Omega$ be a connected open set with $|E| \geq \eta$ whose boundary is a rectifiable Jordan curve coinciding with Σ on a connected, positive $\mathcal{H}^1$-measure set such that*

$$P(E; \mathbb{R}^2 \setminus \Omega) \leq I_\Omega(\eta) + \delta_3. \tag{5.27}$$

Then there is an open, relatively convex set $F \supset E$ such that $\partial F \setminus \Sigma$ meets Σ with interior angles at most $\pi/2$ and

$$P(F; \mathbb{R}^2 \setminus \Omega) \leq P(E; \mathbb{R}^2 \setminus \Omega), \tag{5.28}$$

$$d_H(\partial E, \partial F)^2 + |E \Delta F| \leq C_3(P(E; \mathbb{R}^2 \setminus \Omega) - I_\Omega(\eta)). \tag{5.29}$$

Proof. Let $\gamma : [0,1] \to \mathbb{R}^2$ be a parametrization of $\partial E \setminus \Sigma$. Note that $\gamma(0), \gamma(1) \in \Sigma$ and $\gamma(p) \notin \Omega$ for $p \in (0,1)$ and that by assumption $L(\gamma) \leq \bar{L} := I_\Omega(\eta) + \delta_3$.

Step 1: Let $\tilde{\Sigma} \subset \Sigma$ be the set of points $x \in \Sigma$ for which the normal ray

$$n_x := \{y \in \mathbb{R}^2 : y = x - t\nu_\Sigma(x) : t \geq 0\}. \tag{5.30}$$

has nontrivial intersection with the trace of γ. Recall we orient Σ positively so that ν_Σ is the inner normal of Σ. The set $\tilde{\Sigma}$ is connected thanks to the continuity of γ. Choose $\delta_3 \leq \delta_0$, where δ_0 is from Lemma 2.1, so that $L(\gamma) \leq d_\Sigma/2$ by (5.27), Lemma 2.1, and Remark 2.3. Using this, a basic geometric argument shows that $\tilde{\Sigma}$ is a proper subset of Σ and the image of $\tilde{\Sigma}$ under the Gauss map of Σ is a connected proper subset of a half circle of $\mathbb{S}^1$.

Let $j_0 < j_1$ be the endpoints of the interval J for which $\tilde{\Sigma}$ is the trace of σ restricted to the interval J, and let $x_0 = \sigma(j_0)$ and $x_1 = \sigma(j_1)$. Assume γ is oriented such that $j_0 \leq j_0' < j_1' \leq j_1$ where $\sigma(j_0') = \gamma(0)$ and $\sigma(j_1') = \gamma(1)$. With this orientation, we have $\underline{a} < \overline{a}$ where

$$\underline{a} = \sup\{p \in [0,1] : \gamma(p) \in n_{x_0}\}$$
$$\overline{a} = \inf\{p \in [0,1] : \gamma(p) \in n_{x_1}\}.$$

Note that $\gamma(\underline{a}) \in n_{x_0}$ and $\gamma(\overline{a}) \in n_{x_1}$. Define the curve $\hat{\gamma} : [0,1] \to \mathbb{R}^2 \setminus \Omega$ by letting $\hat{\gamma} = \gamma$ on $[\underline{a}, \overline{a}]$, and on the (possibly trivial) intervals $[0, \underline{a}]$ and $[\overline{a}, 1]$, letting $\hat{\gamma}$ parametrize the segments joining x_0 to $\gamma(\underline{a})$ and joining $\gamma(\overline{a})$ to x_1 respectively. The convexity of Σ and a simple trigonometric argument show

$$L(\hat{\gamma}) \leq L(\gamma). \tag{5.31}$$

Moreover, we claim that

$$d_H(\hat{\gamma}, \gamma)^2 \leq 3\bar{L}\,(L(\gamma) - L(\hat{\gamma}))\,. \tag{5.32}$$

