CONVERGENCE AND REGULARITY OF MANIFOLDS WITH SCALAR CURVATURE AND ENTROPY LOWER BOUNDS

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Abstract. We discuss in this short survey the notion of d_p convergence and its application to studying limits of sequence of Riemannian manifolds (M_i^n, g_i) whose scalar curvatures and entropies are bounded from below by small constants. We will build examples showing how the more classical notions of Gromov-Hausdorff and Intrinsic Flat convergence must fail in the context of lower scalar bounds. More fundamentally, we will see how a notion of a metric space itself is the wrong notion for such limits, as distance functions may degenerate. We will see how to fix these problems by weakening the convergence criteria with respect to the d_p-structure. The d_∞-structure corresponds to the classical metric structure, but for any $p < \infty$ a d_p-structure still comes equipped with a well behaved analysis. Our main result is to show that if a manifold with lower scalar and entropy bounds $R, \mu \geq -\epsilon(n)$, then a ball must be d_p-close to a Euclidean ball for $p = p(n, \epsilon) < \infty$. We also study more general limits, and have applications which include apriori estimates on such spaces.

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1. Introduction and ε-Regularity

A broad theme in geometric analysis aims to understand the structure and a priori regularity of a Riemannian manifold (M^n, g) when restrictions are imposed on its curvature. An essentially equivalent question is to understand the structure of singular limits $M_i^n \to X$, where (M_i^n, g_i) is a sequence of Riemannian manifolds satisfying the same curvature constraint. Naturally, it is important here to identify the appropriate notion of convergence and the class of objects that arise as limits. In the study of manifolds with bounded curvature operator, it suffices to consider manifold limits X under $C^{k,\alpha}$ convergence [Che67, Che70]. In the analysis of spaces with (lower) bounds on Ricci curvature, it is the underlying metric spaces structures of a sequence of Riemannian manifolds that converge, in the Gromov-Hausdorff sense [Gro07], to a limiting metric space. This weaker convergence allows for the necessary formation of singularities in possible limit spaces. In this setting, it is essential to distinguish between *collapsed* and *noncollapsed* limits, where noncollapsing of the sequence M_i^n can be understood as the existence of a uniform lower bound on the volumes of balls. The starting point for the regularity theory for spaces with bounds on Ricci curvature is an ϵ-regularity theorem. This says that if the volume of a unit ball is close to that of the Euclidean ball, then that ball must be close both topologically and geometrically to a Euclidean ball.

When one only assumes bounds on the scalar curvature of a Riemannian manifold, the structure and a priori regularity are significantly less well understood. Here, we survey results from our recent paper [LNN] concerning Riemannian manifolds and their sequential limits under lower bounds on scalar curvature. The correct replacement for *noncollapsing* in this context is a lower bound on the entropy μ of the manifold, introduced below. In [LNN], we formulate and prove a corresponding ϵ-regularity in this context: a statement which should say that if the scalar and entropy lower bounds are nearly-Euclidean, then a ball should be close to a Euclidean ball.

When studying spaces with lower bounds on scalar curvature, it is already understood from [Sor17] that the notion of Gromov-Hausdorff closeness cannot be the correct one. The examples in [Sor17] mimic those from minimal surface theory, and show that small volume tentacles may appear when only a lower scalar curvature bound is assumed. One possible fix for issues like this is the Intrinsic Flat distance [SW11]. In fact, the problem is actually much worse. We will build examples in Section 3 demonstrating that even under small lower bounds on scalar curvature and entropy—where the metric is expected to be close, when understood in the correct sense, to the Euclidean metric—the Gromov-Hausdorff and Intrinsic Flat limits may be completely wild, with jumps in topology, dimension, and the formation of Finsler or worse types of geometries.

Fundamentally, these examples show that lower scalar curvature and entropy bounds simply do not control the behavior of distance functions. Consequently, one cannot expect to prove an ε-regularity theorem or develop a study of limit spaces with respect to any notion of convergence that is based on the distance function. From the correct perspective, this is not entirely surprising; the distance function is closely related to the $W^{1,\infty}$ behavior of functions, and it may simply be too much to ask that this remains uniformly controlled in such a sequence. Indeed, it is now well understood from the study of RCD spaces [AGS13, BE84, Stu05, LV09] that (said correctly) $W^{1,\infty}$-control on the analysis is essentially equivalent to lower bounds on *Ricci* curvature. So, one might expect distance functions to break down in the context of only scalar curvature bounds.

In order to solve this problem, we introduce a new notion of convergence, d_p convergence. Instead of considering the convergence of the underlying (geodesic) metric space structures of a sequence of Riemannian manifolds, the notion of d_p convergence is a based on the convergence of another natural family of distance functions d_p that take the place of the geodesic distance. The effect of this will be to take the required $W^{1,\infty}$-control needed for convergence of distance functions, and reduce it to a required $W^{1,p}$-control for this weaker notion of convergence.

Heuristically, we can think of the difference between the geodesic distance and the d_p distance in analogy to the difference between the L^∞ norm and L^p norm of a function. For example, consider a sequence of smooth functions $f_k : \mathbb{R}^2 \to \mathbb{R}$ that converge pointwise to the characteristic function of the line $\ell = \{(x, y) : x = 0\}$. Such a sequence will not have an L^∞ limit, but can be constructed to converge to zero in L^p_{loc}, smearing out this lower dimensional set on which the pointwise limit is equal to 1. Similarly, for a sequence of metrics on $\mathbb{R}^2$ that converge to the Euclidean metric away from ℓ but become increasingly degenerate along ℓ, the geodesic distance will also degenerate along ℓ. We introduce d_p as a notion of distance so that this singularity is "smeared out" near ℓ, in analogy to L^p convergence of functions.

Let us begin with a definition of the d_p distance functions. Recall that the classical distance on a Riemannian manifold can be equivalently defined by $d(x, y) = \sup\{|f(x) - f(y)| : \|\nabla f\|_{L^\infty} \leq 1\}$; this supremum is always achieved by the distance function itself.

Definition 1.1 (d_p distance on manifolds)**.** *Given a Riemannian manifold (M^n, g) and a real number $p \in (n, \infty]$, we define the $d_{p,g}$ distance between any $x, y \in M$ by*

$$d_{p,g}(x, y) = d_p(x, y) = \sup\left\{|f(x) - f(y)| \ : \ \int_M |\nabla f|^p \, d\mathrm{vol}_g \leq 1\right\}. \tag{1.1}$$

We let $\mathcal{B}_{p,g}(x, r) = \{y \in M : d_p(x, y) < r\}$ denote the ball of radius r with respect to d_p.

On any Riemannian manifold, d_p defines a distance function, and so (M, d_p) is a metric space. The d_p metric space structure of a Riemannian manifold determines its entire geometry in the sense two Riemannian manifolds which are d_p-isometric must in fact be isometric as Riemannian manifolds. This is analogous to the Myers-Steenrod Theorem for the geodesic metric space structure. The d_p distance understands and controls the behavior of the Sobolev space $W^{1,p}$, with $d_\infty = d$ becoming the standard distance function.

While one could define d_p for any $p \geq 1$, we restrict our attention to $p > n$ since $d_p(x, y) = +\infty$ for all $x \neq y$ and $p \leq n$ on any smooth Riemannian manifold. One may analogously define the d_p distance on more general spaces than smooth Riemannian manifolds, as all that is needed is a space equipped with a notion of $W^{1,p}$ Sobolev space. For instance, the d_p distance can be defined on a metric measure space (X, d, m). However, what is important to note in our context is that d_p does not need an underlying metric structure in order to be defined. A rectifiable structure, which gives the ability to differentiate functions and integrate them, will be sufficient. In particular, the functions d_p are well-defined on rectifiable Riemannian spaces (X, g), which are introduced in Definition 2.2, but roughly are topological measure spaces with a compatible rectifiable structure and Riemannian metric on the rectifiable charts.

Let us discuss some examples and basic properties of the d_p distance. To begin with, we study the behavior of the d_p distance on Euclidean space.

Example 1.2 (The d_p distance on Euclidean space)**.** On Euclidean space, for $p > n$ we directly compute that $d_p(x, y) = S|x - y|^{1-n/p}$, where $S = S_{n,p}$ is a normalizing constant with $S_{n,p} \to 1$ as $p \to \infty$. Correspondingly, $\mathcal{B}_{p,g_{euc}}(0, Sr^{1-n/p}) = B(0, r)$ for any $r > 0$.

Another interesting example is the d_p distance on hyperbolic space, which highlights some important distinctions from the geodesic distance.

Example 1.3 (The d_p distance on hyperbolic space)**.** Given $n \geq 2$ and $n < p < \infty$, hyperbolic space $(\mathbb{H}^n, g_{hyp})$ has finite bounded diameter with respect to d_p. Indeed, the Morrey-Sobolev inequality on hyperbolic space (see [Ngu18, MT98]), there exists $C = C(n,p) > 0$ such that $|f(x) - f(y)| \leq C\|\nabla f\|_{L^p(\mathbb{H}^n)}$ for all $f \in W^{1,p}(\mathbb{H}^n)$ and for all$x, y \in \mathbb{H}^n$ Consequently, $d_p(x,y) \leq C$ for all $x, y \in \mathbb{H}^n$.

In the next example, we consider a space that is not a smooth Riemannian manifold, since the Riemannian metric is degenerate. However, one can easily smooth out this example to obtain the featured behavior asymptotically for a sequence of smooth Riemannian manifolds approximating it. The key feature of this example is its illustration of the role of the parameter p and its relationship to the order of singularity of a metric. While d_p defines a distance metric on any smooth Riemannian manifold (M, g), it may only define a pseudometric if the metric g is degenerate—depending on the relationship between the value of p and the order of degeneracy. In the following example, for p large, the d_p distance (as well as the geodesic distance) will reflect the degeneracy of the metric, while for $p > n$ small enough, the degeneracy will be "smeared out" by the d_p distance.

