Chararcterizing minimizers of a constrained planar isoperimetric problem

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(joint work with G. P. Leonardi and G. Saracco)

Given $n \geq 2$ and an open bounded domain $\Omega \subset \mathbb{R}^n$, consider the minimization problem

$$h(\Omega) = \inf \left\{ \frac{P(E)}{|E|} : E \subset \Omega,\ |E| > 0 \right\}, \tag{1}$$

where $P(E)$ and $|E|$ denote the perimeter and volume of E respectively. This constrained isoperimetric problem is known as the *Cheeger problem*, so named for an analogous problem on compact Riemannian manifolds considered by Jeff Cheeger in [2] to establish a lower bound on the first nontrivial eigenvalue of the Laplacian. In the Euclidean setting, this classical isoperimetric problem has roots in the work of Steiner in 1841 in [7].

The infimum value $h(\Omega)$ is known as the Cheeger constant, and a set achieving the infimum is called a Cheeger set. A Cheeger set E exists, and $\partial E \cap \Omega$ has constant mean curvature equal to $h(\Omega)$ and is smooth outside a set of codimension 8. In particular, if $n = 2$, then $\partial E \cap \Omega$ is the countable union of circular arcs of radius $r = 1/h(\Omega)$. While uniqueness fails in general, the union of all Cheeger sets, called the *maximal Cheeger set*, is itself a minimizer of (1).

The Cheeger problem has generated interest in recent years, in part stemming from its connections to numerous other fields including capillarity theory, image processing, and landslide modeling. In each of these settings, it is useful to obtain explicit information about either the Cheeger sets or the value of the Cheeger constant. With this in mind, we are interested in the following general questions:

Given a domain Ω, can one obtain an explicit description of Cheeger sets E in terms of Ω? Can one compute the value of the Cheeger constant?

Some numerical methods based on duality theory have been employed to address these questions, but until recently, Cheeger sets had been precisely characterized for only two classes of domains: convex planar sets [1, 6, 3] and planar strips [5]. In both settings, the Cheeger set E is unique and given by

$$E = \Omega^r \oplus B(0, r)\,. \tag{2}$$

Here, $\Omega^r = \{x \in \Omega : \mathrm{dist}(x, \partial E) \geq r\}$ is the inner parallel set of radius r, and $A \oplus B = \{x + y : x \in A, y \in B\}$ is the Minkowski sum. The value r is given by the unique solution to the equation

$$|\Omega^r| = \pi r^2. \tag{3}$$

As noted above, $h(\Omega) = 1/r$, and thus (3) provides the precise value of the Cheeger constant as well.

The characterization of Cheeger sets given by (2) and (3) cannot be expected to hold for all planar domains. It is not hard to construct counterexamples that fail to be simply connected (for instance, a ball of radius one with a small ball near the boundary removed). Among simply connected domains, one can still construct counterexamples that contain thin necks (for instance, a ball of radius 1 and a ball of radius 2/3 joined by a thin tube). It turns out that, as we show in Theorem 0.1 below, the presence of necks is essentially the only thing that can go wrong for a simply connected planar domain.

To state this property more precisely, we say that a domain Ω has *no necks of radius* r if, given any points $x_1, x_2 \in \Omega$ such that $B(x_i, r) \subset \Omega$ for $i = 1, 2$, there is a continuous path $\gamma : [0, 1] \to \Omega$ with endpoints $\gamma(0) = x_1$ and $\gamma(1) = x_2$ such that $B(\gamma(t), r) \subset \Omega$ for all $t \in (0, 1)$. The main result of [4] is the following:

Theorem 0.1 (Leonardi, Neumayer, Saracco)**.** *Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain with $|\partial\Omega| = 0$, and suppose that Ω has no necks of radius $1/h(\Omega)$. Then the maximal Cheeger set is given by* (2) *and* (3)*.*

It is somewhat unfavorable to have a hypothesis in Theorem 0.1 that involves $h(\Omega)$, as this may not be a priori checkable. What is more useful in practice is the following alternative version of the theorem, with slightly stronger but more easily checkable hypotheses.

Theorem 0.2 (Leonardi, Neumayer, Saracco)**.** *Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain with $|\partial\Omega| = 0$, and suppose that Ω has no necks of radius r for*

$$\frac{inr(\Omega)}{2} \leq r \leq \frac{(|\Omega|/\pi)^{1/2}}{2}. \tag{4}$$

Then the maximal Cheeger set is given by (2) *and* (3)*.*

Here $\mathrm{inr}(\Omega)$ denotes the inradius of Ω, or the radius of the largest ball contained in Ω.

As an application of the theorem, we compute the Cheeger constant of the Koch snowflake K. This fractal is constructed, starting from an equilateral triangle (of side length 3 in our normalization), by iteratively replacing the middle third of each edge by an equilateral triangle. Its boundary is a Jordan curve with infinite length, but with zero Lebesgue measure. It has no necks of any radius. Therefore, Theorem 0.1 applies to K. By computing the Cheeger constant for the polygons in the iterative construction of K and proving error estimates, we establish the following.

Theorem 0.3 (Leonardi, Neumayer, Saracco)**.** *The Cheeger constant of the Koch snowflake is given by $h(K) = 1.89124548...$*

References

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