To see (5.32), first let $\tilde{\gamma} : [0,1] \to \mathbb{R}^2 \setminus \Omega$ be the curve that is equal to γ on $[0,1]$, joins $\gamma(0)$ to $\gamma(\underline{a})$ linearly on $[0, \underline{a}]$, and joins $\gamma(\overline{a})$ to $\gamma(1)$ linearly on $[\overline{a}, 1]$. It is simple to see that $d_H(\hat{\gamma}, \tilde{\gamma})^2 \leq L(\tilde{\gamma})^2 - L(\hat{\gamma})^2 \leq 2\bar{L}(L(\tilde{\gamma}) - L(\hat{\gamma}))$. Next, by Lemma 5.8, $d_H(\tilde{\gamma}, \gamma)^2 \leq \bar{L}(L(\gamma) - L(\tilde{\gamma}))$. Combining these two bounds yields (5.32).

Let $\hat{E}$ be the set bounded by $\hat{\gamma}$ and the segment joining x_0 to x_1. As $|\hat{E} \setminus \Omega| \geq |E \setminus \Omega| \geq \eta$, we have $P(\hat{E}; \mathbb{R}^2 \setminus \Omega) = L(\hat{\gamma}) \geq I_\Omega(\eta)$, so (5.31) and (5.32) imply

$$P(\hat{E}; \mathbb{R}^2 \setminus \Omega) \leq P(E; \mathbb{R}^2 \setminus \Omega), \qquad d_H(\partial E, \partial(\hat{E} \setminus \Omega))^2 \leq C(P(E; \mathbb{R}^2 \setminus \Omega) - I_\Omega(\eta)). \tag{5.33}$$

Note the $E \subset \hat{E}$ and that $\hat{E}$ is contained in the convex region $\mathcal{K}$ bounded by n_{x_0}, n_{x_1}, and the segment joining x_0 to x_1.

Step 2: Next, let $\hat{F}$ be the convex hull of $\hat{E}$ in $\mathbb{R}^2$. Then $\hat{F} \supset \hat{E}$ and $P(\hat{F}) \leq P(\hat{E})$ (this classical fact is shown in the context of indecomposable sets of finite perimeter in [11, Theorems 1 and 6]). So, since $\hat{F} \cap \Omega = \hat{E} \cap \Omega$ by construction, the relatively convex set $F = \hat{F} \setminus \Omega$ satisfies

$$P(F; \mathbb{R}^2 \setminus \Omega) \leq P(\hat{E}; \mathbb{R}^2 \setminus \Omega)\,. \tag{5.34}$$

Moreover, since $\partial F \setminus \Sigma$ is locally linear where it is not contained in $\partial \hat{E}$, an application of Lemma 5.8 shows that

$$d_H(\partial F, \partial(\hat{E} \setminus \Omega))^2 \leq C(P(\hat{E}; \mathbb{R}^2 \setminus \Omega) - P(F; \mathbb{R}^2 \setminus \Omega)) \leq C(P(E; \mathbb{R}^2 \setminus \Omega) - I_\Omega(\eta)) \leq C\delta_3. \tag{5.35}$$

The final inequality comes from (5.27) while the penultimate inequality uses the fact that $|F| \geq \eta$ and thus $P(F; \mathbb{R}^2 \setminus \Omega) \geq I_\Omega(\eta)$. Finally, $\hat{F}$ is also contained in the convex region $\mathcal{K}$. So, the rectifiable curve parametrizing $\partial F \setminus \Sigma$ meets Σ at the points x_0 and x_1 with interior angle at most $\pi/2$. Combining (5.33), (5.34), and (5.35), we obtain (5.28) and the Hausdorff distance estimate of (5.29).

Step 3: It remains to show the bound on the symmetric difference in (5.29) above. Let $\varepsilon = |F\Delta E| = |F| - |E|$. Since the isoperimetric profile is a nondecreasing function of η and $|F| \geq \eta + \varepsilon$, we have $P(E, \mathbb{R}^2 \setminus \Omega) \geq P(F, \mathbb{R}^2 \setminus \Omega) \geq I_\Omega(\eta + \varepsilon) \geq I_\Omega(\eta)$. Combining this with the lower bound from (2.8) yields

$$\Big(\frac{\pi}{2(\eta+\varepsilon)}\Big)^{\frac{1}{2}}\varepsilon \leq I_\Omega(\eta+\varepsilon) - I_\Omega(\eta) \leq P(E, \mathbb{R}^2 \setminus \Omega) - I_\Omega(\eta).$$