Example 1.4. Fix $\alpha > 0$ and consider $(\mathbb{R}^2, g_\alpha)$, where $g_\alpha = dx^2 + |x|^{2\alpha}dy^2$ is a degenerate Riemannian metric. Fix any $x_0, y_0 \in \{(x,y) : x = 0\}$. For any p large enough such that $\alpha p \geq 1$, one can show that $d_{p,g_\alpha}(x_0, y_0) = 0$. In particular, d_{p,g_α} is only a pseudometric on $\mathbb{R}^2$ for such p. On the other hand, if $\alpha p < 1$, we can see that $d_{p,g_\alpha}(x_0, y_0) > 0$ by plugging in a test function that agrees with $f(x,y) = y$ nearby these two points. For this range of p, the d_{p,g_α} defines a metric on $\mathbb{R}^2$ and does not see the degeneracy of g_α

Let us now move toward the statement of our ε regularity theorem. We begin by defining the noncollapsing condition. The Perelman $\mathcal{W}$-functional, introduced in [Per02], is defined for a function $f \in C^\infty(M)$ and real number $\tau > 0$ by

$$\mathcal{W}(g, f, \tau) = \frac{1}{(4\pi\tau)^{n/2}} \int_M \left\{\tau(|\nabla f|^2 + R) + f - n\right\} e^{-f} \, d\mathrm{vol}_g \,. \tag{1.2}$$

The Perelman entropy $\mu(g, \tau)$, which can be viewed as the optimal constant in a log-Sobolev inequality at scale $\tau^{1/2}$, is given by

$$\mu(g,\tau) = \inf\left\{\mathcal{W}(g,f,\tau) : \frac{1}{(4\pi\tau)^{n/2}} \int_M e^{-f} \, d\mathrm{vol}_g = 1, \, e^{-f/2} \in W^{1,2}(M)\right\} . \tag{1.3}$$

Finally, Perelman's ν-functional is given by

$$\nu(g,\tau) = \inf\{\mu(g,\tau') : \tau' \in (0,\tau)\} \,,$$

and just guarantees that we are measuring the entropy at all scales below some point. The Perelman entropy on Euclidean space is equal to zero for any $\tau > 0$, a fact that is equivalent to the Euclidean log-Sobolev inequality. The Perelman entropy $\mu(g,\tau)$ of a complete Riemannian manifold (M, g) with bounded curvature is nonpositive for all $\tau > 0$. Moreover, if the entropy is equal to zero for some $\tau > 0$, then (M, g) is isometric to Euclidean space.

In the context of Riemannian manifolds with lower bounds on scalar curvature, the appropriate noncollapsing assumption is that $\nu(g,1) \geq -\delta$, that is, that the Perelman entropy is close to that of Euclidean space on all scales up to 1. In connection to the volume noncollapsing typically assumed in the context of Ricci curvature lower bounds, it is not hard to show that for a Riemannian manifold with bounded curvature, the assumptions $\nu(g,1) \geq -\delta$ and $R_g \geq -\delta$ imply that all balls below scale one satisfy the almost-Euclidean volume lower bound $\mathrm{vol}_g(B_g(x,r)) \geq (1-\varepsilon)\omega_n r^n$.

The following ε regularity theorem is one of the main results of [LNN]. This theorem shows that if a Riemannian manifold has almost Euclidean lower bound on scalar curvature and entropy, then a p-ball is close to a p-ball in Euclidean space in the measured Gromov-Hausdorff sense of their d_p metric spaces.

Theorem 1.5 (ϵ-Regularity Theorem)**.** *Let (M^n, g) be a complete Riemannian manifold with bounded curvature and fix $\varepsilon > 0$ and $p \geq n+1$. There exists $\delta = \delta(n, \varepsilon, p)$ such that if*

$$R \geq -\delta, \qquad \nu(g,2) \geq -\delta\,, \tag{1.4}$$

then for all $x \in M$, we have

$$d_{GH}\left((\mathcal{B}_{p,g}(x,1), d_{p,g}),\, (\mathcal{B}_{p,g_{euc}}(0,1), d_{p,g_{euc}})\right) \leq \varepsilon \tag{1.5}$$

and for any $0 < r \leq 1$

$$(1-\varepsilon)|\mathcal{B}_{p,g_{euc}}(0,r)| \leq \mathrm{vol}_g(\mathcal{B}_{p,g}(x_0,r)) \leq (1+\varepsilon)|\mathcal{B}_{p,g_{euc}}(0,r)|. \tag{1.6}$$

Here $|\cdot|$ denotes the Euclidean volume. In particular, the measure $d\mathrm{vol}_g$ on the metric measure space $(M, d_{p,g}, d\mathrm{vol}_g)$ is a doubling measure for all scales $r \leq 1$.

Remark 1.6. The assumption of (nonuniformly) bounded curvature is simply to control degeneration at infinity of M, a local version of these statements would drop this condition.

Remark 1.7 (L^1 Sobolev constant)**.** We may replace the entropy lower bound in Theorems 1.5 by a rigid bound on the L^1-Sobolev constant. Namely, we may replace the assumption $\nu(g,2) \geq -\delta$ in (1.4) with the assumption that for all compactly supported $f : B_g(x,1) \to \mathbb{R}$ with $x \in M$ we have

$$\left(\int_M |f|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq (1+\delta)c_n \int_M |\nabla f|\,, \tag{1.7}$$

where c_n is the sharp Sobolev constant on Euclidean space. However, we avoid focusing on this because, as we will see, metric balls are badly behaved objects, and thus any condition which used a metric ball may be more restrictive than it appears. The μ-entropy intrinsically understands the correct d_p distance, and thus $\mu(g,1)$ becomes a condition on the unit d_p-scale, as opposed to the $d = d_\infty$ scale.

Remark 1.8 (Scaling)**.** For any Riemannian manifold (M, g), the rescaled metric $\tilde{g} = r^{-2}g$ satisfies $\mathcal{B}_{p,\tilde{g}}(x_0, \rho) = \mathcal{B}_{g,p}(x_0, \rho r^{1-n/p})$, $R_{\tilde{g}} = r^{-2}R_g$, $\nu(g, 2r^2) = \nu(\tilde{g}, 2)$. If (M, g) is closed or is well behaved at infinity (see [Zha12] for instance), then $\lim_{\tau\to 0}\mu(g,\tau) = 0$. In particular, for any such Riemannian manifold (M, g), the hypotheses of Theorem 1.5 hold at some scale.

The main tool used to establish the proof of Theorem 1.5 is Theorem 1.9 below. This theorem establishes, under the same hypotheses as Theorem 1.5, the existence of $W^{1,p}$ charts on an ϵ-regularity ball. Moreover, it shows that the $W^{1,p}$ norms of functions on this ϵ-regularity ball can be controlled—not quite by the $W^{1,p}$ norm of their pullback to Euclidean space, but by $W^{1,q}$ norms for q close to p.

Theorem 1.9 (L^p estimates for the metric coefficients)**.** *Let (M^n, g) be a complete Riemannian manifold with bounded curvature. Fix, $\varepsilon > 0$, $\kappa > 1$ and $p \in [\kappa, \infty)$. There exists $\delta = \delta(n, p, \kappa, \varepsilon) > 0$ such that if*

$$R \geq -\delta, \qquad \nu(g, 2) \geq -\delta\,, \tag{1.8}$$

then for any $x \in M$ there exist an open set $\Omega \subset M$ containing x and a smooth diffeomorphism $\psi : \Omega \to B(0,1) \subset \mathbb{R}^n$ with $\psi(x) = 0$ satisfying

$$\fint_{B(0,1)} |(\psi^{-1})^* g - g_{euc}|^p \, dy \leq \varepsilon, \qquad \fint_{\Omega} |\psi^* g_{euc} - g|^p \, d\mathrm{vol}_g \leq \varepsilon\,. \tag{1.9}$$

Furthermore, for any $f \in W^{1,p}(B(0,1))$, we have

$$(1-\varepsilon)\|\psi^* f\|_{L^{p/\kappa}(\Omega)} \leq \|f\|_{L^p(B(0,1))} \leq (1+\varepsilon)\|\psi^* f\|_{L^{\kappa p}(\Omega)}, \tag{1.10}$$

$$(1-\varepsilon)\|\nabla\psi^* f\|_{L^{p/\kappa}(\Omega)} \leq \|\nabla f\|_{L^p(B(0,1))} \leq (1+\varepsilon)\|\nabla\psi^* f\|_{L^{\kappa p}(\Omega)}. \tag{1.11}$$

The notation $\fint_\Omega u \, d\mathrm{vol}_g$ is used to denote $\mathrm{vol}_g(\Omega)^{-1} \int_\Omega u \, d\mathrm{vol}_g$. In (1.9), the notation $|\cdot|$ indicates the tensor norm with respect to g_{euc} and g respectively.

2. Structure of Limit Spaces

Thus far, we have only discussed the d_p distance on smooth Riemannian manifolds and convergence in the d_p sense to p-balls in Euclidean space. More generally, *rectifiable Riemannian spaces* are the objects that arise naturally as limits in the d_p sense of Riemannian manifolds with uniform lower bounds on scalar curvature and entropy. Rectifiable Riemannian spaces are introduced precisely in Section 2.1 below.

Heuristically speaking, a rectifiable Riemannian space (X, g) is a topological space X with a rectifiable structure, defined via an atlas of charts, and a possibly degenerate metric g, also defined via charts. A rectifiable Riemannian space may not have a meaningfully defined distance function or metric space structure. Nonetheless, these spaces have sufficient structure to make sense of $W^{1,p}$ Sobolev spaces. We will see in Section 2.2 that the classical approach (see [Haj03]) to defining Sobolev spaces on a metric measure space can be modified to the setting of a rectifiable Riemannian space, despite the lack of metric space structure. In particular, the d_p distance can be defined on a rectifiable Riemannian space.

