

MINIMIZING CLUSTERS WITH PRESCRIBED ASYMPTOTIC GEOMETRY

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ABSTRACT. We construct locally minimizing $(1, 2)$ -clusters whose exterior interfaces are asymptotic to various prescribed singular area-minimizing cones. For $n+1 \leq 7$, Bronsard & Novack characterized all minimizing $(1, 2)$ -clusters as standard lenses, whose exterior interface is planar. For $n+1 \in [8, 2700]$, the authors together with Bronsard showed the existence of a locally minimizing $(1, 2)$ -cluster whose exterior interface blows down to *some* (unknown, possibly non-unique) singular area-minimizing hypercone. For $n+1 = 8$, this was shown independently by Novaga, Paolini & Tortorelli.

Here we develop a refined construction using the Hardt-Simon foliation that realizes prescribed cones. For a singular area-minimizing hypercone C that has an isolated singularity or is cylindrical, we show that if C satisfies an explicit energy bound, then there is a locally minimizing $(1, 2)$ -cluster whose exterior interface is asymptotic to C with quantitative rates. In fact, if C is an area minimizing Lawson cone satisfying this energy bound, we produce a countably infinite family of distinct locally minimizing clusters asymptotic to C , distinguished by their prescribed asymptotic decay to leading order.

We verify this energy bound for the generalized Simons cones $C_{k,k}$ in every even ambient dimension $n+1 = 2k+2 \geq 8$, and for the cylindrical cone $C_{3,3} \times \mathbb{R}$ in $\mathbb{R}^9$, where $C_{3,3}$ is the Simons cone, therefore answering the cone realization problem in these cases. This in particular removes the upper bound of 2700 on the ambient dimension when $n+1$ is even in our preceding work.

1. INTRODUCTION

1.1. Overview. The classical multiple bubble problem asks for least-area partitions of $\mathbb{R}^{n+1}$ into finitely many chambers of prescribed finite volume together with one infinite volume chamber. It has been the subject of extensive work in recent decades [5, 10, 15, 17, 20, 22, 25–28, 35, 37], including the breakthrough work of Milman & Neeman [21]. In recent years, this problem has been extended to clusters with multiple infinite-volume chambers, introducing a new feature absent from the classical multiple bubble problem: the interface between the two infinite chambers may have nontrivial asymptotic geometry.

The simplest instance is the $(1, 2)$ -cluster problem. A $(1, 2)$ -cluster is a partition $\mathcal{X} = (\mathcal{X}(1), \mathcal{X}(2), \mathcal{X}(3))$ of $\mathbb{R}^{n+1}$ into three sets of locally finite perimeter, called chambers, with $0 < |\mathcal{X}(1)| < \infty$ and $|\mathcal{X}(2)| = |\mathcal{X}(3)| = \infty$. A $(1, 2)$ -cluster $\mathcal{X}$ is (locally) minimizing if, for every $\rho > 0$,

$$\mathcal{P}(\mathcal{X}; B_\rho(0)) \leq \mathcal{P}(\mathcal{X}'; B_\rho(0))$$

whenever $\mathcal{X}' = (\mathcal{X}'(1), \mathcal{X}'(2), \mathcal{X}'(3))$ has $\mathcal{X}(i) \Delta \mathcal{X}'(i) \subset\subset B_\rho(0)$ and $|\mathcal{X}(i)| = |\mathcal{X}'(i)|$ for $i = 1, 2, 3$. Here,

$$\mathcal{P}(\mathcal{X}; B_\rho(x)) := \frac{1}{2} \sum_{i=1}^3 P(\mathcal{X}(i); B_\rho(x)), \quad (1.1)$$

where $P(\mathcal{X}(i); B_\rho(x))$ is the relative perimeter of $\mathcal{X}(i)$ in $B_\rho(x)$. The $(1, 2)$ -cluster problem, and the more general (M, N) -cluster problem, was first introduced by Alama, Bronsard & Vriend [1] in relation to the structure tri-block copolymers in the regime when one phase is infinitesimal in volume relative to the other two. There have since been numerous works concerning the existence, classification and stability of such clusters [2, 3, 7–9, 22–24].

When $n+1 \leq 7$, Bronsard and the second author [9] showed that there is a unique locally minimizing $(1, 2)$ -cluster in $\mathbb{R}^{n+1}$ up to homotheties: the *standard lens cluster*. See also [1] when $n = 1$. This cluster $\mathcal{X}_{\text{lens}}$ is characterized by the properties $|\mathcal{X}_{\text{lens}}(1)| = 1$, $\partial\mathcal{X}_{\text{lens}}(2) \cap \partial\mathcal{X}_{\text{lens}}(3) \subset \{x_n = 0\}$ and $\partial\mathcal{X}_{\text{lens}}(1)$ consists of a pair of spherical caps of equal radii meeting on $\{x_n = 0\}$ with equal angles of $\frac{2\pi}{3}$ between the three interfaces.

This uniqueness for $n+1 \leq 7$ is closely related to the fact that planes are the only minimal hypercones in these dimensions. Given the existence of non-planar minimizing hypercones for $n+1 \geq 8$ [6], two natural questions arise. The first raised in was raised in [9] and [23]:

Question 1.1. For $n+1 \geq 8$, do there exist locally minimizing $(1, 2)$ -clusters besides the standard lens?

Since the exterior interface $\partial\mathcal{X}(2) \cap \partial\mathcal{X}(3)$ of a locally minimizing $(1, 2)$ -cluster necessarily blows down to a (possibly non-unique) area-minimizing hypercone, a closely related question is the *cone realization problem*:

Question 1.2. Given an area-minimizing hypercone C in $\mathbb{R}^{n+1}$, is there a locally minimizing $(1, 2)$ -cluster whose exterior interface blows down to C ?

Our previous work with Bronsard [8] addressed Question 1.1. We showed that for $8 \leq n+1 \leq 2700$, uniqueness of locally minimizing $(1,2)$ -clusters fails by demonstrating the existence of local minimizers whose exterior interfaces blow down to *singular* area-minimizing hypercones and are thus not lenses. An analogous result was independently and simultaneously proven by Novaga, Paolini & Tortorelli [24] for the case $n+1 = 8$ with a similar proof.

These constructions, however, do not identify the blow-down cones and thus yield no information about Question 1.2 for any given area-minimizing cone. Indeed, in these results, nothing could be said about the singular blow-down cones apart from an upper bound on their density. Thus, even after the affirmative answer to Question 1.1 for $8 \leq n+1 \leq 2700$, the only area-minimizing cone known to arise as a blow-down of the exterior interface of a minimizing $(1,2)$ -cluster was the plane, realized by the standard lens cluster.

Moreover, the blow-down cones corresponding to the minimizing $(1,2)$ -clusters constructed in these works were not shown to be unique, preventing any possibility of establishing rates of convergence or providing any further description of the shape of the minimizing $(1,2)$ -clusters. And, until the present work, Question 1.1 remained open in dimensions $n+1 > 2700$.

A main goal of this work is to address Question 1.2 for a large family of area-minimizing cones, including both cylindrical cones and cones with isolated singularity, and to give a more precise description of the minimizing $(1,2)$ -clusters produced by the construction. In doing so, we also answer Question 1.1 in the affirmative in every even ambient dimension $n+1 \geq 8$.

The clusters we construct have exterior interfaces that converge quantitatively to the prescribed cone C at infinity. In the isolated-singularity case, this asymptotic control also yields smoothness of the exterior interface outside a sufficiently large ball. In fact, for those cones we can treat that have an isolated singularity, we produce countably many distinct minimizing $(1,2)$ -clusters, distinguished by their leading order of decay to C , standing in contrast to the unique minimizing $(1,2)$ -cluster that is asymptotic to the plane (namely, the lens).

1.2. Main results. For an area-minimizing cone C in $\mathbb{R}^{n+1}$ that either has an isolated singularity or is cylindrical, we reduce the cone realization problem to the validity of a strict inequality for the normalized energy

$$\Lambda_C := \inf \left\{ P(E) - \mathcal{H}^n(C \cap E^{(1)}) : E \text{ is a bounded set of finite perimeter with } |E| = 1 \right\}. \quad (1.2)$$

We write $\Lambda_{\text{plane}}(n+1)$ to denote Λ_C when C is a hyperplane in $\mathbb{R}^{n+1}$; this is explicitly given by

$$\Lambda_{\text{plane}}(n+1) = P(\mathcal{X}_{\text{lens}}(1)) - \omega_n \rho_{n+1}^n$$

where ρ_{n+1} is the radius of the disc $\mathcal{X}_{\text{lens}}(1) \cap \{x_n = 0\}$. Our first main result provides a positive answer to Question 1.2, yielding prescribed conical blowdowns with a rate, for cones with isolated singularities and cylindrical cones satisfying $\Lambda_C < \Lambda_{\text{plane}}(n+1)$. Recall that a (multiplicity one) cone C is *cylindrical* if, up to rotation, it is of the form $C = C' \times \mathbb{R}^m$ for a cone C' with an isolated singularity. We refer the reader to Section 2 for preliminaries about positive Jacobi fields on C .

Theorem 1.3. *If $C \subset \mathbb{R}^{n+1}$ is an area-minimizing hypercone that either has an isolated singularity or is cylindrical, and*

$$\Lambda_C < \Lambda_{\text{plane}}(n+1), \quad (1.3)$$

then there exists a locally minimizing $(1,2)$ -cluster $\mathcal{X}$ such that the blowdown of $\partial\mathcal{X}(2) \cap \partial\mathcal{X}(3)$ is C . Moreover, letting $\gamma_1^\pm$ be the homogeneities of the positive, radially homogeneous, decaying Jacobi fields on C , there exist $R_0 > 0$, $C_0 > 0$, and $C_1 \geq 0$ such that for $r \geq R_0$,

$$\text{dist}_{\mathcal{H}}(\partial\mathcal{X}(2) \cap \partial\mathcal{X}(3) \cap B_r^c, C \cap B_r^c) \leq r^{-\gamma_1^-} (C_0 + C_1 \log r). \quad (1.4)$$

or

$$\text{dist}_{\mathcal{H}}(\partial\mathcal{X}(2) \cap \partial\mathcal{X}(3) \cap B_r^c, C \cap B_r^c) \leq C_0 r^{-\gamma_1^+}, \quad (1.5)$$

where $\text{dist}_{\mathcal{H}}$ denotes Hausdorff distance.

Remark 1.4. The decay rates (1.4), (1.5) arise from the decay of positive Jacobi fields on a cone C with isolated singularity, which determine the asymptotic behavior at infinity of the leaves of the Hardt-Simon foliation (and thus $\partial\mathcal{X}(2) \cap \partial\mathcal{X}(3)$); see Section 2.1 below. Note that the decay (1.4) occurs when C is strictly minimizing, with $C_1 = 0$ if and only if C is strictly stable, namely $\lambda_1 > -(\frac{n-2}{2})^2$.

Remark 1.5. When C is a cone with isolated singularity satisfying the assumptions of Theorem 1.3, then the decay estimates (1.4), (1.5) together with De Giorgi's ε -regularity guarantee that for some $R_1 > 0$, the exterior interface $\partial\mathcal{X}(2) \cap \partial\mathcal{X}(3)$ is smooth outside of $B_{R_1}(0)$.