So, the desired estimate holds provided we bound $(\frac{\pi}{2(\eta+\varepsilon)})^{\frac{1}{2}}$ below by a constant depending only on η and $\kappa_{max}(\Sigma)$. To this end, let $G = F \setminus E$, so $|G| = \varepsilon$. By e.g. [25, Theorem 16.3], we have $P(G; \mathbb{R}^2 \setminus \Omega) \leq P(E; \mathbb{R}^2 \setminus \Omega) + P(F; \mathbb{R}^2 \setminus \Omega)$. So, applying the lower and upper bounds of (2.1) and recalling (5.27) and (5.28), we obtain

$$(2\pi\varepsilon)^{\frac{1}{2}} \leq I_\Omega(\varepsilon) \leq P(G; \mathbb{R}^2 \setminus \Omega) \leq 2(I_\Omega(\eta) + \delta_3) \leq 4(\pi\eta)^{\frac{1}{2}} + 2\delta_3.$$

This completes the proof. □

5.4. **Proof of Proposition 5.2.** We now combine the results of the previous two subsections with a final area-correction step to show Proposition 5.2.

Proof of Proposition 5.2. Let δ_2 and C_2 be as in Lemma 5.7 and let δ_3 and C_3 be as in Lemma 5.9. Let $\delta_1 = \min\{\delta_2, \delta_3/C_2, 1\}$. Applying Lemma 5.7, we obtain a set F_1 satisfying the assumptions of Lemma 5.9 with

$$|E\Delta F_1|^{\frac{1}{2}} \leq C_2\delta_\eta(E) \quad \text{and} \quad P(F_1; \mathbb{R}^2 \setminus \Omega) \leq P(E; \mathbb{R}^2 \setminus \Omega) + C_2\delta_\eta(E)\,. \tag{5.36}$$

Next, applying Lemma 5.9 to F_1, we obtain an open, relatively convex set $F_2 \supset F_1$ such that $\partial F_2 \setminus \Sigma$ meets Σ with interior angles at most $\pi/2$ and

$$P(F_2; \mathbb{R}^2 \setminus \Omega) \leq P(F_1; \mathbb{R}^2 \setminus \Omega), \tag{5.37}$$

$$d_H(\partial F_1, \partial F_2)^2 + |F_1\Delta F_2| \leq C_3(P(F_1; \mathbb{R}^2 \setminus \Omega) - I_\Omega(\eta)) \leq C\delta_\eta(E) \tag{5.38}$$

where $C = C_3(C_2 + 1)$. Combining this with (5.36), we see that

$$|E\Delta F_2| \leq C\delta_\eta(E), \qquad P(F_2; \mathbb{R}^2 \setminus \Omega) \leq P(E; \mathbb{R}^2 \setminus \Omega) + C\delta_\eta(E). \tag{5.39}$$

The set F_2 has $|F_2| \geq \eta$ by construction. If $|F_2| = \eta$, we let $F = F_2$ and see that (5.3) holds. Otherwise, let $j_0 < j_1$ be chosen such that $\sigma([j_0, j_1]) = \partial F_2 \cap \Sigma$. For $j \in [j_0, j_1]$, let F_j be the intersection of F_2 with the convex region $\mathcal{R}_j$ bounded by the normal rays $n_{\sigma(j)}, n_{\sigma(j_1)}$, and the segment joining $\sigma(j)$ to $\sigma(j_1)$. The area of F_j varies continuously in j with $|F_{j_0}| > \eta$ and $|F_{j_1}| = 0$, so we may find $j \in [j_0, j_1]$ such that $F := F_j$ has area η. Thanks to the convexity of Σ, we immediately have

$$P(F; \mathbb{R}^2 \setminus \Omega) \leq P(F_2; \mathbb{R}^2 \setminus \Omega), \tag{5.40}$$

and by construction and (5.39), we have $|F\Delta F_2| = |F_2| - \eta \leq C\delta_\eta(E)$. Combining these estimates with (5.39) yields (5.3).