Generally speaking, rectifiable Riemannian spaces and their Sobolev spaces may be rather poorly behaved. Those rectifiable Riemannian spaces arising as d_p limits of spaces with lower bounds on entropy and scalar curvature, however, have quite a bit more structure— in Section 2.4 we will see that the underlying topological space is a smooth manifold and the Sobolev spaces are "big" and "well behaved" in a sense to be made precise later.

2.1. **Rectifiable Riemannian spaces.** Let us rigorously define a rectifiable Riemannian space. Let X be a Hausdorff topological space equipped with a measure m on X. We will refer to (X, m) as a topological measure space. As the name suggests, a rectifiable Riemannian space is a topological measure space equipped with a *rectifiable structure* and a *Riemannian structure* that is compatible with the measure and the rectifiable structure.

We first address the notion of rectifiability of topological measure space. A topological measure space is not equipped with a distance function, and thus one cannot discuss Lipschitz charts. Instead, the appropriate notion of rectifiability is provided via an atlas of charts with bi-Lipschitz transition maps that cover X up to a set of m-measure zero.

Definition 2.1 (Rectifiable atlas)**.** *Let (X, m) be a topological measure space, and consider a collection of charts $\{(U_a, \phi_a)\}_{a\in\mathcal{I}}$, where $U_a \subseteq \mathbb{R}^n$, $\phi_a : U_a \to X$ is one-to-one and continuous with continuous inverse on its image, and $\mathcal{I}$ is a countable index set. For each $a, b \in \mathcal{I}$, let us denote $U_{a,b} \equiv U_a \cap \phi_a^{-1}(\phi_b(U_b)) \subseteq \mathbb{R}^n$. We say that $\{(U_a, \phi_a)\}_{a\in\mathcal{I}}$ is a rectifiable atlas for (X, m) if:*

1. *For each $a, b \in \mathcal{I}$ such that $U_{a,b}$ is nonempty, every point in $U_{a,b}$ has Lebesgue density one.*
2. *For each $a, b \in \mathcal{I}$ such that $U_{a,b}$ is nonempty, the transition map*
$$\phi_{ba} \equiv \phi_b^{-1} \circ \phi_a : U_{a,b} \to U_{b,a}$$
is bi-Lipschitz.
3. *We have $m\left(X \setminus \bigcup_{a\in\mathcal{I}} \phi_a(U_a)\right) = 0$.*
4. *For each U_a, the measure $(\phi_a^{-1})_* m$ is absolutely continuous with respect to the Lebesgue measure.*

Now, if (X, m) is a topological measure space equipped with a rectifiable atlas $\{(U_a, \phi_a)\}$, we may define a *Riemannian structure* on X by defining a (possibly degenerate) Riemannian metric in the charts U_a. Naturally, we must ask that this metric is suitably compatible with the rectifiable atlas and the measure m. We call the resulting space a rectifiable Riemannian space.

Definition 2.2 (Rectifiable Riemannian space)**.** *Let (X, m) be a topological measure space. We say that (X, m) has a rectifiable Riemannian structure if there is a rectifiable atlas $\{(U_a, \phi_a)\}_{a\in\mathcal{I}}$ on (X, m) and collection of matrix-valued functions $g_a : U_a \to \mathbb{R}^{n\times n}$ satisfying*

1. *For each $x \in U_a$, $g_a(x)$ a positive definite symmetric matrix such that*
$$\sup_{x\in U_a} \|g_a\| + \|g_a^{-1}\| \leq C_a.$$
and g_a is continuous on U_a.
2. *For each nonempty U_{ab}, we have $g_b = \phi_{ba}^* g_a$.*
3. *The measure $\phi_a^* m$ on U_a is given by $\phi_a^* m = \sqrt{\det g_a}\, dx$.*

We say that $\{g_a\}_{a\in\mathcal{I}}$ is the coordinate expression of a rectifiable Riemannian metric g on X and call (X, g) a rectifiable Riemannian space.

One can imagine a variety of ways in which a rectifiable Riemannian space can degenerate. We will first work our way through some basic examples which explore this. This will give some first intuition on what kind of structure is needed to avoid this. The Limit Structure Theorem 2.35

will show that certain degeneracies highlighted in these examples cannot occur in a rectifiable Riemannian space arising as a limit of Riemannian manifolds with lower bounds on entropy and scalar curvature, while the examples of Section 3 show that other types of degeneracies cannot be avoided.

Example 2.3. Any smooth Riemannian manifold (M, g) is a rectifiable Riemannian space.

With regard to Example 2.3, observe that even for a smooth Riemannian manifold (M, g) a given rectifiable atlas may only cover M up to a set of measure zero:

Example 2.4. Let $X = \mathbb{R}^n$ with g_{euc} the Euclidean metric, and let $m = dx$ be the Lebesgue measure. Consider the rectifiable atlas $\{(U_1, \phi_1), (U_2, \phi_2)\}$ where $U_1 = \{(x_1, \ldots, x_n) : x_1 > 0\}$ and $U_2 = \{(x_1, \ldots, x_n) : x_1 < 0\}$ are complementary open half spaces and ϕ_i is the identity chart restricted to U_i for $i = 1, 2$. Then $(\mathbb{R}^n, g_{euc})$ is a rectifiable Riemannian space with respect to this rectifiable atlas.

Example 2.5. Any stratified Riemannian manifold X is a rectifiable Riemannian space.

Example 2.6. As a concrete case of Example 2.5, let $X \subset \mathbb{R}^2$ be a countable union of lines $\{\ell_i\}_{i\in\mathbb{N}}$ passing through the origin and let m be defined by $m|_{\ell_i} = \mathcal{H}^1|_{\ell_i}$. Define $g|_{\ell_i} = g_{\mathbb{R}^2}|_{\ell_i}$ and let $\{(\mathbb{R} \setminus \{0\}, \phi_i)\}_{i\in\mathbb{N}}$ be the rectifiable atlas with $\phi_i : \mathbb{R} \setminus \{0\} \to \ell_i \setminus \{0\}$ defined via the obvious isometric embedding. Then (X, g) is a one dimensional rectifiable Riemannian space.

Due to the flexibility of the definition, the metric tensor of a rectifiable Riemannian space may be mildly singular. Let us consider some examples of this.

Example 2.7. Let $X = \mathbb{R}^n$ and consider the metric defined by $g = \sum_{i=1}^n f_i(x)^2 (dx^i)^2$, where each f_i is a smooth non-negative function on $\mathbb{R}^n$ such that the set $\Sigma = \cup_{i=1}^n \{x : f_i(x) = 0\}$ has Lebesgue measure zero. Let $m = \sqrt{\det g}\, dx$ be the induced measure. Consider the the rectifiable atlas on the topological measure space $(\mathbb{R}^n, m)$ given by $\{(U_a, \phi_a)\}_{a\in\mathbb{N}}$ where $U_a = \cap_{i=1}^n \{x \in \mathbb{R}^n : a^{-1} \le f_i \le a\}$ and ϕ_a is the identity chart on $\mathbb{R}^n$ restricted to U_a. Then, with respect to this rectifiable atlas, $(\mathbb{R}^n, g)$ is a rectifiable Riemannian space.

An important feature of Example 2.7 is that, while the geodesic distance gives rise to a metric space structure (X, d), the metric space may not even be topologically equivalent to $\mathbb{R}^n$. We see this more concretely in the following example.

Example 2.8. Recall Example 1.4 in the previous section, in which we fixed $\alpha > 0$ and considered $(\mathbb{R}^2, g)$, where $g = dx^2 + |x|^{2\alpha} dy^2$. This is a special case of Example 2.7 above and so in particular is a rectifiable Riemannian space. Let d_g denote the distance function with respect to g, i.e. $d_g(x, y) = \inf_\gamma \int_0^1 |\dot\gamma(t)| dt$, where the infimum is taken among all curves γ with $\gamma(0) = x, \gamma(1) = y$. Then, for any two points $p_1, p_2 \in \ell$ where $\ell = \{(x, y) : x = 0\}$, we see that $d_g(p_1, p_2) = 0$. In particular, the metric space $(\mathbb{R}^2, d_g)$ collapses ℓ to a point and is not topologically equivalent to $\mathbb{R}^2$.

In Section 3, we construct rectifiable Riemannian metrics that are qualitatively similar to Example 2.8 that arise as limits of smooth Riemannian manifolds with uniform lower bounds on scalar curvature and entropy. These examples share the feature with Example 2.8 that the distance function with respect to g give rise to a different topology than the underlying space. It will also be the case that the d_p distance gives rise to a different topology than the underlying space for $p >> 1$. Importantly, for $p \leq p_0(g)$, the d_p distance will reflect the "smeared out" behavior the metric as discussed in the previous section and will generate the expected topology.

2.2. **$W^{1,p}$ spaces and d_p distance on rectifiable Riemannian spaces.** In order to do analysis on rectifiable Riemannian spaces, and in particular to understand d_p limits, we need to make sense of $W^{1,p}$ functions in this context. This means being able to take gradients of functions and look at their norms. In particular, understanding $W^{1,p}$ functions on a rectifiable Riemannian space provides a natural extension of this definition of d_p to these singular spaces.

Ideally, we would want to use the rectifiable charts in order understand $W^{1,p}$ functions in coordinates. Realistically, one has to be quite careful about this. A function might be perfectly differentiable in every coordinate chart, but not really be a $W^{1,p}$ function as its gradient may have a distributional component, as we see in the following example.