For any area-minimizing hypercone C that has any isolated singularity or is cylindrical, Theorem 1.3 in particular gives an affirmative answer to Question 1.2 provided C satisfies the strict inequality

$$\Lambda_C < \Lambda_{\text{plane}}(n+1). \quad (1.6)$$

Thus the next major question becomes whether (1.6) holds for given cones. In our previous work [8], we verified (1.6) for certain *quadratic* (or Lawson) cones

$$C_{k,l} = \{(x, y) \in \mathbb{R}^{k+1} \times \mathbb{R}^{l+1} : n+1 = k+l+2, |x|^2 = \frac{k}{l}|y|^2\}, \quad (1.7)$$

in dimensions $8 \leq n+1 \leq 2700$. There, the proof in $n+1 = 8$ is done through explicit by-hand computation, while in higher dimension the proof is computer-assisted via fully rigorous interval arithmetic. Our next main result verifies (1.6) for generalized Simons cones in every even dimension $n+1 \geq 8$ and for the cylindrical cone $C_{3,3} \times \mathbb{R}$.

Theorem 1.6. *The strict inequality (1.6) and thus the conclusions of Theorem 1.3 hold for $C = C_{k,k}$ when $n+1 = 2k+2$ for any $n+1 \geq 8$, and for $C_{3,3} \times \mathbb{R}$ in $\mathbb{R}^9$, the latter providing a first example of a locally minimizing (1,2)-cluster modeled on a cone with a non-isolated singularity.*

Observe that Theorem 1.8 gives a positive answer to Question 1.1 in every even ambient dimension $n+1 \geq 8$.

Remark 1.7. We believe that the conclusion of Theorem 1.6 should remain true for the quadratic cones $C_{k,l}$ with $l = k+1$ in odd dimensions $n+1 = k+l+2 \geq 9$, using the competitors constructed in [8] and the method of proof of Theorem 1.6. Since our primary goal is to demonstrate the method that allows us to remove the unnecessary upper bound on ambient dimension in our previous argument, we do not pursue this here.

Our final main result produces an abundance of distinct minimizing (1,2)-clusters that are asymptotic to area-minimizing Lawson cones. This includes one minimizer whose exterior interface decays to C at a *strictly faster* rate than the rate guaranteed by (1.4) in Theorem 1.3, and countably many clusters with precisely the decay (1.4) with distinct leading-order coefficients. This flexibility is very much in contrast to the symmetry and rigidity properties exhibited by local minimizers both in lower dimensions for this problem, and in the classical multiple bubble problem.

Theorem 1.8. *Suppose $C_{k,l} \subset \mathbb{R}^{n+1}$ is an area-minimizing Lawson cone satisfying*

$$\Lambda_{C_{k,l}} < \Lambda_{\text{plane}}(n+1)$$

Then there are infinitely many distinct minimizing (1,2)-clusters $\mathcal{X}$ with $|\mathcal{X}(1)| = 1$ that are asymptotic to $C_{k,l}$.

More precisely, there is a minimizing (1,2)-cluster $\mathcal{X}$ with $|\mathcal{X}| = 1$ with

$$\text{dist}_{\mathcal{H}}(\partial\mathcal{X}(2) \cap \partial\mathcal{X}(3) \cap B_r^c, C_{k,l} \cap B_r^c) = o(r^{-\gamma_1^-}) \quad (1.8)$$

and a countable collection of non-zero numbers $\{a_i\} \subset \mathbb{R}$, accumulating to 0, and corresponding local minimizers $\mathcal{X}_i$ with $|\mathcal{X}_i(1)| = 1$ such that

$$\text{dist}_{\mathcal{H}}(\partial\mathcal{X}_i(2) \cap \partial\mathcal{X}_i(3) \cap B_r^c, C_{k,l} \cap B_r^c) = a_i|x|^{-\gamma_1^-} + o(r^{-\gamma_1^-}) \quad (1.9)$$

In view of Theorem 1.6 and our earlier work [8], Theorem 1.8 applies to all generalized Simons cones $C_{k,k}$ for $2k+2 = n+1 \geq 8$ and to all quadratic cones $C_{k,k+1}$ for $2k+3 = n+1 \in \{9, \dots, 2699\}$.

1.3. Outline of proofs. The starting point for the proofs of Theorems 1.3 and 1.8 is a refinement of the construction we introduced in [8]. The new idea is to use the leaves of the Hardt-Simon foliation of the prescribed cone as barriers, forcing the minimizing (1,2)-clusters we construct to have the prescribed asymptotic behavior.

Let C be a fixed singular area-minimizing hypercone in $\mathbb{R}^{n+1}$ that has isolated singularity (the cylindrical case is similar), and satisfies (1.6). For a fixed large scale $R > 0$, we use a local variational problem to produce a (1,2)-cluster $\mathcal{X}_R$ with $|\mathcal{X}_R(1)| = 1$ that locally minimizes the cluster perimeter $\mathcal{P}$ (up to a small error) among volume-constrained competitors in B_{3R} and whose exterior interface coincides with C outside B_{4R} . For a sequence of such scales $R_k \nearrow +\infty$, we let $\mathcal{X}_k = \mathcal{X}_{R_k}$ be a corresponding sequence of such minimizers.

One would like to consider the limit of the $\mathcal{X}_k$, but compactness issues arise as the finite-volume chamber $\mathcal{X}_k(1)$ could lose mass in the limit. A concentration-compactness argument shows that, up to a subsequence, $\mathcal{X}_k$ splits into finitely many (independent of k) asymptotically non-interacting concentrations $\{\mathcal{X}_{k,i}\}_{i=1}^N$. After translating by some vector $x_{k,i}$, each concentration converges to a locally minimizing (1,2)-cluster $\mathcal{X}_{\infty,i}$ in $\mathbb{R}^{n+1}$ whose finite-volume chamber $\mathcal{X}_{\infty,i}(1)$ has a definite amount mass.

At this stage, there is in principle enormous flexibility for the geometry of the exterior interface for each limiting concentration $\mathcal{X}_{\infty,i}$. A blow-down of the exterior interface of $\mathcal{X}_{\infty,i}$ could be a plane, e.g. if the concentration escapes along an asymptotically flat region and resembles a standard lens. It could be the prescribed cone C , e.g.

if the corresponding translations $x_{k,i}$ remain bounded. Or it could be any other area-minimizing hypercone with density at most that of C , e.g. if the translations $x_{k,i}$ tend to infinity but $|x_{k,i}|/R_k \rightarrow 0$ and the exterior interface slowly changes geometry at intermediate scales.¹

In order to prove Theorem 1.3, let T_λ be the Hardt-Simon foliation associated to C ; see section 2. By construction, for each $k \in \mathbb{N}$, the finite-volume chamber $\mathcal{X}_k(1)$ and the exterior interface $\partial\mathcal{X}_k(2) \cap \partial\mathcal{X}_k(3)$ of $\mathcal{X}_k$ are trapped between two leaves $T_{\pm\lambda}$ of the foliation for sufficiently large $\lambda > 0$, so let λ_k be the smallest $\lambda > 0$ for which this is true. If $\lambda_* := \liminf \lambda_k$ is finite, this means that (up to a subsequence), the finite-volume chambers $\mathcal{X}_k(1)$ are trapped between the leaves $T_{\pm 2\lambda_*}$ for all $k \in \mathbb{N}$. Together with uniform density estimates for $\mathcal{X}_k(1)$, this prevents loss of mass and gives compactness of the full sequence. The limiting (1, 2)-cluster then satisfies the conclusions of Theorem 1.3.

On the other hand, if $\lambda_* = +\infty$, then the sequence is not trapped between a fixed pair of leaves of the foliation and instead some concentration $\mathcal{X}_{k,i}$ has center $x_{k,i}$ drifting off to infinity. But, this means the sequence of translated nearest leaves $T_{\lambda_k} - x_{k,i}$ converges locally smoothly to a plane. The resulting limiting cluster $\mathcal{X}_{\infty,i}$ has exterior interface lying to one side of a plane, and hence, by the maximum principle and the characterization in [9], is a standard lens cluster (and the mass v_1 of its finite-volume chamber has a definite lower bound). Going back to sequence $\mathcal{X}_k$, we excise this lens-like piece by hand to obtain competitors $\mathcal{X}'_k$ for a family of minimization problems as above but with $|\mathcal{X}'_k(v)| = 1 - v_1$. This reduction can be repeated. Each time $\lambda_* = +\infty$, we extract and remove a definite amount of mass escaping as an asymptotic lens. Since the extracted volumes have a uniform positive lower bound, the process terminates after finitely many steps. The assumption (1.6) and the concavity of the scaling ensure that some definite amount mass remains after the lens-like pieces are removed; this final sequence must have $\lambda_* < \infty$, allowing us to conclude Theorem 1.3 as above.

Theorem 1.8 is based on a similar construction, but the initial family of variational problems is modeled on a fixed leaf T_λ of the Hardt-Simon foliation of the cone C rather than on C itself. Then, a competitor argument and the fact that it is too expensive to transition from the existing leaf T_λ to any other leaf in the foliation (including the cone $C_{k,l} = T_0$ itself) forces the exterior interface to behave exactly as T_λ to lowest order; see Lemma 5.1. Here we are relying heavily on the work [33], which allows us to characterize the asymptotic behavior of the interface $\partial\mathcal{X}(2) \cap \partial\mathcal{X}(3)$ as that of a leaf at lowest order.

Finally, the proofs of both parts of Theorem 1.6 build on competitor constructions from our previous paper with Bronsard [8]. There, we constructed competitors E_C for the variational problem Λ_C defined in (1.2) for generalized Simons cones $C = C_{k,k}$ in $\mathbb{R}^{n+1}$ for even ambient dimensions $n + 1 = 2k + 2$ (and more generally for Lawson cones in all dimensions). In order to show (1.6) for these generalized Simons cones C (and thus the first part of Theorem 1.6, it suffices to show $\mathcal{E}(E_C) < \Lambda_{\text{plane}}(n + 1)$ where we let $\mathcal{E}(E) = P(E) - \mathcal{H}^n(C \cap E^{(1)})$. Since $\mathcal{E}(E_C)$ can be expressed explicitly as sums, products, and quotients of dimensionally-dependent integrals of single variable functions, we are able to carefully estimate $\mathcal{E}(E_C)$ using Laplace's method with quantitative error terms to establish the inequality analytically. To prove the second part of Theorem 1.6, that is, that (1.6) holds for $C = C_{3,3} \times \mathbb{R}$, we construct a competitor E_C for Λ_C as a warped product of the competitor $E_{C_{3,3}}$ from our previous paper and compute $\mathcal{E}(E_C)$ using the coarea formula.

In both parts of Theorem 1.6, we expect the methods here could be generalized to other cones, e.g. to other Lawson cones (c.f. Remark 1.7) or cylinders over other generalized Simons cones. In view of Theorem 1.3, doing so would solve the cone realization problem, Question 1.2, in these cases. We leave these generalizations as an interesting open problem.

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2. NOTATION AND PRELIMINARIES

We use the notation $P(E)$ to denote the perimeter of a set $E \subset \mathbb{R}^{n+1}$ of finite perimeter, and $P(E; U)$ to denote the relative perimeter of a set E of locally finite perimeter in an open set $U \subset \mathbb{R}^{n+1}$. We additionally use the notation ∂^*E for the reduced boundary of such a set, and we will always assume that E satisfies $\overline{\partial^*E} = \partial E$; see [19, Proposition 12.19]. We point the reader to [19] for all relevant background on sets of finite perimeter.

¹In our prior work [8], at this point in the construction, we ruled out the possibility that *every* concentration has a planar blow-down by using (1.6), the characterization of minimizing (1, 2)-clusters with planar area growth [9], and the concavity of the scaling. Hence there was at least one concentration blowing down to some singular cone. While this affirmatively answered Question 1.1 in any dimension for which (1.6) holds for some cone, little can be said about the geometry of the resulting minimizing (1, 2)-cluster.