Finally, assume that $\partial E\setminus\Sigma$ is a rectifiable curve with endpoints on Σ. The same argument used in the proof of Lemma 5.7 shows that ∂E intersects Σ on a positive $\mathcal{H}^1$-measure set. Thus, there was no need to apply Lemma 5.7 because the set E already met the hypotheses of Lemma 5.9. Hence we may take $F_1 := E$ in the above argument, still have (5.39) and now additionally obtain from Lemma 5.9 that $d_H(\partial E, \partial F_2)^2 \leq C_3\delta_\eta(E)$. Next, the same argument used in step 1 of Lemma 5.9 using Lemma 5.8 shows that

$$d_H(\partial F, \partial F_2)^2 \leq C(P(F_2; \mathbb{R}^2 \setminus \Omega) - P(F; \mathbb{R}^2 \setminus \Omega)) \leq C\delta_\eta(E).$$

Together with (5.39), this shows (5.4). This completes the proof. □

Appendix A. Proof of (2.12)

Proof of (2.12). Fix $\gamma \in \mathcal{B}_\eta$. It suffices to consider the case when γ is oriented so that $A_\Sigma(\eta) > 0$. We further assume without loss of generality that γ is parametrized by arclength. Fix any small $\epsilon > 0$. Since γ stays outside of Ω away from the two endpoints and is defined on a compact set, we may obtain an approximation $\gamma_\epsilon \in \mathcal{B}$ of γ such that

(1) $|L(\gamma) - L(\gamma_\epsilon)| < \epsilon$ and $|A_\Sigma(\gamma) - A_\Sigma(\gamma_\epsilon)| < \epsilon$, and moreover

(2) the image of γ_ϵ is the union of finitely many piecewise C^2 curves $\gamma_0, \gamma_1, \cdots \gamma_k$ where $\gamma_0 \in \mathcal{B}$ is embedded and γ_i are closed and embedded for $i = 1, \cdots, k$, where for $i, j \in \{0, \cdots, k\}$, γ_i and γ_j do not intersect except possibly meeting transversally at their endpoints.

As γ is of class H^2, and hence C^1, such a γ_ϵ can be easily constructed to be in fact even piecewise linear; taking a fine enough subdivision $t_0 = 0 < t_1 < \cdots < t_i = i/k < \cdots < t_k = L(\gamma)$ of $[0, L(\gamma)]$ for a large integer k and replacing $\tilde{\gamma}|_{[t_i, t_{i+1}]}$ with the line segment connecting $\tilde{\gamma}(t_i)$ and $\tilde{\gamma}(t_{i+1})$, we can get a piecewise linear curve which satisfies (1). Then, up to slightly perturbing the vertices of the piecewise linear γ_ϵ to avoid any overlapping segments, γ_ϵ can be taken to satisfy (2). Denote by E_0 the region bounded by γ_0 and Σ, and E_k the region bounded by γ_i for $i = 1, \cdots, k$. By the usual isoperimetric inequality for closed curves we have $L(\gamma_i)^2 \geq 4\pi|E_i|$, and $L(\gamma_0) \geq I_\Omega(|E_0|)$. Therefore,

$$L(\gamma_\epsilon) = \sum_{i=0}^{k} L(\gamma_i) \geq I_\Omega(|E_0|) + \sum_{i=1}^{k} \sqrt{4\pi|E_i|}.$$

On the other hand, note that $|E_0| + |E_1| + \cdots + |E_k| \geq \eta - \epsilon$ since the E_i's are counted with a sign in the algebraic area of γ_ϵ. By (2.1) we have that $\sqrt{4\pi a} \geq I_\Omega(a)$ for any $a > 0$. We can now use that the isoperimetric profile is sub-additive in the sense that $I_\Omega(a) + I_\Omega(b) \geq I_\Omega(a + b)$ for any $a, b > 0$; see Lemma 5.6 for a more general and quantitative version of this statement. Because $I_\Omega(a)$ is nondecreasing in a, combining all of the above we have that

$$L(\gamma_\epsilon) \geq I_\Omega(|E_0|) + \sum_{i=1}^{k} \sqrt{4\pi|E_i|} \geq \sum_{i=1}^{k} I_\Omega(|E_i|) \geq I_\Omega(|E_0| + \cdots + |E_k|) \geq I_\Omega(\eta - \epsilon).$$