Example 2.9. Consider $(\mathbb{R}^n, g_{euc})$ with the rectifiable atlas $\{(U_1, \phi_1), (U_2, \phi_2)\}$ comprising two open half spaces as in Example 2.4. The function $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) = 0$ if $x_1 < 0$ and $f(x) = 1$ if $x_1 \geq 0$ clearly does not have gradient in $L^p(\mathbb{R}^n)$, since its distributional gradient is a singular measure supported on $\{x_1 = 0\}$. However, letting $f_a = \phi_a^* f$ for $a = 1, 2$, we have $g^{ij}\partial_i f_a \partial_j f_a \equiv 0$ for all $x \in U_a$ for $a = 1, 2$.

To deal with this subtlety, we follow a classical approach from metric measure spaces (see for instance [Haj03, Sections 5-7]) to build the Sobolev space theory by considering the behavior of functions along curves. Roughly speaking, the approach in the metric measure space context goes as follows: (1) consider the collection of rectifiable curves on (X, m, d) and the notion of sets of rectifiable curves of p-measure zero; (2) define a p-weak upper gradient of a function u by way of the fundamental theorem of calculus on p-a.e. curve; (3) understand $W^{1,p}$ Sobolev spaces on (X, m, d) as the collection of L^p functions with p-weak upper gradient in L^p.

On the surface, this approach appears to fundamentally rely on the metric space structure of (X, m, d), since the notion of a rectifiable curve requires a distance function. In reality, however, after some basic modifications, this approach adapts very well to the context of rectifiable Riemannian spaces. Indeed, on a metric measure space (or Riemannian manifold), every rectifiable curve admits an absolutely continuous (in fact, Lipschitz) parametrization, namely the arc length parametrization. In practice, it is this absolutely continuous parametrization that is used in the Sobolev space theory. The key point is that even when no distance function is available, the notion of an absolutely continuous curve is available and the behavior of a function along it can be studied.

So, when determining the appropriate class of curves along which to study the behavior of functions in the context of rectifiable Riemannian spaces, two major factors that must be taken into account: (a) curves must be suitably compatible with the rectifiable atlas on the space in order to deal with the difficulty illustrated in Example 2.9; and (b) the absence of a distance function prohibits us considering rectifiable curves and instead, we must restrict our attention to absolutely continuous parametrizations of curves.

Let (X,g) be a rectifiable Riemannian space with rectifiable atlas $\{(U_a,\phi_a)\}_{a\in\mathcal{I}}$. Denote the singular part of X by $X^s = X\setminus \cup_a U_a$, which may or may not correspond to topological singularities of the space.

Definition 2.10 (Absolutely Continuous Curves). *Let $\gamma:[\alpha,\beta]\to X$ be a continuous curve and set $I_a \equiv \gamma^*(\phi_a(U_a))$ for each $a\in\mathcal{I}$. We say that γ is absolutely continuous if the following properties hold.*

(a) *$\gamma^*(X^s)\subset[\alpha,\beta]$ is a countable set;*
(b) *For every $\varepsilon>0$, there exists $\delta>0$ such that if $\{(s_i,t_i)\}_{i=1}^\infty$ is a collection of disjoint intervals in $[\alpha,\beta]$ such that for each i, we have $s_i,t_i\in I_{a_i}$ for some $a_i\in\mathcal{I}$ and $\sum_{i=1}^\infty |s_i-t_i|<\delta$, then*
$$\sum_{i=1}^\infty |\gamma_{a_i}(s_i)-\gamma_{a_i}(t_i)|_{g_a(\gamma(s_i))}<\varepsilon. \tag{2.1}$$

Part (a) of the definition guarantees that the behavior of an absolutely continuous curve γ can be entirely reflected in the charts of its rectifiable atlas, even though the rectifiable charts may only cover (X,g) up to a set of measure zero, by ensuring that there is no contribution to the singular part of the distributional derivative of γ on the set $\gamma^*(X^s)$. In particular, this eliminates the issue illustrated in Example 2.9:

Example 2.11. In Example 2.9, the curve $\gamma(t)=(t,0,\dots,0)$ is not an absolutely continuous curve in the sense of Definition 2.10 because it violates condition (a).

Part (b) of Definition 2.10 replaces the classical notion of a curve with finite length, as one does not have a notion of a distance function with which to measure the length. The following lemma shows how condition (b) guarantees that the curve is absolutely continuous in each chart in a suitably uniform sense and further clarifies how this notion of absolutely continuous curve fits into the classical notion on smooth spaces.

Lemma 2.12. *Let $\gamma:[\alpha,\beta]\to X$ be an absolutely continuous curve in the sense of Definition 2.10 above. Then the following properties hold.*

(1) *For each $a\in\mathcal{I}$, the function $\gamma_a=\phi_a^{-1}\circ\gamma: I_a\to U_a$ is differentiable for a.e. $s\in I_a$. Here we again let $I_a=\gamma^*(\phi_a(U_a))\subset[\alpha,\beta]$.*
(2) *For all $\varepsilon>0$, there exists $\delta>0$ such that if $S\subset[\alpha,\beta]$ with $|S|<\delta$, then $\int_S|\dot\gamma|_g<\varepsilon$.*
(3) *If (X,g) is a smooth Riemannian manifold, then the length of γ is given by $L(\gamma)=\int_\alpha^\beta|\dot\gamma|_g\,dt$.*

Remark 2.13. In Lemma 2.12 (2) and (3) and in the sequel, we let $|\dot\gamma|_g\equiv\sqrt{g(\dot\gamma,\dot\gamma)}$, which is well-defined for a.e. $t\in[\alpha,\beta]$ by Lemma 2.12(1) and via the rectifiable atlas $\{(U_a,\phi_a)\}_{a\in\mathcal{I}}$.

Remark 2.14. Having in mind Lemma 2.12(3), we define the length of an absolutely continuous curve $\gamma:[\alpha,\beta]\to X$ in a rectifiable Riemannian space by $L(\gamma)=\int_\alpha^\beta|\dot\gamma|_g$. One can check that this notion is independent of Lipschitz reparametrizations. More generally, we will use the notation $\int_\gamma f$ to mean $\int_\gamma f := \int_\alpha^\beta f(\gamma(t))\,|\dot\gamma|_g\,dt$ for a function $f:X\to\mathbb{R}$.

Remark 2.15. Definition 2.10 is parametrization dependent, as it requires an absolutely continuous parametrization. This is not restrictive, as on a smooth Riemannian manifold with a classical atlas of charts, every rectifiable curve in the classical sense admits a reparametrization, namely its arc length parametrization, that is an absolutely continuous curve in the sense of Definition 2.10.

Having in hand the appropriate class of curves along which to study the behavior of functions, we follow the classical approach in metric measure spaces to use our absolutely continuous curves to define the notion of a p-weak upper gradient of a function. Most of the Sobolev theory is built up in an identical fashion to the metric measure space setting.

Let $\mathfrak{M}$ denote the collection of all absolutely continuous curves on (X, g). For $1 \leq p < \infty$, we say that a family of curves $\Gamma \subset \mathfrak{M}$ has $\mathrm{Mod}_p(\Gamma) = 0$ if there exists a nonnegative Borel measurable function $f \in L^p(X)$ such that $\int_\gamma f = +\infty$ for every $\gamma \in \Gamma$. A property is said to hold for p-a.e absolutely continuous curve if it holds for every curve in $\mathfrak{M} \setminus \Gamma$ where $\mathrm{Mod}_p(\Gamma) = 0$.[1] It follows directly from the definition that for any nonnegative Borel measurable function $f \in L^p(X)$, then $\int_\gamma f < \infty$ for p-a.e. absolutely continuous curve.

Next, we define upper gradients and p-weak upper gradients of functions $u : X \to \mathbb{R}$.[2]

Definition 2.16 (Upper gradients and p-weak upper gradients)**.** *Let $u : X \to \mathbb{R}$ and $G : X \to [0, \infty]$ be Borel measurable functions. We say that G is an upper gradient for u if*

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma G \tag{2.2}$$

for every absolutely continuous curve $\gamma : [a, b] \to X$. For $1 \leq p < \infty$, we say that G is a p-weak upper gradient for u if the upper gradient condition (2.2) *holds for p-a.e. absolutely continuous curve $\gamma : [a, b] \to X$.*

The following example shows that this is a natural notion of gradient.

Example 2.17. Consider Euclidean space as a rectifiable Riemannian space with the rectifiable atlas comprising only the identity chart. For a smooth function $u : \mathbb{R}^n \to \mathbb{R}$, the classical gradient $|\nabla u|$ is an upper gradient for u.

Furthermore, we note that the potential issues highlighted by Example 2.9 are eliminated with respect to this definition.

Example 2.18. Consider the rectifiable Riemannian space and the function f defined in Example 2.9. We see clearly that $G = 0$ is not a p-weak upper gradient for f, since the upper gradient condition 2.2 fails for any curve that crosses the hyperplane $\{x_1 = 0\}$. In fact, considering the family Γ of absolutely continuous curves of the form $\gamma(t) = (0, x') + te_1$ for $t \in (-\varepsilon, \varepsilon)$, we easily see that f has no p-weak upper gradient in $L^p(X)$.

[1]The notion of Mod_p measure on families of curves was first introduced by Ahlfors and Beurling in [AB50] and further developed by Fuglede in [Fug57] in the Euclidean and Riemannian settings; for the corresponding definition of families of curves with $\mathrm{Mod}_p(\Gamma) = 0$ in the metric measure space context, see [Haj03, Definition 5.1] and the equivalent formulation of the definition given in [Haj03, Theorem 5.5].

[2]The notion of weak upper gradient was first introduced by Heinonen and Koskela in [HK98], and the definition we give here is analogous to [Haj03, Definition 6.1].