Let C be an area-minimizing hypercone in $\mathbb{R}^{n+1}$. By this we mean that $C = \partial E$ for a conical set of locally finite perimeter $E \subset \mathbb{R}^{n+1}$ with $P(E; U) \leq P(F; U)$ for any open, bounded set $U \subset \mathbb{R}^n$ and any set $F \subset \mathbb{R}^{n+1}$ with $E \Delta F \Subset U$. By conical, we mean that $\iota_{0,r}(E) = E$ for the map $\iota_{0,r}(y) := \frac{y}{r}$.²

Recall that we have monotonicity of surface area ratios $r \mapsto \frac{\mathcal{H}^n(\Sigma)}{r^n}$ for any area-minimizing hypersurface Σ in $\mathbb{R}^{n+1}$. This in particular implies that if Σ is in addition entire, any subsequential limit³

$$C = \lim_{j \rightarrow \infty} \frac{\Sigma \cap B_{\rho_j}}{\rho_j}, \rho_j \uparrow \infty$$

is an area-minimizing hypercone satisfying

$$\frac{\mathcal{H}^n(C \cap B_\rho)}{\omega_n \rho^n} = \Theta_\Sigma := \lim_{R \uparrow \infty} \frac{\mathcal{H}^n(\Sigma \cap B_R)}{\omega_n R^n} \quad \forall \rho > 0. \quad (2.1)$$

Thanks to an exterior version (see e.g. [8, Lemma A.1]) of the classical monotonicity formula for area-minimizing hypersurfaces, any blowdown

$$K = \lim_{j \rightarrow \infty} \frac{\mathcal{X}(2) \cap B_{\rho_j}}{\rho_j}, \quad \rho_j \uparrow \infty$$

of a minimizing (1,2)-cluster $\mathcal{X}$ yields a set of locally finite perimeter K such that ∂K is an area-minimizing hypercone.

2.1. Cones with isolated singularities. In this subsection C has an isolated singularity. Recall that Hardt & Simon proved the following.

Theorem 2.1 ([16, Theorem 2.1]). *Let C be an area-minimizing hypercone with $\text{sing}(C) = \{0\}$ in $\mathbb{R}^{n+1}$ and let K_+, K_- be connected components of $\mathbb{R}^{n+1} \setminus \overline{C}$, so that $K_\pm$ are local perimeter minimizers. There exists a unique open set $F_\pm \subset K_\pm$ such that $F_\pm$ is a local perimeter minimizer, $\partial F_\pm$ is analytic, asymptotic to C at infinity, and $\text{dist}(\partial F_\pm, \{0\}) = 1$.*

In particular, $\{\lambda \partial F_\pm\}_{\lambda > 0}$ yields a (unique) foliation of $K_\pm$ by analytic area-minimizing hypersurfaces.

For simplicity of notation, we will henceforth write $T_{\pm\lambda} = \lambda \partial F_\pm$ for $\lambda > 0$ and $T_0 = C$, and $F_{\pm\lambda}$ for the sets of locally finite perimeter $\lambda F_\pm$. In other words, we may write $\{T_\lambda\}_{\lambda \in \mathbb{R}}$ for the foliation of $\mathbb{R}^{n+1}$ associated to C , with $T_0 = C$, which is singular only at $\lambda = 0$, and $\{F_\lambda\}_{\lambda \in \mathbb{R}}$ for the corresponding family of sets of locally finite perimeter. Note that the subfamilies $\{F_\lambda\}_{\lambda > 0}$ and $\{F_\lambda\}_{\lambda < 0}$ are nested. We will henceforth refer to this foliation as the *Hardt-Simon foliation* associated to C .

Remark 2.2. We will only make use of the existence and regularity part of Theorem 2.1. We do not require the uniqueness, although this is the part of the conclusion that is the most difficult to generalize to area-minimizing cones without isolated singularities, and remains open in general; see for instance [14, 32, 36].

Recall (see e.g. [16] or [31]) that stability guarantees the existence of positive Jacobi fields (namely solutions of $\Delta_C f + |A_C|^2 f = 0$) given in polar coordinates by $f_\pm(\xi, r) = \phi_1(\xi) r^{-\gamma_1^\pm}$ on $C \setminus \{0\}$, for $\phi_1 : C \cap \partial B_1 \rightarrow (0, \infty)$ a positive first eigenfunction of $\Delta_{C \cap \partial B_1} + |A_{C \cap \partial B_1}|^2$ with eigenvalue $\lambda_1 \geq -(\frac{n-2}{2})^2$ and $\gamma_1^\pm$ solving

$$\gamma_1^\pm(n-2-\gamma_1^\pm) = -\lambda_1,$$

i.e.

$$\gamma_1^\pm = \frac{n-2}{2} \pm \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_1}.$$

We normalize ϕ_1 to be positive with $\|\phi_1\|_{L^2(\partial B_1)} = 1$. After this normalization, note that if C is *strictly stable*, i.e. $\lambda_1 > -(\frac{n-2}{2})^2$, then there will be two distinct positive Jacobi fields $f_\pm$, and $\gamma_1^- < \frac{n-2}{2} < \gamma_1^+$. Otherwise, if $\lambda_1 = -(\frac{n-2}{2})^2$, there will be just one such positive Jacobi field, and $\gamma_1^\pm = \frac{n-2}{2}$.

We recall in addition (see, for instance, [16, Theorem 3.2]) that C is said to be *strictly minimizing* if⁴ there exists $c_0(n, C) > 0$ and $\rho_0(n, C, \lambda) > 0$ such that for any $\rho_0 < R$ the leaves $T_{\pm\lambda}$ of the Hardt-Simon foliation given by Theorem 2.1 satisfy

$$T_{\pm\lambda} \setminus B_R = c_0 \lambda \text{graph}_C(\phi_1 \cdot |x|^{-\gamma_1^-} + o(|x|^{-\gamma_1^-})) \quad (2.2)$$

²Recall, thanks to identification between codimension one boundaryless integral currents and boundaries of sets of locally finite perimeter in $\mathbb{R}^{n+1}$, that this is equivalent to C being an area-minimizing hypercone in the framework of integral currents.

³This is made sense of weakly as currents, or equivalently, boundaries of sets of locally finite perimeter.

⁴In light of [16], this is equivalent to saying that there exists $\kappa > 0$ such that for any $R > 0$ and any area-minimizing hypersurface (i.e. codimension 1 integral current) T with $\partial T = \partial C$ in $\overline{B_R}$ satisfying $\text{spt } T \subset B_R^c$, one has $\mathcal{H}^n(C \cap B_R) \leq \mathcal{H}^n(T \cap B_R) - \kappa \rho^n$.

in the case that C is strictly stable, and

$$T_{\pm\lambda} \setminus B_R = c_0 \lambda \operatorname{graph}_C \left(\phi_1 \cdot |x|^{-\frac{n-2}{2}} \log |x| + o\left(|x|^{-\frac{n-2}{2}} \log |x|\right) \right) \quad (2.3)$$

otherwise (i.e. if $\lambda_1 = -(\frac{n-2}{2})^2$). Here, we are using the notation

$$\operatorname{graph}_C u = \{x + u(x)\nu_C(x) : x \in C \setminus \{0\}\}.$$

for a function $u : C \setminus \{0\} \rightarrow \mathbb{R}$, where ν_C denotes a unit normal to $C \setminus \{0\}$ with fixed orientation.

Instead, if C is not strictly minimizing, then

$$T_{\pm\lambda} \setminus B_R = c_0 \lambda \operatorname{graph}_C(\phi_1 \cdot |x|^{-\gamma_1^+}) + o(|x|^{-\gamma_1^+}), \quad (2.4)$$

see [16, (1.9)].

2.2. Cylindrical cones. If $C = C' \times \mathbb{R}^m$ where C' is an area minimizing cone with isolated singularity, we will (in a slight abuse of notation) use $F_{\pm\lambda}$ and $T_{\pm\lambda} = \partial F_{\pm\lambda}$ to refer to the cylindrical extension to $\mathbb{R}^n$ of subsets of $\mathbb{R}^m$ associated to C' and defined in the previous subsection.

2.3. Currents and the Plateau problem. We will be considering the Plateau problem with boundary data that is graphical over C . We adopt the notation of [29], including the mass $\mathbb{M}(T)$ of a current T , the multiplicity one current $\llbracket S \rrbracket$ associated to an oriented submanifold, and the spherical radius s slice $\langle T, |\cdot|, s \rangle$ of a current T . To set notation, we fix a normal vector ν_C for C and consider the associated current $\llbracket C \rrbracket$. We also orient the leaves T_λ of the foliation so that $\llbracket T_\lambda \rrbracket / r \rightarrow \llbracket C \rrbracket$ as $r \rightarrow \infty$. Given a function $h : C \cap \partial B_s \rightarrow \mathbb{R}$, the current $\llbracket \operatorname{graph}_{C \cap \partial B_s} h \rrbracket$ is taken to have the orientation that agrees with the slice $\langle \llbracket C \rrbracket, |\cdot|, s \rangle$ when $h = 0$. Graphs over C are similarly oriented.

3. COMPACTNESS REVISITED

In this section, we revisit the compactness procedure laid out in our previous paper [8]. There, we fixed an area-minimizing hypercone C and considered a sequence of minimization problems μ_{R_k} “modeled on C ” for clusters with a volume constraint prescribing the first chamber to have volume 1. We used the resulting sequence of minimizers of μ_{R_k} to construct locally minimizing (1,2)-clusters whose exterior interfaces blow down to singular cones. To carry out the more refined analysis of the present paper, we need to generalize this setup slightly in two directions. First, we consider minimization problems modeled on a fixed area-minimizing hypersurface, not necessarily a cone, and second, given a range of volume constraints $v \in [v_0, 1]$ for the μ_{R_k} problems, we obtain a lower bound depending only on $v_0 > 0$ for the volume of the first chamber of any limiting local minimizer. Neither of these modifications plays a decisive role in the compactness procedure. The only change is ensuring that lower volume density estimates for the first chamber (Lemma 3.1 below) are uniform across $v \in [v_0, 1]$, which in turn is a consequence of uniform estimates in the concentration compactness-type result [19, Lemma 29.10] under lower perimeter and volume bounds on the sets in question. Thus, we recall the main steps from [8] and provide a sketch of how the proofs therein adapt to the present setting.

Throughout this section, Σ is a fixed, possibly singular, entire area-minimizing hypersurface in $\mathbb{R}^{n+1}$ on which all estimates will depend. Let

$$\Lambda_\Sigma = \inf \{P(E) - \mathcal{H}^n(\Sigma \cap E^{(1)}) : E \text{ is a bounded set of finite perimeter with } |E| = 1\}.$$

Here, by bounded, we mean that there exists $R > 0$ such that $E \subset B_R$.