Taking $\epsilon \to 0$ finishes the proof. □

Appendix B. Proof of Lemma 5.3

Proof of Lemma 5.3. Step 1: Let $\gamma : [0, 1] \to \mathbb{R}^2 \setminus \Omega^\circ$ be a constant speed parametrization of $\partial F \setminus \Sigma$, so that $\gamma(0), \gamma(1) \in \Sigma$ and $\gamma(t) \notin \Sigma$ for $t \in (0, 1)$. For $N \in \mathbb{N}$ large to be fixed later, let $p_j = j/N$ for $j = 0, \ldots, N$ and let $\gamma_1 : [0, 1] \to \mathbb{R}^2$ be the polygonal curve defined as follows. For $j = 1, \ldots, N-2$, define $\gamma_1|_{[p_j, p_{j+1}]}$ to be the constant speed linear interpolation from $\gamma(p_j)$ to $\gamma(p_{j+1})$. Let $\gamma_1(0)$ and $\gamma_1(1)$ be the nearest point projections of $\gamma(p_1)$ and $\gamma(p_{N-1})$ on Σ respectively. Define $\gamma_1|_{[p_0, p_1]}$ as the constant speed linear interpolation from $\gamma_1(0)$ to $\gamma(p_1)$ and $\gamma_1|_{[p_{N-1}, p_N]}$ as the constant speed linear interpolation from $\gamma(p_{N-1})$ to $\gamma(1)$. By construction, γ_1 is a piecewise linear convex curve whose endpoints meet Σ orthogonally. Provided N is chosen sufficiently large, $\gamma_1(p)$ lies outside Ω for all $p \in (0, 1)$. Together with Σ, γ_1 bounds an open and relatively convex set $F_1 \subset F$. The errors

$$d_H(\partial F, \partial F_1), \quad |F \setminus F_1|, \quad |L(\gamma) - L(\gamma_1)| \tag{B.1}$$

can be made arbitrarily small by choosing N sufficiently large.

Step 2: The curve γ_1 is smooth away from the corners at $p_1, \ldots, p_{N-1}$. We can smooth each of these corners in a $C^{2,1}$ fashion as follows. Choosing $\sigma \ll 1/N$, let $\gamma_2 : [0, 1] \to \mathbb{R}^2$ be the curve that is equal to γ_1 outside of $\cup_{j-1}^{N-1}[p_j - \sigma, p_j + \sigma]$ and such that for each $j = 1, \ldots N - 1$, $\gamma_2|_{[p_j - \sigma, p_j + \sigma]}$ is defined as the cubic Bézier curve with parameters chosen so that, at the endpoints $p_j - \sigma$ and $p_j + \sigma$, the tangents match those of γ_1 and the curvature vanishes. By construction, γ_2 is a convex curve, and provided σ is chosen sufficiently small, $\gamma_2(p)$ lies outside of Ω for each $p \in (0, 1)$. Since γ_1 and γ_2 agree in neighborhoods of their endpoints, γ_2 meets Σ orthogonally at its endpoints. Together with Σ, γ_2 bounds an open, relatively convex set $F_2 \subset F_1$, and the errors

$$d_H(\partial F_2, \partial F_1), \quad |F_1 \setminus F_2|, \quad |L(\gamma_2) - L(\gamma_1)| \tag{B.2}$$

can be made arbitrarily small by choosing σ sufficiently small.