Now, let $\tilde{W}^{1,p}(X)$ be the collection of all Borel measurable functions $u : X \to \mathbb{R}$ such that u is L^p integrable and u has a p-weak upper gradient $G \in L^p(X)$. The Sobolev space $W^{1,p}(X)$ on a rectifiable Riemannian space is defined in the following way.[3]

Definition 2.19. *For any $u \in \tilde{W}^{1,p}(X)$, we define*

$$\|u\|_{W^{1,p}(X)} = \|u\|_{L^p(X)} + \inf_G \|G\|_{L^p(X)}, \tag{2.3}$$

where the infimum is taken over all p-weak upper gradients G of u. We define the space $W^{1,p}(X) = \tilde{W}^{1,p}(X)/\sim$, where $u \sim v$ for $u, v \in \tilde{W}^{1,p}(X)$ if $\|u - v\|_{W^{1,p}(X)} = 0$.

From this point, we can establish a number of basic properties of the space $W^{1,p}(X)$ showing that this space possesses many of the important features of Sobolev spaces in smooth settings, which we collect in the following proposition. The analogous properties are established in the metric measure space setting in [Haj03, Section 7]. In fact, the proofs there can be carried over almost verbatim, with only the modification being the distinction between the use of absolutely continuous curves in our setting as opposed to rectifiable curves in the setting of metric measure spaces.[4]

Proposition 2.20 (Basic properties of the Sobolev space $W^{1,p}(X)$)**.** *Let (X, g) be a rectifiable Riemannian space and fix $1 \le p < \infty$. Then the following properties hold.*

1. *(Closedness) Suppose $\{u_i\}_{i=1}^\infty, \{G_i\}_{i=1}^\infty$ are sequences in $L^p(X)$ such that u_i and G_i converge weakly to $u \in L^p(X)$ and $G \in L^p(X)$ respectively. If G_i is a p-weak upper gradient of u_i for each $i \in \mathbb{N}$, then there is a representative of u in L^p such that G is a p-weak upper gradient of u.*
2. *(Lower semi-continuity) Let $p \in (1, \infty)$ and let $u_i \in W^{1,p}$ be a bounded sequence converging weakly in $L^p(X)$ to u. Then there is a representative of u such that $u \in W^{1,p}(X)$ and*
$$\|u\|_{W^{1,p}(X)} \le \liminf_{i\to\infty} \|u_i\|_{W^{1,p}(X)}. \tag{2.4}$$
3. *(Banach space) The space $W^{1,p}(X)$ is a Banach space.*
4. *(Minimal p-weak upper gradient) There exists a minimal p-weak upper gradient $G_u \in L^p(X)$ in the sense that $G_u \le G$ m-a.e. for every p-weak upper gradient $G \in L^p(X)$.*
5. *(Smooth spaces) Suppose (X, g) is a smooth Riemannian manifold, then $W^{1,p}(X)$ coincides with the standard Sobolev space of X. Moreover, the norm of gradient vector $|\nabla u|_g$ is the least p-weak upper gradient for $u \in W^{1,p}(X)$.*

Thanks to Proposition 2.20, for any $u \in W^{1,p}(X)$ we may write

$$\|u\|_{W^{1,p}(X)} = \|u\|_{L^p(X)} + \|G_u\|_{L^p(X)}, \tag{2.5}$$

where G_u is the least p-weak upper gradient of u.

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[3] We follow the definition first introduced by Shanmugalingam in [Sha00] in the context of metric measure spaces and presented in Definition 7.1 of [Haj03]. We note that a closely related definition of Sobolev spaces on a metric measure space was given by Cheeger in [Che99].

[4] In the metric measure space setting, properties (1)-(3), (5) were originally proven by Shanmugalingam [Sha00], and property (4) was established by Cheeger in [Che99] for $p > 1$ and Hajłasz [Haj03] for $p = 1$.

Given the definition of $W^{1,p}$ spaces on a rectifiable Riemannian space, the notion of the d_p distance extends naturally to this setting. That is, for a rectifiable Riemannian space (X, g), a real number $p \in (n, \infty)$ and two points $x, y \in X$, the d_p distance $d_{p,g,X}(x, y)$ between x and y is defined to be

$$d_{p,g,X}(x,y) = \sup\left\{ |f(x) - f(y)| : \int_X |\nabla f|^p \, dm \leq 1, \, f \in W^{1,p}_{loc}(X) \cap C^0_{loc}(X) \right\}. \tag{2.6}$$

As we have seen in the examples above, rectifiable Riemannian spaces, along with their Sobolev spaces and d_p distances, can exhibit various types of degenerate behavior. To close this section, we introduce two definitions formalizing when certain degenerate behaviors do not occur. These are $W^{1,p}$-rectifiable completeness and d_p-rectifiable completeness of a rectifiable Riemannian space.

We begin with $W^{1,p}$-rectifiable completeness. Without imposing any additional structure, the space $W^{1,p}$ on a rectifiable Riemannian space may be trivial. Moreover, as we saw in Example 2.9, the usual coordinate expression for the norm of the gradient may not be meaningful. For this reason, we introduce the notion of rectifiable Riemannian spaces that are $W^{1,p}$*-rectifiably complete* as spaces for which the space $W^{1,p}$ is sufficiently large and the minimal p-weak upper gradient coincides with the derivative in charts almost everywhere.

Definition 2.21 ($W^{1,p}$-rectifiable completeness)**.** *Fix $p > n$. We say that (X, g) is $W^{1,p}$-rectifiably complete if the following holds:*

(a) *$W^{1,p}(X)$ is dense in $L^p(X)$;*
(b) *For all $u \in W^{1,p}(X)$ and $a \in I$, the function $u_a = \phi_a^* u : U_a \to \mathbb{R}$ is differentiable a.e. and*
$$G_u(\phi_a(x)) = |\nabla u|_g \equiv \sqrt{g_a^{-1}(\partial u_a(x), \partial u_a(x))}$$
for $\phi_a^* m$-a.e. $x \in U_a$. Here, ∂u_a denotes the Euclidean gradient of u_a.

Example 2.22. A smooth Riemannian manifold is $W^{1,p}$-rectifiably complete for any $p \in (n, \infty)$.

Example 2.23. It is easy to check that the rectifiable Riemannian space of Example 2.6 is $W^{1,p}$-rectifiably complete for any $p \in (n, \infty)$.

Example 2.24. Once more, for $\alpha > 0$ consider the rectifiable Riemannian space $(\mathbb{R}^2, g_\alpha)$, where $g_\alpha = dx^2 + |x|^{2\alpha} dy^2$ introduced in Examples 1.4 and 2.8. Fix $p > 2$. There exists $\alpha = \alpha(p) \in (0, 1/2p)$ such that $(\mathbb{R}^2, g_\alpha)$ is $W^{1,p}$-rectifiably complete.

Example 2.25. We will see in Theorem 2.35 below that d_p limits of sequences of smooth Riemannian manifolds satisfying uniform lower bounds on scalar curvature and entropy are $W^{1,p}$-rectifiably complete for suitably chosen p.

Next, we introduce the notion of d_p-rectifiable completeness. In Example 1.4, we saw an example of a degenerate metric on a rectifiable Riemannian space for which d_p only defined a pseudometric (for p large), and so the topology generated by d_p was not the same as the topology as the underlying space. A d_p*-rectifiably complete* rectifiable Riemannian space is one for which this does not happen:

Definition 2.26 (d_p-rectifiable completeness)**.** *Given a rectifiable Riemannian space (X, g), we say that (X, g) is d_p-rectifiably complete if d_p defines a metric on X and the topology induced by d_p coincides with the topology of X.*

Example 2.27. A smooth Riemannian manifold is d_p-rectifiably complete for any $p \in (n, \infty)$.

Example 2.28. Once more, for $\alpha > 0$ consider the rectifiable Riemannian space $(\mathbb{R}^2, g_\alpha)$, where $g_\alpha = dx^2 + |x|^{2\alpha} dy^2$ introduced in Examples 1.4 and 2.8. Fix $p > 2$. There exists $\alpha = \alpha(p) \in (0, 1/2p)$ such that $(\mathbb{R}^2, g_\alpha)$ is d_p-rectifiably complete.

Example 2.29. We will see in Theorem 2.35 below that d_p limits of sequences of smooth Riemannian manifolds satisfying uniform lower bounds on scalar curvature and entropy are d_p-rectifiably complete for suitably chosen p.

2.3. d_p convergence.

Our primary interest in the d_p distance is to give rise to a notion of convergence which captures the convergence of $W^{1,p}$ Sobolev spaces. Now that we have the d_p distance defined and the correct category of spaces to consider it on, namely rectifiable Riemannian spaces (X, g), we may precisely define this convergence.

We begin with d_p convergence of compact sequences. Observe that a compact rectifiable Riemannian space (X, g) that is d_p-complete can be viewed as a metric measure space $(X, d_{p,g}, d\mathrm{vol}_g)$. For such spaces, we say that (X_i, g_i) converges to (X, g) in the d_p sense if the corresponding metric measure spaces $(X_i, d_{p,g_i}, d\mathrm{vol}_{g_i})$ converge to the metric measure space $(X, d_{p,g}, d\mathrm{vol}_g)$ in the measured Gromov-Hausdorff sense.