Let F_Σ be one of the two sets of locally finite perimeter in $\mathbb{R}^{n+1}$ with $\partial F_\Sigma = \Sigma$. For a fixed number $R > 0$, define the confinement functional

$$\mathcal{G}_R(E) := \int_E g_R(|y|) dy \quad \text{where} \quad g_R(t) := \begin{cases} 0 & t \in [0, \sqrt{R}), \\ \frac{t - \sqrt{R}}{\sqrt{R}} & t \geq \sqrt{R} \end{cases} \quad (3.1)$$

and consider the energy functional

$$\mathcal{E}_R(\mathcal{X}) = \mathcal{P}(\mathcal{X}; B_{4R}) + \mathcal{G}_R(\mathcal{X}(1))$$

for a (1,2)-cluster $\mathcal{X}$. Let $\bar{R} \geq 1$ be such that $|B_{\bar{R}}| \geq 2$. For $v \in (0, 1]$ and $R \geq \bar{R}$, consider the variational problem

$$\tau_{R,\Sigma}(v) = \inf \{ \mathcal{E}_R(\mathcal{X}) : |\mathcal{X}(1)| = v, \mathcal{X}(2) \setminus B_{3R} = F_\Sigma \setminus B_{3R}, \mathcal{X}(3) \setminus B_{3R} = F_\Sigma^c \setminus B_{3R} \}. \quad (3.2)$$

It is not difficult to check that the function $R \mapsto \tau_{\Sigma,R}(v) - \mathcal{H}^n(\Sigma \cap B_{4R})$ is nonnegative (by the minimality of Σ) and nonincreasing (thanks to nested competitor sets and the fact that $g_{R_2} - g_{R_1} \leq 0$ for $R_2 \geq R_1$), and thus the asymptotic normalized energy

$$\tilde{\Lambda}_\Sigma(v) = \lim_{R \rightarrow \infty} (\tau_{\Sigma,R}(v) - \mathcal{H}^n(\Sigma \cap B_{4R})) \quad (3.3)$$

is well defined. Observe that $\tilde{\Lambda}_\Sigma(1) \leq \Lambda_\Sigma$. Note that the constraints in (3.2) ensure that any competitor $\mathcal{X}$ has $\mathcal{X}(1) \subset B_{3R}$, and the direct method guarantees the existence of a minimizer $\mathcal{X}_R^v$ of $\tau_{R,\Sigma}(v)$ for each $v \in (0, 1]$ and $R \geq \bar{R}$. As in [8, Section 3], a basic competitor argument yields some key initial estimates for such a minimizer. Let $\mathcal{X}'$ be the $(1, 2)$ -cluster where $\mathcal{X}'(1)$ is a volume- v ball B^v centered at the origin, $\mathcal{X}'(2) = F_\Sigma \setminus \mathcal{X}'(1)$, and $\mathcal{X}'(3) = F_\Sigma^c \setminus \mathcal{X}'(1)$. Taking $\mathcal{X}'$ as a competitor in (3.2), we find

$$\mathcal{P}(\mathcal{X}_R^v; B_{4R}) \leq \mathcal{E}_R(\mathcal{X}_R^v) \leq \mathcal{E}_R(\mathcal{X}') \leq P(B^v) + P(F_\Sigma; B_{4R}). \quad (3.4)$$

Since the local perimeter minimality of F_Σ means that $P(\mathcal{X}_R^v(h); B_{4R}) \geq P(F_\Sigma; B_{4R})$ for $h = 2, 3$, we in turn find that

$$P(\mathcal{X}_R^v(1)) + 2\mathcal{G}_R(\mathcal{X}_R^v(1)) = 2\mathcal{E}_R(\mathcal{X}_R^v) - P(\mathcal{X}_R^v(2); B_{4R}) - P(\mathcal{X}_R^v(3); B_{4R}) \leq 2P(B^v). \quad (3.5)$$

Since $g_R(|y|) \geq s/\sqrt{R}$ for any $y \in \mathbb{R}^{n+1} \setminus B_{\sqrt{R}+s}$, the bound (3.5) guarantees that

$$|\mathcal{X}_R^v(1) \setminus B_{\sqrt{R}+s}| \leq \frac{\sqrt{R}P(B^v)}{s} \quad (3.6)$$

for any $s > 0$. Using (3.6) and arguing as in [8, Lemma 4.1] shows that for v bounded uniformly away from 0, the first chambers of minimizers of $\tau_{R,\Sigma}(v)$ satisfy the following uniform lower volume density estimates.

Lemma 3.1. *Fix $v_0 \in (0, 1]$ and $R \geq \bar{R}$. There is a constant $c_0 = c_0(v_0, n, \Sigma) > 0$ such that if $\mathcal{X}_R^v$ is a minimizer of $\tau_{R,\Sigma}(v)$ with $v \in [v_0, 1]$, then*

$$\frac{|\mathcal{X}_R^v(1) \cap B_\rho(y)|}{\omega_{n+1}\rho^{n+1}} \geq c_0 \quad (3.7)$$

for each $x \in \partial\mathcal{X}_R^v(1)$ and $\rho \in (0, 1]$.

Proof. The proof is similar to [8, Lemma 4.1]; we sketch the proof here for completeness. Suppose by way of contradiction that (3.7) does not hold. Then we may find sequences $\{R_k\}$, v_k , and ρ_k and minimizers $\mathcal{X}_k = \mathcal{X}_{R_k}^{v_k}$ of $\tau_{R_k,\Sigma}(v)$ such that

$$\rho_k^{-(n+1)} |\mathcal{X}_k(1) \cap B_{\rho_k}(y)| \rightarrow 0. \quad (3.8)$$

Take $c_{n+1} > 0$ to be the dimensional constant from the nucleation lemma [19, Lemma 29.10] and fix $0 < \varepsilon < \min\{v_0, \frac{P(B^{v_0})}{2(n+1)c_{n+1}}\}$. Let $\bar{N} = (\frac{2P(B^1)}{c_{n+1}\varepsilon})^{n+1}$. Applying the nucleation lemma to each $\mathcal{X}_k(1)$ together with the bound (3.5), we obtain points $\{x_{k,1}, \dots, x_{k,N_k}\}$ with $N_k \leq v_k(\frac{P(\mathcal{X}_k(1))}{c_{n+1}\varepsilon})^{n+1} \leq \bar{N}$, such that $|\mathcal{X}_k(1) \setminus \cup_{i=1}^{N_k} B_1(x_{k,i})| < \varepsilon$ and

$$|B_1(x_{k,i}) \cap \mathcal{X}_k(1)| \geq \left(\frac{c_{n+1}\varepsilon}{P(\mathcal{X}_k(1))}\right)^{n+1} \geq \bar{N}^{-1} \quad \text{for each } i = 1, \dots, N_k.$$

Applying (3.6) with $s = s_k = 2\sqrt{R_k}P(B^1)\bar{N}$ ensures that $B_1(x_{k,i}) \subset B_{\sqrt{R_k}+s_k+2}$ for each $i = 1, \dots, N_k$, and in particular $g_k(|x_{k,i}|)$ is bounded above independently of i and k . From here, by repeating Step 2 and Step 3 of [8, Proof of Lemma 4.1] (which in turn are adaptations of arguments in the proofs of [19, Lemma 17.2] and [9, Lemma 4.1]), we perform volume fixing variations and use the classical differential inequality argument to show there exist ρ_1 and c_1 , independent of k , such that $|\mathcal{X}_k(1) \cap B_\rho(y)| \geq c_1\rho^n$ for each $k \in \mathbb{N}$, $y \in \partial\mathcal{X}_k(1)$, and $\rho \in (0, \rho_1)$. Up to replacing c_1 by $c_1\rho_1^{n+1}$, we see that the same estimate extends to all $\rho \in (0, 1]$. This contradicts (3.8) and completes the proof. $\square$

From Lemma 3.1, we can show uniform density estimates for minimizers of $\tau_{R,\Sigma}(v)$ with $v \in [v_0, 1]$ and that, for a fixed v , along (sub)sequences $R_k \rightarrow \infty$, the first chambers $\mathcal{X}_k^v(1)$ concentrate in a uniform number of well-separated balls.

Lemma 3.2. *Fix $v_0 \in (0, 1]$, $v \in [v_0, 1]$, and $R \geq \bar{R}$. There are constants $c_1, C_0 > 0$ and $N_0 \in \mathbb{N}$ depending on v_0 , n , and Σ such that the following holds. Any minimizer $\mathcal{X}_R^v$ of $\tau_{R,\Sigma}(v)$ has*

$$\mathcal{X}_R^v(1) \subset B_{C_0\sqrt{R}} \quad (3.9)$$

and, for each $h \in \{1, 2, 3\}$, $y \in \partial\mathcal{X}_R^v(h) \cap B_{2R}$, and $\rho \in (0, 1]$, satisfies the density estimates

$$c_1 \leq \frac{|\mathcal{X}_R^v(h) \cap B_\rho(y)|}{\omega_{n+1}\rho^{n+1}} \leq 1 - c_1. \quad (3.10)$$

Moreover, consider a sequence $\mathcal{X}_k$ of minimizers to $\tau_{R_k, \Sigma}(v)$ with $R_k \rightarrow \infty$. Up to a subsequence, there exist $R_0 \geq \bar{R}$, $N \leq N_0$ and points $\{x_{k,1}, \dots, x_{k,N}\} \subset B_{C_0 \sqrt{R_k}}$ such that

$$\mathcal{X}_k(1) \subset \bigcup_{i=1}^N B_{R_0}(x_{k,i}), \quad (3.11)$$

$B_{R_0}(x_{k,i})$ intersects $\mathcal{X}_k(1)$ nontrivially, and $\lim_k |x_{k,i} - x_{k,j}| = +\infty$.

Proof. Let $\mathcal{X}_R^v$ be a minimizer of $\tau_{R, \Sigma}(v)$. Using the uniform lower density estimate (3.7) and the nucleation lemma [19, Lemma 29.10], we repeat the argument of [8, Lemma 4.1, Step 4] essentially verbatim to obtain points $\{y_1, \dots, y_M\}$ with $M \leq N_0$ such that

$$\{y_i\}_{i=1}^M \subset B_{C_1 \sqrt{R}} \quad \text{and} \quad \mathcal{X}(1) \subset \bigcup_{i=1}^M B_4(y_i), \quad (3.12)$$

where $C_1 > 0$ and $N_0 \in \mathbb{N}$ depend on n , v_0 , and the constant c_0 from (3.7), and thus on n , v_0 , and Σ .

This in particular guarantees (3.9) with $C_0 = C_1 + 4\bar{R}^{-1/2}$. With (3.9) in hand, the proof of [8, Corollary 4.2] can be repeated verbatim to obtain the density estimates (3.10).