Step 3: Reparametrize γ_2 on $[0, 1]$ with orientation such that the normal ν_{γ_2} to γ_2 coincides with the outward unit normal to F_2. For $\rho \geq 0$ to be chosen later, define $\gamma_{3,\rho} : [0, 1] \to \mathbb{R}^2$ as follows. For $p \in [\sigma, 1 - \sigma]$, let

$$\gamma_{3,\rho}(p) = \gamma_2(p) + \rho\nu_{\gamma_2}(p), \tag{B.3}$$

which is a $C^{2,1}$, embedded convex curve. Let $\gamma_{3,\rho}(0)$ be the nearest point projection of $\gamma_{3,\rho}(\rho)$ on Σ and define $\gamma_{3,\rho}|_{[0,\sigma]}$ to be the constant speed linear interpolation from $\gamma_{3,\rho}(0)$ to $\gamma_{3,\rho}(\sigma)$. Define $\gamma_{3,\rho}(1)$ and $\gamma_{3,\rho}|_{[1-\sigma,1]}$ analogously. By construction, $\gamma_{3,\rho}(p) \in \mathbb{R}^2 \setminus \Omega$ for all $p \in (0, 1)$ and $\gamma_{3,\rho}$ meets Σ orthogonally.

Let $F_3 = F_{3,\rho}$ be the open, connected region bounded by $\gamma_{3,\rho}$ and Σ. We claim that F_3 is relatively convex, provided δ_0, ρ, σ, and $1/N$ are sufficiently small. To this end, we will show the set $G_{3,\rho}$ bounded by γ_3 and the segment joining $\gamma_3(0)$ and $\gamma_3(1)$ is convex. First notice that, by the convexity of Ω and orthogonal contact angle between Σ and the segment $\gamma_2|_{[0,1/N-\sigma]}$, the linear extension of the shifted segment $\gamma_{3,\rho}|_{[\sigma,1/N-\sigma]}$, which intersects Σ if ρ is small, has contact angle at most $\pi/2$, and likewise for the other side. Consequently, the interior angles of $\gamma_{3,\rho}$ at $p = \sigma$ and $p = 1 - \sigma$ are at most π. Up to replacing $\gamma_{3,\rho}$ by the curve obtained by running the corner-smoothing procedure (with a smaller σ') of Step 2 at $p = \sigma$ and $p = 1 - \sigma$, we may also assume $\gamma_{3,\rho}$ is $C^{2,1}$.

Consequently, $G_{3,\rho}$ is convex provided the turning angle of $\gamma_{3,\rho}$ is at most 2π, or equivalently (given the orthogonal contact angle of $\gamma_{3,\rho}$), if the set of normals $A_\rho = \{\nu_\Sigma(x) : x \in \Sigma \cap \partial F_{3,\rho}\}$ to Σ lies in a half-circle of $\mathbb{S}^1$. Choose $\delta_0 > 0$ according to Lemma 2.1. Since $\delta_\eta(F) \leq \delta_0$, Lemma 2.1 and Remark 2.3 guarantee that the endpoints $\gamma(0)$ and $\gamma(1)$ of γ cannot be antipodal points of Σ, and that the set of normals $\{\nu_\Sigma(x) : x \in \Sigma \cap \partial F\}$ lies in a *strict* subset of a half-circle of $\mathbb{S}^1$. By the continuity of the construction, there exist $\bar{N}$ and $\bar{\rho}$ such that the same holds for A_ρ provided $\rho \leq \bar{\rho}$, $N \geq \bar{N}$, and σ is small enough depending on N as in Step 2. This yields the desired convexity.

Now, the area $|F_{3,\rho}|$ varies continuously and is monotonically increasing with respect to ρ. Moreover, there exists $\hat{N}$ such that for any choice of parameters $N > \hat{N}$ and $\sigma \ll 1/N$, there exists $\rho \leq \bar{\rho}$ such that $|F_{3,\rho}| > \eta$. Since $F_{3,0} = F_2 \subset F$ has area at most η, for each N (and σ depending on N as in step 2) we may choose ρ_0 such that $|F_{3,\rho_0}| = \eta$. Let $\gamma_3 = \gamma_{3,\rho_0}$. From the construction we see that $\rho_0 \to 0$ as $1/N \to 0$ and that

$$d_H(\gamma_2, \gamma_3), \quad |F_2 \Delta F_3|, \quad |L(\gamma_2) - L(\gamma_3)| \tag{B.4}$$

can be made arbitrarily small by choosing $1/N$ (and thus ρ_0) sufficiently small. Choosing $1/N$ small enough depending on ε and taking $F_\varepsilon = F_3$, the proof follows by combining (B.1), (B.2), and (B.4). □

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