Definition 2.30 (d_p convergence)**.** *Let (X, g) and (Y, h) be compact d_p complete rectifiable Riemannian spaces. Given $\varepsilon > 0$, we say that*

$$d_p((X, g), (Y, h)) \leq \varepsilon \tag{2.7}$$

if there exist collections of points $\{x_i\}_{i=1}^N \subset X$ and $\{y_i\}_{i=1}^N \subset Y$ such that each collection is ε-dense with respect to d_p and

$$|d_{p,g,X}(x_i, x_j) - d_{p,h,Y}(y_i, y_j)| \leq \varepsilon \tag{2.8}$$

and for all $r \in [\varepsilon, 1]$, we have

$$1 - \varepsilon \leq \frac{\mathrm{vol}_g(\mathcal{B}_p(x_i, r))}{\mathrm{vol}_h(\mathcal{B}_p(y_i, r))} \leq 1 + \varepsilon. \tag{2.9}$$

In other words, two compact spaces are ε close in the d_p sense if their d_p metric spaces are ε Gromov-Hausdorff close and the volumes of balls above scale ε are close. With this definition, we may rephrase the conclusion of Theorem 1.5 by saying that $d_p((\mathcal{B}_{p,g}(x, 1), (\mathcal{B}_{p,g_{euc}}, 1)) \leq \varepsilon$ for every $x \in M$.

Remark 2.31. In (2.9), we require volumes of balls to be ε-close up to scale 1. Up to scaling, we may replace 1 with any other fixed number.

Remark 2.32. One could easily replace mGH with Intrinsic Flat distance in the above and the results of this paper are the same. The main key for us is the weakening of the usual distance with $p = \infty$ to $p < \infty$.

We now move toward defining pointed d_p convergence for noncompact spaces. This notion is defined in a similar spirit as in the Gromov-Hausdorff case, but there is a subtle point due to the behavior of d_p at large distances. Recall that a sequence of pointed proper metric spaces (X_i, d_i, x_i) is said to converge to a pointed proper metric (X, d, x) in the pointed Gromov-Hausdorff topology if $(\overline{B}_{d_i}(x_i, R), d_i) \to (\overline{B}_d(x, R), d)$ in the Gromov-Hausdorff topology for every $R > 0$.

One is initially tempted to define pointed d_p convergence in direct analogue of this definition by asking for d_p convergence on the closures of p-balls of increasingly large radius. However, in view of Example 1.3, we see that this cannot be the correct notion. Indeed, we have seen that the hyperbolic space equipped with the d_p metric is *not* a proper metric space for $p > n$, since balls of sufficiently large radius have noncompact closure.

Instead, we make use of d_p completeness to construct an exhaustion $\{Cov(x, N)\}_{N\in\mathbb{N}}$ that plays the role of balls of large radius. Roughly speaking, $Cov(x, N)$ is the set of points that are linked to x by a sequence of N precompact p-balls of radius at most 1, and, importantly, has compact closure. More specifically, let (X, g, x) be a d_p-complete pointed rectifiable Riemannian space. By d_p completeness, for any $y \in X$, there is some radius $r \leq 1$ sufficiently small such that $\mathcal{B}_p(y, 4r) \Subset X$ has compact closure. Define $Cov(x, N)$ be to the collection of points y such that there is $\{(z_i, r_i)\}_{i=1}^N$ satisfying

1. (1) $r_i \leq 1$;
2. (2) $x, y \in \cup_{i=1}^N \mathcal{B}_p(z_i, r_i)$;
3. (3) $\mathcal{B}_p(z_i, r_i) \cap \mathcal{B}_p(z_{i+1}, r_{i+1}) \neq \emptyset$ for all $i = 1, ..., N-1$;
4. (4) $\mathcal{B}_p(z_i, 4r_i)$ is pre-compact.

Note that $Cov(x, N)$ is an open set, and that by the triangle inequality, we always have the containment $Cov(x, N) \subseteq \mathcal{B}_p(x, 2N)$. To get an intuitive idea for how the sets $Cov(x, N)$ behave, if we define the analogue of $Cov(x, N)$ with respect to the geodesic distance instead of the d_p distance, then on any Riemannian manifold (or more generally, on any proper length space), this set is simply a geodesic ball of radius $2N$.

The important advantage of working with the sets $Cov(x, N)$ instead of p-balls of increasing radius is that the set $Cov(x, N)$ has compact closure for any $N \in \mathbb{N}$, and $\{Cov(x, N)\}_{N\in\mathbb{N}}$ exhausts X in the sense that for any compact connected set $\Omega \subset X$ containing x, there exists $N \in \mathbb{N}$ such that $\Omega \subset Cov(x, N)$. We can now define pointed d_p convergence.

Definition 2.33. *Let (X_i, g_i, x_i) and (X, g, x) be d_p-complete pointed rectifiable Riemannian spaces. We say that*

$$(X_i, g_i, x_i) \to (X, g, x) \tag{2.10}$$

in the pointed d_p sense if the following holds. For all $N \in \mathbb{N}$, there exists $N' \geq N$ and compact sets $\Omega \subset X$ and $\Omega_i \subset X_i$ such that

1. (1) $Cov(x, N) \subset \Omega \subset Cov(x, N')$,

(2) $Cov_i(x_i, N) \subset \Omega_i \subset Cov_i(x_i, N')$ *for* i *sufficiently large,*
(3) $d_p((\Omega_i, g_i), (\Omega, g)) \to 0$.

Remark 2.34. Observe that in part (3) of Definition 2.33 above, the d_p convergence on the compact sets Ω_i, Ω corresponds to the relative d_p distances d_{p,g_i,Ω_i} and $d_{p,g,\Omega}$. This is necessary as d_p is not a local object.

2.4. **Compactness and Structure of Limit Spaces.** We have introduced rectifiable Riemannian spaces and their $W^{1,p}$ spaces because these are the objects that arise naturally as (pointed) d_p limits of spaces with lower bounds on scalar curvature and entropy, as we see in the following theorem. Indeed, a sequence of Riemannian manifolds satisfying small uniform lower bounds on scalar curvature is precompact with respect to pointed d_p convergence and subsequentially converges to a rectifiable Riemannian space. This limiting rectifiable Riemannian space, having arisen as the limit of "nice" objects, enjoys additional topological and geometric structure.

Theorem 2.35 (Structure of limit spaces)**.** *Let* $\{(M_i, g_i, x_i)\}$ *be a sequence of complete pointed Riemannian manifolds with bounded curvature and let* $p \geq n+1$*. Then there exists* $\delta = \delta(n,p) > 0$ *such that if*

$$R_{g_i} \geq -\delta, \qquad \nu(g_i, 2) \geq -\delta\,, \tag{2.11}$$

then there exists a pointed rectifiable Riemannian space (X, g, x)*, with* X *topologically but not necessarily metrically a smooth manifold, such that the following holds.*

(1) *We have* $d_p((M_i, g_i, x_i), (X, g, x)) \to 0$ *in the pointed sense of Definition 2.33.*
(2) *The space* (X, g, x) *is* $W^{1,p}$*-rectifiably complete in the sense of Definition 2.21.*
(3) *The space* (X, g, x) *is* d_p*-rectifiably complete in the sense of Definitions 2.26.*

The first part of the theorem tells us that there exists a rectifiable space X to which the M_i converge, and the underlying topological space X is a smooth topological manifold. The second part of the theorem tells us that limit space is $W^{1,p}$ complete, which in particular guarantees that the gradient of a function is indeed the coordinate gradient that one would compute in rectifiable charts. The third part tells us that the underlying topological space is a smooth topological manifold, and the metric d_p generates the topology of X.

As we have emphasized, it may be that X does not have a well behaved metric structure and this convergence may not be in the Gromov-Hausdorff or Intrinsic Flat sense. Even if the sequence does have a (geodesic) Gromov-Hausdorff limit (Y, d), the spaces X and Y need not even be topologically equivalent. In fact, it may be the case that for $q >> p$, the space has a d_q limit that is not d_q-rectifiably complete and thus may not be topologically equivalent to the smooth topological space X. We see these points explicitly demonstrated in the examples of the next section.

3. Examples

The concepts of the d_p distance and corresponding d_p convergence were introduced as necessary relaxations of the notions of the geodesic distance and corresponding distance function-based notions of convergence in the context of sequences of spaces with lower scalar curvature and entropy bounds.

In particular, we have discussed how the distance function itself is almost entirely uncontrollable for such sequences. Let us now make this precise, and in the process see that the d_p convergence in Theorem 1.5 cannot be replaced with Gromov-Hausdorff convergence or Intrinsic Flat convergence.

In this section, we will collect and outline various examples of sequences of complete Riemannian manifolds (M_i, g_i) with bounded curvature that satisfy the almost non-negative entropy and scalar curvature assumptions of our main theorems. In each example, the d_p limits of our spaces will be either Euclidean space or a flat torus, and these limits do not agree with their Gromov-Hausdorff and Intrinsic Flat limits. Constructing examples with almost non-negative scalar curvature is not too challenging, but showing that the entropies are well behaved takes quite a bit more work. Philosophically, this example will be similar to situations studied very recently by Allen-Sormani [AS20], without the lower scalar curvature and entropy requirements, see also [AS19].

For instances, we will construct a sequence of metric g_i on torus $\mathbb{T}^n, n \geq 4$ such that $R_{g_i} \geq -i^{-1}$, $\nu(g_i, 2) \geq -\varepsilon$ and g_i converges to a point in Gromov-Hausdorff topology or to a zero current in the Intrinsic Flat topology while its d_p limit is the flat torus for all $p \geq n+1$. This example will be given precisely in Example 3.7. In fact, we will construct both compact or complete noncompact examples with almost non-negative scalar curvature and entropy and at the same time the the Gromov-Hausdorff and Intrinsic Flat limits are not locally Euclidean. For example, we are able to construct sequence of Riemannian metric g_i on $\mathbb{R}^n, n \geq 4$ with bounded curvature such that the scalar curvature R_{g_i} and entropy $\nu(g_i, 2)$ both converging to 0 so that pointed Gromov-Hausdorff limit is the taxicab metric on Euclidean space while the d_p limit is the Euclidean space for all $p \geq n+1$. This example will be given more precisely in Example 3.5. These examples demonstrate that one cannot replace d_p closeness with Gromov-Hausdorff or Intrinsic Flat closeness in Theorem 1.5. Furthermore, in Example 3.4, we will show that the value of p for which we establish d_p convergence in Theorem 1.5 cannot be taken arbitrarily large for fixed δ.