Next, take a sequence $\mathcal{X}_k^v$ of minimizers to $\tau_{R_k, \Sigma}(v)$ with $R_k \rightarrow \infty$ and the corresponding points $\{y_{k,1}, \dots, y_{k,M_k}\}$ with $M_k \leq N_0$ satisfying (3.12) with $\mathcal{X}_k^v$ in place $\mathcal{X}$, we can repeat the argument of [8, Lemma 4.4] to obtain the final conclusion of the lemma. $\square$

Now, for the remainder of the section, fix $v_0 \leq v \leq 1$, a sequence $R_k \rightarrow \infty$, and a corresponding sequence of minimizers $\mathcal{X}_k$ of $\tau_{R_k, \Sigma}(v)$. Suppose that we have passed to a subsequence (without relabeling) to obtain R_0 , N , and $\{x_{k,1}, \dots, x_{k,N}\}$ as in Lemma 3.2. For each $k \in \mathbb{N}$ and each $i = 1, \dots, N$, define the *concentration* $\mathcal{X}_{k,i}$ to be the translated $(1, 2)$ -cluster

$$\mathcal{X}_{k,i} = \mathcal{X}_k - x_{k,i}. \quad (3.13)$$

As a direct corollary of the uniform lower density estimates of Lemma 3.1 is a lower bound on the volume of the first chamber of any concentration:

Lemma 3.3. *There exists $\bar{v} = \bar{v}(v_0, n, \Sigma) > 0$ such that $|\mathcal{X}_{k,i}(1) \cap B_{R_0}| \geq \bar{v}$ for each $i = 1, \dots, N$ and $k \in \mathbb{N}$.*

Proof. Since $R_0 \geq \bar{R}$ and B_{R_0} intersects $\mathcal{X}_{k,i}(1)$ nontrivially by Lemma 3.2, there is some $x \in \partial \mathcal{X}_{k,i}(1) \cap B_{R_0}$. By Lemma 3.1 applied with $\rho = 1$, we know that $|\mathcal{X}_{k,i}(1) \cap B_1(x)| \geq c_0 \omega_{n+1}$. Since we can moreover assume (up to a subsequence) that the balls $B_{R_0+1}(x_{k,i})$ are disjoint, we know from (3.11) that $\mathcal{X}_{k,i}(1) \cap B_1(x) \subset B_{R_0}$. Hence the lemma holds with $\bar{v} = c_0 \omega_{n+1}$. $\square$

The concentrations $\mathcal{X}_{k,i}$ converge to limiting clusters that are themselves locally minimizing:

Lemma 3.4. *For each $i = 1, \dots, N$, there exist partitions $\mathcal{X}_{\infty,i} = (\mathcal{X}_{\infty,i}(1), \mathcal{X}_{\infty,i}(2), \mathcal{X}_{\infty,i}(3))$ of $\mathbb{R}^n$ by sets of locally finite perimeter with the following properties:*

(i) *setting $v_i := |\mathcal{X}_{\infty,i}(1)|$, we have $v_i \geq \bar{v}$ (with $\bar{v}$ as in Lemma 3.3) and*

$$\sum_{i=1}^N v_i = v; \quad (3.14)$$

(ii) *for each $h = 1, 2, 3$, $\mathcal{X}_{k,i}(h)$ and $\partial \mathcal{X}_{k,i}(h)$ converge locally in the Hausdorff distance to $\mathcal{X}_{\infty,i}(h)$ and $\partial \mathcal{X}_{\infty,i}(h)$ respectively;*

(iii) *if $|\mathcal{X}_{\infty,i}(2)| = |\mathcal{X}_{\infty,i}(3)| = \infty$, then $\mathcal{X}_{\infty,i}$ is a locally minimizing $(1, 2)$ -cluster with $\mathcal{X}_{\infty,i}(1) \subset B_{R_0}$. If $|\mathcal{X}_{\infty,i}(2)|$ (resp. $|\mathcal{X}_{\infty,i}(3)|$) is finite, then $\mathcal{X}_{\infty,i}(2)$ (resp. $\mathcal{X}_{\infty,i}(3)$) is empty and*

$$\frac{1}{2} \sum_{j=1}^3 P(\mathcal{X}_{\infty,i}(j); B_{2R_0}) \geq (n+1) \omega_{n+1}^{1/(n+1)} v_i^{n/(n+1)}. \quad (3.15)$$

Proof. All but the claim that $v_i \geq \bar{v}$ in (i) and the second claim of (iii) are direct analogues of those in [8, Lemma 4.5] and the proofs are the same. The additional statement that $v_i \geq \bar{v}$ follows from Lemma 3.3. The additional statement that the finiteness of $|\mathcal{X}_{\infty,i}(2)|$ implies that $\mathcal{X}_{\infty,i}(2)$ is empty can be deduced as follows. Suppose not. Arguing just as in [8, Lemma 4.5(iii)] shows that $\mathcal{X}_{\infty,i}(2)$ is locally perimeter minimizing in $\mathbb{R}^{n+1} \setminus B_{R_0}$. Thus $\mathcal{X}_{\infty,i}(2)$ satisfies volume density estimates in $\mathbb{R}^{n+1} \setminus B_{R_0+1}$ and a standard argument (as in the proof of (3.11)) shows that $\mathcal{X}_{\infty,i}(2)$ is bounded; say $\mathcal{X}_{\infty,i}(2) \subset B_s$ for some $s > 0$. Then by the Hausdorff convergence of the chambers and their boundaries, $\mathcal{X}_{k,i}(2)$ has non-empty intersection with $B_{2s}(x_{k,i})$ and has empty intersection with

the annulus $B_{4s}(x_{k,i}) \setminus B_{2s}(x_{k,i})$. Taking the competitor $\mathcal{X}'_k$ in τ_{Σ, R_k} for $R_k > s$ with $\mathcal{X}'_k = (\mathcal{X}_k(1), \mathcal{X}_k(2) \setminus B_{3s}(x_{k,i}), \mathcal{X}_k(3) \cup (\mathcal{X}_k(2) \cap B_{3s}(x_{k,i})))$ violates the minimality of $\mathcal{X}_k$, a contradiction. $\square$

Applying Lemma 3.4 and repeating the arguments of [8, Sections 5 and 6] produces a locally minimizing (1,2)-cluster modeled on *some* singular area minimizing cone:

Proposition 3.5. *Suppose $\Lambda_\Sigma < \Lambda_{\text{plane}}$ and $v \in (0, 1]$. For some $i \in \{1, \dots, N\}$, $\mathcal{X} := X_{\infty, i}$ is a locally minimizing (1,2)-cluster and any blowdown of $\partial^* \mathcal{X}(2) \cap \partial^* \mathcal{X}(3)$ is a singular area-minimizing cone with density at most Θ_Σ .*

Recall Θ_Σ was defined in (2.1). In the sections that follow, we give a refined analysis that produces—for suitably chosen Σ —a locally minimizing (1,2)-cluster whose (unique) blowdown is precisely the blowdown of Σ .

4. PROOF OF THEOREM 1.3

Throughout this section, we fix an area-minimizing hypercone C as in Theorem 1.3, so that C either has an isolated singularity or is cylindrical, and satisfies

$$\alpha_0 := \Lambda_{\text{plane}}(n+1) - \Lambda_C > 0.$$

Let $\{T_\lambda\}_{\lambda \in \mathbb{R}}$ be the foliation associated with C as defined in Sections 2.1-2.2, with the convention that $T_0 = C$. Observe that there exists $\lambda_0 > 0$ (depending on C) such that for every λ with $|\lambda| \leq \lambda_0$ we have

$$\Lambda_{\text{plane}}(n+1) - \Lambda_{T_\lambda} =: \alpha_\lambda \geq \frac{\alpha_0}{2} > 0. \quad (4.1)$$

For the remainder of the section, we let $\lambda \in [-\lambda_0, \lambda_0]$ be fixed. Up to swapping the labels of the components $K^\pm$ of $\mathbb{R}^{n+1}$ with boundary C , we can and will assume that $\lambda \geq 0$.

We use the shorthand $\tau_{R, \lambda}(v) = \tau_{R, T_\lambda}(v)$ for the variational problem (3.2) introduced in the previous section, i.e.

$$\tau_{R, \lambda}(v) = \inf \{ \mathcal{E}_R(\mathcal{X}) : |\mathcal{X}(1)| = v, \mathcal{X}(2) \setminus B_{3R} = F_\lambda \setminus B_{3R}, \mathcal{X}(3) \setminus B_{3R} = F_\lambda^c \setminus B_{3R} \}. \quad (4.2)$$

Lemma 4.1. *Fix $v_0 \in (0, 1]$ and $\lambda_- < 0 \leq \lambda < \lambda_+$. There exist $\bar{\rho}, \hat{R} > 0$ depending on v_0, n, C, λ , and $\lambda_\pm$ such that if $v \in [v_0, 1]$, $R \geq \hat{R}$, and $\mathcal{X}$ is a minimizer of $\tau_{R, \lambda}(v)$ satisfying*

$$\mathcal{X}(1) \subseteq \mathbb{R}^{n+1} \setminus (\bar{F}_{\lambda_+} \cup \bar{F}_{\lambda_-}), \quad (4.3)$$

then there is $\bar{\rho} > 0$ such that $\mathcal{X}(1) \subset B_{\bar{\rho}}(0)$ if the singularity of C is isolated and $\mathcal{X}(1) \subset B_{\bar{\rho}}(0) \times \mathbb{R}^m$ if C is cylindrical and

$$F_{\lambda_+} \subset \mathcal{X}(2) \quad \text{and} \quad F_{\lambda_-} \subset \mathcal{X}(3). \quad (4.4)$$

In particular, $\partial \mathcal{X}(2) \cap \partial \mathcal{X}(3) \subset \mathbb{R}^{n+1} \setminus (F_{\lambda_+} \cup F_{\lambda_-})$.

Remark 4.2. The assumptions that $\lambda_- < 0$ and $\lambda_+ > 0$ are simply made to streamline the statement using the notation for the sets $F_{\lambda_\pm}$. While the statement above is sufficient for our needs, it is immediate from the proof that the analogous result holds when $0 \leq \lambda_- < \lambda < \lambda_+$.

Proof. To see the first conclusion, let $\bar{R}$ be the dimensional constant from Section 3, assume $R \geq \bar{R}$ and fix $x \in \partial \mathcal{X}(1)$. By Lemma 3.1, $|\mathcal{X}(1) \cap B_1(x)| \geq c_0 \omega_{n+1}$ where $c_0 = c_0(v_0, n, C, \lambda)$. On the other hand, in the case that C has an isolated singularity, the assumption (4.3) guarantees that $|\mathcal{X}(1) \cap B_1(x)| \leq |B_1(x) \setminus (\bar{F}_{\lambda_+} \cup \bar{F}_{\lambda_-})|$, which tends to zero as $|x| \rightarrow \infty$. Hence $|x| < \bar{\rho}$ for a constant $\bar{\rho}$ depending on $v_0, n, C, \lambda, \lambda_+$, and λ_- . Similarly, if $C = C' \times \mathbb{R}^m$ is cylindrical, then we still have $|\mathcal{X}(1) \cap B_1(x)| \leq |B_1(x) \setminus (\bar{F}_{\lambda_+} \cup \bar{F}_{\lambda_-})|$, and this still tends to zero as $|x| \rightarrow \infty$ for any $x \in \mathbb{R}^{n+1}$ lying outside a tubular neighborhood of the linear subspace $\{0\} \times \mathbb{R}^m$, so $\mathcal{X}(1) \subset B_{\bar{\rho}}(\{0\} \times \mathbb{R}^m)$ where $\bar{\rho}$ has the same dependencies.

Next we show (4.4). We may assume $P(\mathcal{X}(3); T_{\lambda_+}) = P(\mathcal{X}(3); T_{\lambda_-}) = 0$; otherwise apply the following argument along a sequence $\lambda_+^j \downarrow \lambda_+$ and $\lambda_-^j \uparrow \lambda_-$ where this holds. We also assume for simplicity that $\mathcal{X}(1), \mathcal{X}(2)$ are open, which we may do up to modifying on a Lebesgue null set using minimality. Note that $\mathcal{X}(3)$ has trivial intersection with $F_{\lambda_+} \setminus \bar{B}_{3R}$ thanks to the constraints of the minimization problem (4.2) and the fact that $F_{\lambda_+} \subset F_\lambda$. We claim that $\mathcal{X}(3)$ has trivial intersection with $F_{\lambda_+} \cap \bar{B}_{3R}$ as well, thus establishing the first conclusion of (4.4).

Indeed, consider the (1,2)-cluster $\mathcal{X}' = (\mathcal{X}'(1), \mathcal{X}'(2), \mathcal{X}'(3))$ with

$$\mathcal{X}'(1) = \mathcal{X}(1), \quad \mathcal{X}'(2) = \mathcal{X}(2) \cup F_{\lambda_+}, \quad \mathcal{X}'(3) = \mathcal{X}(3) \setminus F_{\lambda_+}.$$

Taking $\mathcal{X}'$ as a competitor for $\tau_{R, \lambda}(v)$ and using that $\mathcal{X}(1) = \mathcal{X}'(1)$, we find

$$\mathcal{H}^n(\partial \mathcal{X}(2) \cap \partial \mathcal{X}(3)) \leq \mathcal{H}^n(\partial \mathcal{X}'(2) \cap \partial \mathcal{X}'(3))$$

which in particular shows that

$$P(\mathcal{X}(3); F_{\lambda_+} \cap B_{4R}) \leq P(T_{\lambda_+}; \mathcal{X}(3) \cap B_{4R}). \quad (4.5)$$

On the other hand, consider the set $G := F_{\lambda_+} \setminus (\mathcal{X}(3) \cap B_{3R})$. Since $\mathcal{X}(3) \setminus B_{3R} \subset F_{\lambda_+}^c \setminus B_{3R}$, we see that $G = F_{\lambda_+} \setminus \mathcal{X}(3)$. Using that F_{λ_+} is locally perimeter minimizing, [19, Theorem 16.3], and (4.5), we have

$$\begin{aligned} P(F_{\lambda_+}; B_{4R}) &\leq P(G; B_{4R}) = P(F_{\lambda_+}; B_{4R} \setminus \mathcal{X}(3)) + P(\mathcal{X}(3); F_{\lambda_+} \cap B_{4R}) \\ &\leq P(F_{\lambda_+}; B_{4R} \setminus \mathcal{X}(3)) + P(T_{\lambda_+}; \mathcal{X}(3) \cap B_{4R}) = P(F_{\lambda_+}; B_{4R}). \end{aligned}$$

So, G , along with F_{λ_+} , is locally perimeter minimizing in B_{4R} , and $G \subset F_{\lambda_+}$. This, however, violates the maximum principle. Indeed, by construction, there is a point $x \in \partial G \cap \partial F_{\lambda_+} \cap B_{4R}$. Then [29, Lemma 37.10] or [34, Theorem 2.2] (the latter of which is a consequence of [30, Corollary 1]) guarantees ∂G and ∂F_{λ_+} coincide in a neighborhood of x . Since $S := \partial F_{\lambda_+} \cap B_{4R} = T_{\lambda_+} \cap B_{4R}$ is connected, this means the set $A := S \cap \partial G$ is a relatively open and closed subset of S , hence $A = T_{\lambda_+} \cap B_{4R}$. In other words, $G = F_{\lambda_+}$, i.e. $F_{\lambda_+} \cap \mathcal{X}(3)$ is empty, establishing the claim.

Repeating this argument with F_{λ_-} in place of F_{λ_+} and with the roles of $\mathcal{X}(2)$ and $\mathcal{X}(3)$ swapped shows that $\mathcal{X}(2)$ has trivial intersection with F_{λ_-} . From the boundary data of (4.2), we then deduce that $F_{\lambda_-} \subset \mathcal{X}(3)$. This completes the proof. $\square$

Now, to set up the statement of the next lemma and the constructions giving rise to Theorems 1.3 and 1.8, fix $0 < v_0 \leq v \leq 1$, and choose a sequence $R_k \rightarrow \infty$. Let $\mathcal{X}_k$ be corresponding minimizers of $\tau_{R_k, \lambda}(v)$. Pass to a subsequence, without relabeling, to obtain R_0 , N , and $\{x_{k,i}\}_{i=1}^N$ as in Lemma 3.2, so that in particular

$$\mathcal{X}_k(1) \subset \bigcup_{i=1}^N B_{R_0}(x_{k,i})$$

by (3.11). For $i = 1, \dots, N$, let $v_i = |\mathcal{X}_{\infty,i}(1)|$ for the limits $\mathcal{X}_{\infty,i}$ of the corresponding concentrations (3.13) as obtained in Lemma 3.4. For each $k \in \mathbb{N}$ and $i = 1, \dots, N$, choose $\lambda_{k,i} \in \mathbb{R}$ so that

$$B_{R_0}(x_{i,k}) \subset \mathbb{R}^{n+1} \setminus F_{\lambda_{k,i}} \quad \text{and} \quad \text{dist}(x_{i,k}, T_{\lambda_{k,i}}) = R_0 + 1.$$

After passing to a subsequence and re-indexing in i , we may assume $|\lambda_{k,1}| = \max_{1 \leq i \leq N} \{|\lambda_{k,i}|\}$ for each k , and that each $\lambda_{k,1}$ has the same sign. Without loss of generality we may assume $\lambda_{k,1} \geq 0$. Define the constant $\lambda_* \in \mathbb{R}_+ \cup \{+\infty\}$ by $\lambda_* := \liminf_{k \rightarrow \infty} \lambda_{k,1}$. We pass to a further subsequence, without relabeling, so that this $\liminf$ is a limit, i.e.

$$\lambda_* := \lim_{k \rightarrow \infty} \lambda_{k,1}. \quad (4.6)$$

It follows from the first conclusion of Lemma 4.1 that $\lambda_* < \infty$ if the sequence $\{x_{k,1}\}$ is bounded, while $\lambda_* = +\infty$ if the sequence is unbounded. We prove the following dichotomy using Lemma 4.1. The argument is similar to the proof of [8, Theorem 1.2]. Here $\bar{\rho} > 0$ is the constant from Lemma 4.1.

Lemma 4.3. *The following dichotomy holds:*

- If $\lambda_* < +\infty$, then $\mathcal{X}_k(1) \subset B_{\bar{\rho}}(0)$ if the singularity of C is isolated and $\mathcal{X}_k(1) \subset B_{\bar{\rho}}(0) \times \mathbb{R}^m$ if C is cylindrical and

$$F_{2\lambda_*} \subset \mathcal{X}_k(2), \quad \text{and} \quad F_{-2\lambda_*} \subset \mathcal{X}_k(3) \quad (4.7)$$

for all k sufficiently large. Thus $\mathcal{X}_{\infty,1}(1) \subset B_{\bar{\rho}}(0)$, $F_{2\lambda_*} \subset \mathcal{X}_{\infty,1}(2)$, and $F_{-2\lambda_*} \subset \mathcal{X}_{\infty,1}(3)$.

- If $\lambda_* = +\infty$, then

$$\tilde{\Lambda}_{\Sigma}(v) \geq \Lambda_{\text{plane}}(n+1)v_1^{n/(n+1)} + \tilde{\Lambda}_{\Sigma}(v - v_1). \quad (4.8)$$

Here $v_1 \geq \bar{v}$ is first-chamber volume of the limit $\mathcal{X}_{\infty,1}$ and $\tilde{\Lambda}_{\Sigma}(v)$ is the asymptotic renormalized energy defined in (3.3). We adopt the convention that $\tilde{\Lambda}_{T_{\lambda}}(0) = 0$.

Proof. If $\lambda_* < +\infty$, then for all k large enough, the clusters $\mathcal{X}_k$ satisfy the assumption (4.3) with $\lambda_{\pm} = \pm 2\lambda_*$. The conclusion (4.7) then follows directly from Lemma 4.1. The remaining conclusions are then immediate consequences of Lemma 3.4.

Next suppose $\lambda_* = +\infty$. Each shifted leaf $\hat{T}_k := T_{\lambda_{k,1}} - x_{k,1}$ intersects B_{R_0+2} nontrivially. Moreover, $\sup_{x \in T_{\pm}} |A_{T_{\pm}}(x)| < +\infty$, so by scaling we have $\sup_{x \in \hat{T}_k} |A_{\hat{T}_k}(x)| \rightarrow 0$ as $k \rightarrow \infty$. Here A denotes the second fundamental form. As a consequence, $\hat{T}_k$ converges locally in C^2 to a hyperplane Π .

Furthermore, each $\mathcal{X}_k$ satisfies (4.3) with $\lambda_{\pm} = \pm \lambda_{k,1}$, and hence Lemma 4.1 guarantees that

$$\partial \mathcal{X}_k(2) \cap \partial \mathcal{X}_k(3) \subset \mathbb{R}^{n+1} \setminus (F_{\lambda_{k,1}} \cup F_{-\lambda_{k,1}}).$$

In particular, the concentration $\mathcal{X}_{k,1}$ has all interfaces lying to one side of $\hat{T}_k$, and thus its limiting partition $\mathcal{X}_{\infty,1} = (\mathcal{X}_{\infty,1}(1), \mathcal{X}_{\infty,1}(2), \mathcal{X}_{\infty,1}(3))$ obtained in Lemma 3.4 has interfaces lying to one side of Π , and

$|\mathcal{X}_{\infty,1}(1)| = v_1 \geq \bar{v}$. In view of Lemma 3.4(iii), there are two possible scenarios: either $|\mathcal{X}_{\infty,1}(2)| = |\mathcal{X}_{\infty,1}(3)| = +\infty$, or $|\mathcal{X}_{\infty,1}(h)| < \infty$ for one of $h = 2, 3$.

Case 1: First suppose $|\mathcal{X}_{\infty,1}(2)| = |\mathcal{X}_{\infty,1}(3)| = +\infty$. By Lemma 3.4(iii), $\mathcal{X}_{\infty,1}$ is a locally minimizing (1,2)-cluster. By the considerations above, $\mathcal{X}_{\infty,1}(2)$ contains a half space bounded by Π . This means any blowdown of $\partial\mathcal{X}_{1,\infty}(2) \cap \partial\mathcal{X}_{1,\infty}(3)$ is a minimal surface contained in a half space, and hence is a hyperplane by the maximum principle. By [9, Theorem 2.9], this implies that $\mathcal{X}_{\infty,1}$ is a standard lens cluster of volume v_1 centered at a point $y \in \mathbb{R}^{n+1}$.

Let us use this information to bound the energy of $\mathcal{X}_k$ from below. Let $\hat{B}_k = B_{3R_0}(x_{k,1} + y)$, and note that $B_{R_0}(x_{k,1}) \subset \hat{B}_k$. Assuming without loss of generality that R_0 is large enough so that $\mathcal{X}_{\text{lens}}(1) \subset B_{R_0}$, we have

$$\mathcal{P}(\mathcal{X}_{\infty,1}; B_{3R_0}(y)) = (3R_0)^n \omega_n + \Lambda_{\text{plane}}(n+1)v_1^{n/(n+1)}, \quad (4.9)$$

and therefore

$$\mathcal{P}(\mathcal{X}_k; \hat{B}_k) = (3R_0)^n \omega_n + \Lambda_{\text{plane}}(n+1)v_1^{n/(n+1)} + o_k(1). \quad (4.10)$$

By Lemma 3.4(ii), each interface of the concentration $\mathcal{X}_{k,1}$ converges locally in the Hausdorff distance to the corresponding interface of the standard lens cluster above. In particular, the interface $\partial\mathcal{X}_k(2) \cap \partial\mathcal{X}_k(3)$ becomes arbitrarily close to a plane in the annulus $B_{4R_0}(x_{k,1} + y) \setminus \hat{B}_k$. So, we may find sets⁵ $\mathcal{X}'_k(2)$ and $\mathcal{X}'_k(3)$ that coincide with $\mathcal{X}_k(2)$ and $\mathcal{X}_k(3)$ respectively in $B_{4R_k} \setminus \hat{B}_k$ and are complementary in $\hat{B}_k$ so that the cluster $\mathcal{X}'_k = (\mathcal{X}_k(1) \setminus \hat{B}_k, \mathcal{X}'_k(2), \mathcal{X}'_k(3))$ satisfies

$$\mathcal{P}(\mathcal{X}'_k; B_{4R_k}) = \mathcal{P}(\mathcal{X}_k; B_{4R_k} \setminus \hat{B}_k) + \omega_n(3R_0)^n + o_k(1). \quad (4.11)$$