3.1. The basic building block: a two-parameter family of metrics.

We begin by outlining the construction of a two-parameter family of metrics on $\mathbb{R}^{n+1}$ for $n \geq 3$ that serve as the basic building block for constructing all of our examples. Let h denote the standard metric on $\mathbb{S}^{n-1}$. We define the two-parameter family of metrics $g_{\delta,\varepsilon}$ on $M = \mathbb{R}_+ \times \mathbb{S}^{n-1} \times \mathbb{R}$ by

$$g_{\delta,\varepsilon} = dr^2 + f^2_{\delta,\varepsilon}(r)h + \varphi^2_{\delta,\varepsilon}(r)dx^2\,. \tag{3.1}$$

The warping factor $f_{\delta,\varepsilon}$ is used to identify $\mathbb{R}^+ \times \mathbb{S}^{n-1}$ topologically with $\mathbb{R}^n$, however geometrically this will be done in a way to add a large amount of positive curvature to the space. The warping factor $\varphi_{\delta,\varepsilon}$ is constructed so that it will *slowly* degenerate as $r \to 0$. If this degeneration is sufficiently slow, we can preserve the lower bound on scalar curvature, and, much more challenging, the lower bound on entropy as well. If $\varphi(0) = 0$, then this would imply that the line $\{0^n\} \times \mathbb{R}$ has a fully degenerate metric g along it, in particular $d_g((0^n, s), (0^n, t)) = 0$ for any two points along the line $\{0^n\} \times \mathbb{R}$. The parameters $\epsilon, \delta > 0$ are built so that we may approach such a degenerate limit smoothly and in different ways, depending on our end goal.

Crucially, this two-parameter family of metrics satisfies a lower bound on entropy and scalar curvature that is uniform for all ε and δ sufficiently small. Geometrically, what is happening is that the warping factor is changing so slowly that even though the actual metric geometry may be behaving very poorly, in some weaker sense the geometry looks very Euclidean at all points and scales. This sense of closeness to Euclidean space will be good enough to force the small lower entropy bound on the example.

Theorem 3.1. *Fix $n \geq 3$, $\eta > 0$ and $L > 0$. There exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ depending on n, η and L such that the following holds. For all $\varepsilon \leq \varepsilon_0$ and $\delta \leq \delta_0$, the metric $g_{\delta,\varepsilon}$ defined in* (3.1) *satisfies*

$$R_{g_{\delta,\varepsilon}} \geq -\eta, \qquad \nu(g_{\delta,\varepsilon}, L) \geq -\eta. \tag{3.2}$$

Remark 3.2. Note that the metrics $g_{\delta,\varepsilon}$ are defined on an $n+1$ dimensional space, so fixing $n \geq 3$ means that our examples are of dimension 4 or higher.

Let us again discuss the examples geometrically, this time with more of a focus on how each parameter behaves in the construction. One can think of the metric $g_{\delta,\varepsilon}$ defined in (3.1) in the following way. The portion $dr^2 + f^2_{\delta,\varepsilon}(r)h$ of the metric $g_{\delta,\varepsilon}$ agrees with the Euclidean metric on $\mathbb{R}^n$ far from $0 \in \mathbb{R}^n$, while in a neighborhood of $0 \in \mathbb{R}^n$, it is a smoothed-out cone metric on $\mathbb{R}^n$ with cone angle proportional to δ. The parameter ε governs the scale at which this cone metric is smoothed out. This component can roughly be thought of as Euclidean $\mathbb{R}^n$, although taking the smoothed cone in place of $\mathbb{R}^n$ provides a crucial positive scalar curvature contribution in order to guarantee that the scalar curvature lower bound (3.2) holds as long as $n \geq 3$. We let $\varphi_{\delta,\varepsilon} : \mathbb{R}_+ \to \mathbb{R}_+$ be the function which is roughly defined as $\varphi_{\delta,\varepsilon}(r) \approx \min\{r^\delta, 1\}$ in the interval (ε, ∞) but has $\varphi_{\delta,\varepsilon}(0) = \varepsilon^\delta$ and is smoothed out at scale ε. In this way, the component $\varphi^2_{\delta,\varepsilon}(r)dx^2$ adds a fiber at each point on $(\mathbb{R}^n, dr^2 + f^2_{\delta,\varepsilon}(r)h)$. Away from $0 \in \mathbb{R}^n$, these fibers are Euclidean, but for r small, the fibers become increasingly degenerate.

Due to the prescribed behavior, the sequence of metric will converge to the Euclidean metric away from the ray $\ell = \{x : x_i = 0 \text{ for } i = 1, \dots, n\}$ in $\mathbb{R}^{n+1}$ if we pass $\varepsilon, \delta \to 0$. Since $\varphi_{\delta,\varepsilon}(r) = \varepsilon^\delta$ nearby $r = 0$, by choosing the parameters ε and δ to be coupled in different ways and sending one or both parameters to zero, we obtain different limiting rectifiable Riemannian spaces. For instances, if we choose $\varepsilon \approx \delta$ and let $\varepsilon \to 0$, then the metric tensors converge smoothly to the Euclidean metric $g_\infty = \lim_{\varepsilon\to 0} g_{\varepsilon,\varepsilon} = \sum_{i=1}^{n+1}(dx^i)^2$. While if we choose $\varepsilon << \delta$, then the limiting metric will be degenerating along ℓ. In both of these two examples, the constructed sequence converges to the Euclidean metric in L^p_{loc} for all $p > 1$, while in the latter case the Gromov-Hausdorff limit is very different; see Example 3.3 below. This will correspond to our general d_p ε-regularity theorem when the entropy and scalar curvature have lower bound converging to 0.

3.2. **Examples constructed from the main building block.** In the following, we will make use of the metrics $g_{\delta,\varepsilon}$ with the parameters ε and δ coupled appropriately to produce sequences of metrics whose d_p limits and Gromov-Hausdorff limits are entirely different.

Example 3.3 (Collapsing along a line in Euclidean space)**.** Let $n \geq 3$. By choosing $\delta = \delta(\varepsilon) \to 0$ so that $\varepsilon^\delta \to 0$ in (3.1), we obtain a sequence of metrics which degenerate along a ray in $\mathbb{R}^{n+1}$ and remain the flat metric away from it. In the Gromov-Hausdorff limit, the ray collapses to a point. On the other hand g_ε converges to the Euclidean metric in $L^p_{loc}(\mathbb{R}^{n+1})$ for all $p > 1$ and one can show that the pointed d_p limit is the flat Euclidean space. In particular, $\mathrm{vol}_{g_\varepsilon}(\mathcal{B}_{p,g_\varepsilon}(0,1)) \to \mathrm{vol}_{g_{euc}}(\mathcal{B}_{p,euc}(0,1))$ as $\varepsilon \to 0$, while the volumes of metric balls are tending to infinity:

$$\mathrm{vol}_{g_\varepsilon}(B_{g_\varepsilon}(0,1)) \to +\infty\,. \tag{3.3}$$

Indeed, for ε sufficiently small, $B_{g_\varepsilon}(0,1)$ contains the Euclidean strip $\{(x,y) \in \mathbb{R}^n \times \mathbb{R} : \frac{1}{4} \leq |x| \leq \frac{1}{2}, |y| \leq \frac{1}{2\varepsilon}\}$. Since g_ε converges smoothly uniformly to the Euclidean metric away from $|x| = 0$, we see that (3.3) holds.

We see from the above example that the metric degeneration which causes the metric collapse occurs along a line in $\mathbb{R}^4$.

Example 3.4 (d_p convergence does not hold for all p)**.** In contrast to Example 3.3, in this example we only pass the smoothing parameter $\varepsilon \to 0$ but fix $\delta > 0$ small in the construction of (3.1). The corresponding sequence of metrics converges pointwise, and in L^p_{loc} for p less than some $p_1(\delta)$, to $g_\infty = g_{cone} + r^\delta dx^2$, which degenerates at $r = 0$. One can prove that the sequence converges in the pointed d_p sense to $(\mathbb{R}^{n+1}, g_\infty, 0^n)$ for $p \in [n+2, p_0(\delta)]$. However, this d_p convergence to $(\mathbb{R}^{n+1}, g_\infty, 0^n)$ does not hold for all $p \in [n+2, \infty)$. Indeed, for p sufficiently large, the metric space $(\mathbb{R}^{n+1}, d_{p,g_\infty}, 0^n)$ is topologically distinct from the underlying topology on $\mathbb{R}^{n+1}$, and in particular is not d_p-complete. This illustrates that δ must be taken to depend on p in our ε-regularity theorems, and if we only assume a lower bound on the entropy lower bound and scalar curvature along the sequence, then the limiting rectifiable Riemannian metric g_∞ may have an inverse that is only bounded in $L^{p_0}_{loc}$ for some $p_0(\delta) > 1$ but not all $p > 1$.

Example 3.5 (Collapsing lines in Euclidean space)**.** In this example, we use the building block of Example 3.3 to construct a sequence of metrics on $\mathbb{R}^{n+1}$ for $n \geq 3$ whose Gromov-Hausdorff limit is the taxicab metric, while the d_p limit is the flat metric on $\mathbb{R}^{n+1}$. The basic idea of the construction is to cut off the building block of Example 3.3 to obtain a degenerating metric on a tubular neighborhood of a line in Euclidean space, and to glue this metric into tubular neighborhoods of an increasing dense collection of lines in $\mathbb{R}^{n+1}$. In this way, the same argument in the proof of Theorem 3.1 will infer that the sequence have $R_{\tilde{g}_{r_0}} \geq -r_0$ and $\nu(\tilde{g}_{r_0}, 2) \geq -r_0$ as $r_0 \to 0$ by relabeling the indices. Direct computation shows that $\tilde{g}_{r_0}$ converges to $g_{\mathbb{R}^{n+1}}$ in $L^p_{loc}(\mathbb{R}^{n+1})$ for all $p \geq 1$. In particular, this implies that $\mathrm{vol}_{\tilde{g}_{r_0}}(\Omega) \to \mathrm{vol}_{euc}(\Omega)$ for any compact set $\Omega \subset \mathbb{R}^{n+1}$. One can show that the sequence converges to $(\mathbb{R}^n, g_{euc}, 0^n)$ in the pointed d_p sense for all $p \in [n+1, \infty)$.

However, in the pointed Gromov-Hausdorff topology, this sequence converges to the taxicab metric. To roughly explain this, consider the metrics $(\mathbb{R}^{n+1}, \hat{g}_\epsilon) \equiv (\mathbb{R}^{n+1}, \epsilon^{-2}\tilde{g}_{r_0})$. Here, $\varepsilon = \varepsilon(r)$ is the parameter chosen above. Clearly the metrics $(\mathbb{R}^{n+1}, \hat{g}_\epsilon)$ are isometric to $(\mathbb{R}^{n+1}, \tilde{g}_{r_0})$ by a Euclidean dilation. Let $\ell_{ij} \equiv \pi_j^{-1}(z_{ij})$ denote the lines we have glued around. Then, roughly, we have on each such line ℓ_{ij} that $\hat{g}_\epsilon = 1$ in the direction of this line, and that $\hat{g}_\epsilon \approx \epsilon^{-2}$ in all other directions and at all other points. We also have, in coordinates, that these lines are $o(\epsilon)$ dense. Clearly, a path of minimal length from x to y is now one which stays on these lines as long as possible, and moving from one line to another now causes an error which is approximately $\epsilon^{-1}o(\epsilon) = o(1)$. In particular, we see that $d_{\hat{g}_\epsilon}(x,y) = \sum |x_i - y_i| + o(1)$. Further, as $\epsilon \to 0$ we see a minimal path is any path which is always moving in coordinate directions (specifically, along our increasing dense collection of lines ℓ_{ij}). Hence, $(\mathbb{R}^n, \hat{g}_\epsilon)$ is limiting to the taxi-cab metric.

Next, we construct some examples in the compact setting by taking quotient on $\mathbb{R}^n$.

Example 3.6 (Collapsing circle in torus)**.** By taking a quotient of the sequence constructed in Example 3.3, we construct a sequence of metrics $\{g_i\}_{i \in \mathbb{N}}$ on the torus $\mathbb{T}^{n+1}$ for $n \geq 3$ such that each

g_i coincides with the flat metric away from a shrinking tubular neighborhood of a fixed $S^1 \subset \mathbb{T}^{n+1}$. This sequence has a scalar curvature lower bound tending to zero, and the entropy lower bound $-\delta$ can be made arbitrarily small, uniformly along the sequence. The sequence g_i becomes degenerate along this S^1, and in the Gromov-Hausdorff limit, the S^1 collapses to a point. In particular, the metric space arising as the Gromov-Hausdorff limit is not topologically a torus. The d_p limit will be the flat torus for any $p \geq n+2$.

By replicating the construction of the degeneracy, we can construct examples so that the sequence of metrics on the torus $\mathbb{T}^{n+1}$ converges to a metric space Y^k with $k < n+1$ or even $k = 0$ (a point) in the Gromov-Hausdorff topology. The following is an example showing that the Gromov-Hausdorff limit can be fully degenerated to a single point.

Example 3.7 (Collapsing $\mathbb{T}^{n+1}$ to a point)**.** By taking a quotient in Example 3.5, it produces a sequence of metrics on $\mathbb{T}^{n+1}$ such that the sequence collapses to a point in the Gromov-Hausdorff and Intrinsic Flat topologies. The basic idea of the construction is to choose an increasingly dense collection of strips around copies of $S^1 \subset \mathbb{T}^{n+1}$ with all different orientations, in a similar fashion to Example 3.5, and then to paste the degenerating metrics of Examples 3.6.This sequence has a scalar curvature lower bound tending to zero, and the entropy lower bound $-\delta$ can be made arbitrarily small, uniformly along the sequence.

Due to the existence of increasing dense fibre in all different orientations, the metrics will converge in the Gromov-Hausdorff topology to a point. Furthermore by [SW11, Corollary 3.21], we have $(\mathbb{T}^{n+1}, \tilde{g}_{r_0})$ converging to the zero current in the Intrinsic Flat sense. However, $(\mathbb{T}^{n+1}, \tilde{g}_{r_0})$ converges to the flat $n+1$-dimensional torus in the d_p sense for each $p \in [n+2, \infty)$. In fact, by implementing the above gluing except for certain orientation, it is not difficult to construct examples such that the Gromov-Hausdorff limit has Hausdorff dimension in between $n+1$ and 0.

4. Applications and Open Problems

Let us conclude by discussing some applications of the results discussed here to the underlying structure of spaces with lower scalar curvature and entropy bounds, as well as some conjectures and open problems. First, as an application of the proof of Theorem 1.5, we prove that Riemannian manifolds satisfying a uniform lower bound on entropy and scalar curvature satisfy a Morrey-Sobolev embedding with a uniform constant.

Theorem 4.1 (L^∞ Sobolev Embedding)**.** *Let (M^n, g) be a complete Riemannian manifold with bounded curvature and let $p \geq n+1$ and $q > n$. There exists $\delta = \delta(n,p,q) > 0$ and $C_{n,q} > 0$ such that if*

$$R \geq -\delta, \qquad \nu(g,2) \geq -\delta\,, \tag{4.1}$$

then for all $f \in W^{1,q}(M)$, we have

$$\|f\|_{L^\infty(M)} \leq C_{n,q}\left(\|\nabla f\|_{L^q(M)} + \|f\|_{L^q(M)}\right). \tag{4.2}$$

More locally, for all $x_0 \in M$ and $f \in W_0^{1,q}(\mathcal{B}_{p,g}(x_0,1))$, we have

$$\|f\|_{L^\infty(\mathcal{B}_{p,g}(x_0,1))} \leq C_{n,q}\|\nabla f\|_{L^q(\mathcal{B}_{p,g}(x_0,1))}\,. \tag{4.3}$$

In terms of the d_p distance we can upgrade this to a Hölder embedding: there exists $\alpha = \alpha(n, q) \in (0, 1)$

$$|f(x) - f(y)| \leq C_{n,q,p} d_p(x, y)^{\alpha} \|\nabla f\|_{L^q(\mathcal{B}_{p,g}(x_0,1))} \tag{4.4}$$

for all $x, y \in \mathcal{B}_{p,g}(x_0, 1)$.

Remark 4.2. The examples of Section 3 demonstrate that the Hölder embedding of (4.3) cannot hold with the geodesic distance in place of the d_p-distance.

The proof of Theorem 1.5 can also be applied to obtain an an a priori L^q bound for the scalar curvature of closed Riemannian manifolds with lower bounds on scalar curvature and entropy, for $q < 1$:

Theorem 4.3 (L^q scalar curvature estimates)**.** *Let (M^n, g) be a closed Riemannian manifold and let $\epsilon > 0$ and $q \in (0, 1)$ be fixed. There exists $\delta = \delta(n, q, \varepsilon) > 0$ such that if*

$$R \geq -\delta, \qquad \nu(g, 2) \geq -\delta\,, \tag{4.5}$$

then we have

$$⨍_M |R|^q \, d\mathrm{vol}_g \leq \varepsilon\,. \tag{4.6}$$

Motivated by Theorem 4.3, we expect that the scalar curvature satisfies a priori bounds in L^1.

Conjecture 4.4. *Let (M^n, g) be a closed Riemannian manifold with $R, \nu(g, 2) \geq -A$, then there exists $B(n, A) > 0$ such that*

$$⨍_M |R| \, d\mathrm{vol}_g \leq B\,. \tag{4.7}$$

More generally one should expect a full structure theory to hold for spaces with lower bounds on scalar curvature and entropy, not just small lower bounds. The ε-regularity of this paper should play the usual role of controlling the regular set.

The examples of Section 3 show that, in dimension $n \geq 4$, the distance functions of Riemannian manifolds with lower bounds on scalar curvature and entropy cannot be controlled in any meaningful way. It remains an open question whether similar examples exist in dimension 3, or alternatively, whether one strengthen the ε-regularity results of Theorem 1.5, known to be sharp for $n \geq 4$, in the case $n = 3$.

Open Problem 4.5. *Show one of the two following hold:*

1. *Build a sequence of metrics g_i on $\mathbb{R}^3$ with $R_i \geq -1/i$ and $\nu(g_i, 2) \geq -1/i$ such that the metric space structures do not converge to Euclidean space in the Gromov-Hausdorff or Intrinsic Flat sense.*
2. *Strengthen Theorem 1.5 in the case $n = 3$ so that the conclusion holds with respect to Gromov-Hausdorff or Intrinsic Flat convergence.*

4.1. **Acknowledgements.** M.-C. Lee was partially supported by the National Science Foundation under Grant No. DMS-1709894 and EPSRC grant number EP/T019824/1. R. Neumayer was partially supported by the National Science Foundation under Grant No. DMS-1901427, as well as Grant No. DMS-1502632 "RTG: Analysis on manifolds" at Northwestern University. Aaron Naber was partially supported by the National Science Foundation Grant No. DMS-1809011. We thank Brian Allen, Dan Lee, and Christina Sormani for helpful comments.

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