Rearranging and summing (4.10) and (4.11) shows that

$$\mathcal{P}(\mathcal{X}_k; B_{4R_k}) = \mathcal{P}(\mathcal{X}'_k; B_{4R_k}) + \Lambda_{\text{plane}}(n+1)v_1^{n/(n+1)} + o_k(1). \quad (4.12)$$

Let $v_{k,1} = |\mathcal{X}_k(1) \cap \hat{B}_k| \rightarrow v_1$, so that the cluster $\mathcal{X}'_k$ above is an admissible competitor for $\tau_{\lambda, R_k}(v - v_{k,1})$. Here we adopt the convention that $\tau_{\lambda, R_k}(0) = 0$, and in a slight abuse of notation consider a partition with empty $\mathcal{X}(1)$ -chamber as an admissible competitor for this problem. Adding the term $\mathcal{G}_{R_k}(\mathcal{X}_k(1)) - \mathcal{H}^n(\Sigma \cap B_{4R_k})$ to both sides of (4.12) and noting that $\mathcal{G}_{R_k}(\mathcal{X}'_k(1)) \leq \mathcal{G}_{R_k}(\mathcal{X}_k(1))$, we find

$$\begin{aligned} \tilde{\Lambda}_{T_\lambda}(v) &= \mathcal{E}_k(\mathcal{X}_k; B_{4R_k}) - \mathcal{H}^n(\Sigma \cap B_{4R_k}) + o_k(1) \\ &\geq \Lambda_{\text{plane}}(n+1)v_1^{n/(n+1)} + \mathcal{E}_k(\mathcal{X}'_k; B_{4R_k}) - \mathcal{H}^n(\Sigma \cap B_{4R_k}) + o_k(1) \\ &\geq \Lambda_{\text{plane}}(n+1)v_1^{n/(n+1)} + [\tau_{R_k, \lambda}(v - v_k) - \mathcal{H}^n(\Sigma \cap B_{4R_k})] + o_k(1). \end{aligned}$$

The term in brackets on the right-hand side tends to $\tilde{\Lambda}_{T_\lambda}(v - v_1)$ since $v \mapsto \tau_{R, \lambda}(v)$ is continuous in v , uniformly in R , a fact that can be readily verified through simple competitor arguments. Thus

$$\tilde{\Lambda}_{T_\lambda}(v) \geq \Lambda_{\text{plane}}(n+1)v_1^{n/(n+1)} + \tilde{\Lambda}_{T_\lambda}(v - v_1), \quad (4.13)$$

completing the proof in this case.

Case 2: Now suppose $|\mathcal{X}_{\infty,1}(h)| < \infty$ for either $h = 2$ or 3 . Without loss of generality we may assume this holds for $h = 2$. By Lemma 3.4(iii), $\mathcal{X}_{\infty,1}(2)$ is empty, and in particular from the Hausdorff convergence, $\mathcal{X}_{k,1}(2) \cap B_{4R_0}(x_{k,1})$ is empty for k sufficiently large. Notice that $\Lambda_{\text{plane}}(n+1) < (n+1)\omega_{n+1}$ since the standard lens cluster is the unique local minimizer with planar area growth by [9, Theorem 2.9]. In conjunction with (3.15), this means $\mathcal{P}(\mathcal{X}_{\infty,1}; B_{3R_0}) > \Lambda_{\text{plane}}(n+1)v_1^{n/(n+1)}$, and thus, now letting $\hat{B}_k = B_{3R_0}(x_{k,1})$,

$$\mathcal{P}(\mathcal{X}_k; \hat{B}_k) > \Lambda_{\text{plane}}(n+1)v_1^{n/(n+1)} + o_k(1). \quad (4.14)$$

The cluster $\mathcal{X}'_k = (\mathcal{X}_k(1) \setminus \hat{B}_k, \mathcal{X}_k(2), \mathcal{X}_k(3) \cup \hat{B}_k)$ satisfies

$$\mathcal{P}(\mathcal{X}'_k; B_{4R_k}) = \mathcal{P}(\mathcal{X}_k; B_{4R_k} \setminus \hat{B}_k) \quad (4.15)$$

Rearranging and summing these together yields

$$\mathcal{P}(\mathcal{X}_k; B_{4R_k}) > \Lambda_{\text{plane}}(n+1)v_1^{n/(n+1)} + \mathcal{P}(\mathcal{X}'_k; B_{4R_k}) + o_k(1). \quad (4.16)$$

We now repeat the argument of case 1 starting directly after (4.12) to conclude the proof in this case as well. $\square$

⁵Concretely, one can obtain (the interface between) such sets by gluing a correctly oriented plane in $B = B_{cR_0}(x_{k,1} + y)$, for a suitable $c \in [1, 3/2]$, to the interface $\partial\mathcal{X}_k(2) \cap \partial\mathcal{X}_k(3)$ in $B_{4R_0}(x_{k,1} + y) \setminus B$, incurring a $o_k(1)$ error of energy along a portion of ∂B .

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. Fix $v_0 > 0$ small enough so that

$$\Lambda_{\text{plane}}(n+1) \cdot (1-v_0)^{n/(n+1)} > \Lambda_{T_\lambda} > 0 \quad (4.17)$$

for all $|\lambda| < \lambda_0$; this is possible thanks to (4.1). Fix $\lambda \in (-\lambda_0, \lambda_0)$ and take a sequence R_k tending to $+\infty$. For each $k \in \mathbb{N}$, let $\mathcal{X}_k^1$ be a minimizer of the variational problem $\tau_{R_k, \lambda}(1)$ as defined in (4.2). Let λ_* be the constant associated to the sequence $\{\mathcal{X}_k^1\}$ as defined in (4.6). Here we have tacitly passed to a subsequence without relabeling and will continue to do so in the remainder of the proof.

Apply Lemma 4.3. If $\lambda_* < +\infty$, then the conclusions of the theorem are satisfied by the limiting cluster $\mathcal{X}$ of the first concentration. The decay rates (1.4) and (1.5) follow from the decay of the leaves $T_{\pm 2\lambda_*}$ of the foliation, see Remark 1.4. Suppose $\lambda_* = +\infty$. Then according to Lemma 4.3,

$$\tilde{\Lambda}_{T_\lambda}(1) \geq \Lambda_{\text{plane}} v_1^{n/(n+1)} + \tilde{\Lambda}_{T_\lambda}(1-v_1). \quad (4.18)$$

By (4.17), we see that $v_1 \leq 1-v_0$, i.e. $1-v_1 \geq v_0$, hence we may run this argument again: let $\mathcal{X}_k^{1-v_1}$ be minimizers of $\tau_{R_k, \lambda}(1-v_1)$. Once again, take the λ_* associated to this sequence as in (4.6). By Lemma 4.3, if $\lambda_* < +\infty$, the proof is complete. If $\lambda_* = +\infty$, use the notation v_2 in place of v_1 to denote the volume of the first chamber of the limiting concentration $\mathcal{X}_{\infty, 1}$ from this sequence. Then Lemma 4.3 together with (4.18) and the concavity of $t \mapsto t^{n/(n+1)}$ shows

$$\tilde{\Lambda}_{T_\lambda}(1) \geq \Lambda_{\text{plane}}(v_1 + v_2)^{n/(n+1)} + \tilde{\Lambda}_{T_\lambda}(1-v_1-v_2).$$

Again invoking (4.17), we have $1-v_1-v_2 \geq 0$, allowing us to iterate the procedure again. In the j th iteration, the associated sequence of minimizers of $\tau_{R_k, \lambda}(1-v_1-\dots-v_{j-1})$ either satisfies $\lambda_* < +\infty$, in which case the proof is complete by Lemma 4.3, or else $\tilde{\Lambda}_{T_\lambda}(1) \geq \Lambda_{\text{plane}}(v_1+\dots+v_j)^{n/(n+1)}$, which by (4.17) guarantees $v_1+\dots+v_j \leq 1-v_0$. By Lemma 3.3, we have $v_i \geq \bar{v}$ for each i , hence we must have $\lambda_* < +\infty$ for the j th iteration of the argument for some $j \leq \lceil 1/\bar{v} \rceil$, thus allowing us apply Lemma 4.3 and conclude the existence of a minimizing cluster with blowdown C . Again, the decay estimates (1.4) and (1.5) are immediate consequences of the decay of the leaves of the foliation. $\square$

Remark 4.4. For each $\lambda \in [-\lambda_0, \lambda_0]$, the proof above produces a minimizing $(1, 2)$ -cluster $\mathcal{X}_\lambda$ with $|\mathcal{X}_\lambda(1)| = v_\lambda \geq \bar{v}$. Up to rescaling by a factor $\rho_\lambda = |\mathcal{X}_\lambda(1)|^{-1/n} \leq \bar{v}^{-1/n}$, we may assume $|\mathcal{X}_\lambda(1)| = 1$.

A contradiction argument akin to the one in the proof of Lemma 3.1 shows that the constant c_0 in the lower density estimate (3.7) can be taken to be uniform for all T_λ with $\lambda \in [-\lambda_0, \lambda_0]$. In turn, the volume lower bound $\bar{v} > 0$ for each concentration produced in this construction (coming from Lemma 3.3) is uniform in $\lambda \in [-\lambda_0, \lambda_0]$. In particular the rescaling factor ρ_λ has a uniform upper bound for all $\lambda \in [-\lambda_0, \lambda_0]$.

5. PROOF OF THEOREM 1.8

In this section, we will be frequently working with the positive Jacobi fields $\phi_1(\xi)r^{-\gamma_1^\pm}$ from Section 2 (or $\phi_1(\xi)r^{-\gamma_1}$ with $\gamma_1 = \frac{n-2}{2}$ if C is strictly stable). To simplify notation, we will let

$$\beta := n-2-2\gamma_1^- \geq 0, \quad (5.1)$$

and we notice that $\gamma_1^+ + \gamma_1^- = n-2$, and thus $\gamma_1^+ - \gamma_1^- = \beta$.

For $0 < s < r$, let $A_{s,r}$ denote the annulus $B_r \setminus B_s$. Given $u : C \cap \overline{A_{s,r}} \rightarrow \mathbb{R}$, we will write

$$\text{graph}_C^{s,r} u = \{x + u(x)\nu_C(x) : x \in C \cap A_{s,r}\}.$$

Note that the graph is not necessarily contained in the annulus. If $(s, r) = (0, \infty)$, we will omit the superscript and simply write graph_C .

Lemma 5.1. *Let C be a strictly minimizing cone with an isolated singularity and let $\{T_\lambda\}_{\lambda \in \mathbb{R}}$ be the Hardt-Simon foliation given by Theorem 2.1. Fix $\lambda \in \mathbb{R}$ and suppose there exist $R_0 > 0$, $q(r) : (0, \infty) \rightarrow (0, \infty)$, and $a \in \mathbb{R}$ such that*

$$T_\lambda \setminus B_{R_0} = \text{graph}_C q(|x|)\phi_1(x/|x|), \quad (5.2)$$

with $q(r) = ar^{-\gamma_1^-} + o(r^{-\gamma_1^-})$, and ϕ_1 is as in Section 2.1. For $R_0 \leq \rho_0 < S < R/4$, suppose that we have a hypersurface $M = M_1 \sqcup M_2$ where $M_1 = \text{graph}_C^{\rho_0, S} u$ for a function $u : C \cap A_{\rho_0, S} \rightarrow \mathbb{R}$ satisfying

$$\|u - \tilde{a}r^{-\gamma_1^-}\phi_1\|_{L^\infty(C \cap A_{\rho_0, S})} + \|r(\nabla_C u - \nabla_C(\tilde{a}r^{-\gamma_1^-}\phi_1))\|_{L^\infty(C \cap A_{\rho_0, S})} = o(S^{-\gamma_1^-}) \quad (5.3)$$

if C is strictly stable, and

$$\|u - \tilde{a}r^{-\gamma_1^-} \log r \cdot \phi_1\|_{L^\infty(C \cap A_{\rho_0, S})} + \|r(\nabla_C u - \nabla_C(\tilde{a}r^{-\gamma_1^-} \log r \cdot \phi_1))\|_{L^\infty(C \cap A_{\rho_0, S})} = o(S^{-\gamma_1^-} \log S) \quad (5.4)$$

otherwise, while $\llbracket M_2 \rrbracket$ is an area-minimizing integral current with $\partial \llbracket M_2 \rrbracket = \llbracket \text{graph}_{C \cap \partial B_R} q\phi_1 \rrbracket + \llbracket \text{graph}_{C \cap \partial B_S} u \rrbracket$. If $\tilde{a} \neq a$, then

$$\mathcal{H}^n(M) - \mathcal{H}^n(\text{graph}_C^{\rho_0, R}(q\phi_1)) \rightarrow \infty \quad (5.5)$$

as $S \rightarrow \infty$ and $R/S \rightarrow \infty$.

Remark 5.2. The hypothesis (5.2) is required in order to gain improved error estimates on the area comparison between M and T_λ that are higher order in the smaller scale S rather than the larger scale R . This is crucial, since the dominant term in the area comparison is of order S^β in the strictly stable case and order $\log S$ in the non-strictly stable case. Within the proof of the lemma, this can be seen in the derivation of (5.29) and in step four.

It is worth pointing out that the hypothesis (5.2) is not necessary in the specific case of $n = 8$ and $\tilde{a} = 0$, since one may show that all higher order errors are $O(R^{n-3\gamma_1^- - 3})$ and $\gamma_1^- \geq \frac{n-2}{2} - \sqrt{\left(\frac{n-2}{2}\right)^2 - (n-1)}$ so in particular $n - 3\gamma_1^- - 3 < 0$ when $n = 7$, cf. the area expansion in Appendix A.1 and step one of the proof, but we omit the details here since the conclusion is significantly stronger under hypothesis (5.2).

Proof. We perform the computation in the case when C is in addition strictly stable, namely (2.2) holds for T_λ . The only difference in the argument for the non-strictly stable case is that, in light of the amended asymptotic behavior (2.3), all terms of the form S^β and R^β become respectively $\log S$ and $\log R$, with error terms being of the order $o(\log S)$. We therefore omit the details of the modifications in the non-strictly stable case and simply state the final estimate obtained in that case in the concluding step of the proof below.

Before giving the proof, which is divided into steps, let us set some notation. We will use polar coordinates (r, ω) for $x \in \mathbb{R}^{n+1}$. We set $Z_u(x) := |\nabla_C u(x)| + |u(x)|/|x|$, and write

$$Q_{s,r}(u_1, u_2) := \int_{C \cap A_{s,r}} \langle \nabla_C u_1, \nabla_C u_2 \rangle - |A_C|^2 u_1 u_2 d\mathcal{H}^n$$

for the second variation of C restricted to an annulus. We let $F_{s,r}(u) = \mathcal{H}^n(\text{graph}_C^{s,r} u) - \mathcal{H}^n(C \cap A_{s,r})$. Note also that by Lemma A.1, for R/S large enough, M_2 is graphical over C , and so we can extend u so that the entire surface M is the graph of u .

Step zero (expansion of area). We claim that there are $\varepsilon_0, C_0 > 0$ such that if $\|Z_u\|_{L^\infty(C \cap A_{s,r})} \leq \varepsilon_0$, then

$$F_{s,r}(u) = \frac{1}{2} Q_{s,r}(u, u) + \mathcal{R}_{s,r}(u), \quad (5.6)$$

where the remainder $\mathcal{R}_{s,r}(u)$ satisfies

$$|\mathcal{R}_{s,r}(u)| \leq C_0 \int_{C \cap A_{s,r}} Z_u^3 d\mathcal{H}^n, \quad (5.7)$$

and for any smooth, compactly supported $\psi : C \cap A_{s,r} \rightarrow \mathbb{R}$, the first variation $\delta \mathcal{R}_{s,r}(u)$ of the remainder at u satisfies

$$|\delta \mathcal{R}_{s,r}(u)[\psi]| \leq C_0 \int_{C \cap A_{s,r}} Z_u^2 Z_\psi d\mathcal{H}^n. \quad (5.8)$$

Lastly, up to decreasing ε_0 and increasing C_0 , we claim that

$$|\delta^2 F_{s,r}(u)[v, w] - Q_{s,r}(v, w)| \leq C_0 \|Z_u\|_{L^\infty} \left(\int_{C \cap A_{s,r}} |\nabla_C v|^2 + \frac{v^2}{r^2} \right)^{1/2} \left(\int_{C \cap A_{s,r}} |\nabla_C w|^2 + \frac{w^2}{r^2} \right)^{1/2}. \quad (5.9)$$

The proofs of these claims can be found in Appendix A.1.

Step one (area difference on $A_{\rho_0, S}$). Here we claim that

$$\mathcal{H}^n(M \cap A_{\rho_0, S}) - \mathcal{H}^n(T_\lambda \cap A_{\rho_0, S}) = \frac{1}{2} [a^2 - \tilde{a}^2] \gamma_1^- S^\beta + o(S^\beta). \quad (5.10)$$

Let $L_C := \Delta_C + |A_C|^2$ denote the Jacobi operator on C . Thanks to step zero combined with an integration by parts, we may write

$$\begin{aligned} \mathcal{H}^n(M \cap A_{\rho_0, S}) - \mathcal{H}^n(T_\lambda \cap A_{\rho_0, S}) &= \frac{1}{2} Q_{\rho_0, S}(u, u) - \frac{1}{2} Q_{\rho_0, S}(v, v) + \mathcal{R}_{\rho_0, S}(u) - \mathcal{R}_{\rho_0, S}(v) \\ &= \frac{1}{2} \int_{C \cap A_{\rho_0, S}} (v L_C v - u L_C u) d\mathcal{H}^n \end{aligned}$$

$$+ \frac{1}{2} \int_{C \cap \partial B_S} (u \partial_r u - v \partial_r v) d\mathcal{H}^{n-1} + C_1(\rho_0) \\ + \mathcal{R}_{\rho_0, S}(u) - \mathcal{R}_{\rho_0, S}(v).$$

Now, (5.3) and the definition of v guarantee that

$$\mathcal{R}_{\rho_0, S}(u) = O(S^{n-3\gamma_1^- - 3}) = o(S^\beta) \quad \text{and} \quad \mathcal{R}_{\rho_0, S}(v) = O(S^{n-3\gamma_1^- - 3}) = o(S^\beta). \quad (5.11)$$

Moreover, by elliptic estimates we have

$$L_C(v) = o(S^{-\gamma_1^- - 2}) \quad \text{and} \quad L_C u = o(S^{-\gamma_1^- - 2}), \quad (5.12)$$

and

$$\partial_r v = -a\gamma_1^- S^{-\gamma_1^- - 1} \phi_1 + o(S^{-\gamma_1^- - 1}) \quad \text{and} \quad \partial_r u = -\tilde{a}\gamma_1^- S^{-\gamma_1^- - 1} \phi_1 + o(S^{-\gamma_1^- - 1}). \quad (5.13)$$

Inserting (5.11)-(5.13) into the calculation above and applying the coarea formula yields (5.10) as desired.

Step two (reduction of area estimate on $A_{S,R}$ to case with Jacobi field data on $C \cap \partial B_S$). Consider the Plateau problem (again in the sense of integral currents) with boundary data given by the graph of $\tilde{a}S^{-\gamma_1^-} \phi_1$ on $C \cap \partial B_S$ and given by T_λ on $C \cap \partial B_R$. By Lemma A.1, the solution is a normal graph of a function $\tilde{u}$ on $C \cap A_{S,R}$. Similarly, let $\tilde{v}$ be the function whose graph solves the Plateau problem with boundary data given by the graph of $aS^{-\gamma_1^-} \phi_1$ on $C \cap \partial B_S$ and given by T_λ on $C \cap \partial B_R$. Note that

$$Z_{\tilde{u}} + Z_{\tilde{v}} = O(|x|^{-\gamma_1^- - 1}). \quad (5.14)$$

by Lemma A.1. In this step we show that

$$F_{S,R}(u) = F_{S,R}(\tilde{u}) + o(S^\beta) \quad \text{and} \quad F_{S,R}(v) = F_{S,R}(\tilde{v}) + o(S^\beta). \quad (5.15)$$

We prove the first estimate in (5.15); the second is the same. We first show that

$$F_{S,R}(u) \leq F_{S,R}(\tilde{u}) + o(S^\beta) \quad (5.16)$$

Let $H = (u - \tilde{u})|_{C \cap \partial B_S}$, which has $\|H\|_{C^1(C \cap \partial B_S)} = o(S^{-\gamma_1^-})$ by (5.3). By an elementary extension argument, we may extend H to some $W \in W^{1,\infty}(C; \mathbb{R})$ such that

$$\text{spt } W \subset C \cap B_{2S}, \quad \|W\|_{L^\infty(C)} = o(S^{-\gamma_1^-}), \quad \text{and} \quad \|\nabla_C W\|_{L^\infty(C \cap B_{2S} \setminus B_S)} = o(S^{-\gamma_1^- - 1}). \quad (5.17)$$

Since W agrees with H on ∂B_S , $\text{graph}_C^{S,R}(\tilde{u} + W)$ is an admissible competitor for the Plateau problem corresponding to u . Thus, Taylor expanding and using (5.9) and the fact that the second fundamental form A_C of C satisfies $|A_C| \simeq |x|^{-1}$ on $C \setminus \{0\}$, we obtain

$$F_{S,R}(u) \leq F_{S,R}(\tilde{u} + W) \quad (5.18)$$

$$\leq F_{S,R}(\tilde{u}) + \delta F_{S,R}(\tilde{u})[W] + C_0 \|Z_v\|_{L^\infty} Q_{S,R}(W, W) \quad (5.19)$$

Define $G : C \cap \partial B_S \rightarrow \mathbb{R}^{n+1}$ by $G(x) = x + \tilde{u}(x)\nu_C(x)$. Since $N := \text{graph}_C^{S,R} \tilde{u}$ is a minimal surface with boundary and $\text{spt } W \subset B_{2S} \subset B_R$, we have

$$\delta F_{S,R}(\tilde{u})[W] = \int_{G(C \cap \partial B_S)} (W \circ G^{-1})(\nu_C \circ G^{-1}) \cdot \nu_N^{\text{co}} d\mathcal{H}^{n-1},$$

where ν_N^{co} denotes the unit conormal to $G(C \cap \partial B_S)$ in $G(C \cap B_S)$. Towards estimating the dot product $(\nu_C \circ G^{-1}) \cdot \nu_N^{\text{co}}$, we recall that since $|\nabla_C \tilde{u}| \leq Z_{\tilde{u}} = O(S^{-\gamma_1^- - 1})$ on $C \cap \partial B_S$. Again using that $|A_C| \simeq |x|^{-1}$ on $C \setminus \{0\}$, a direct computation of $|(\nu_C \circ G^{-1}) \cdot \nu_N^{\text{co}}|$ shows that it is bounded by $(1 + o(1))|\nabla_C \tilde{u}|$, yielding

$$(\nu_C \circ G^{-1}) \cdot \nu_N^{\text{co}} = O(S^{-\gamma_1^- - 1}) \quad \text{on } G(C \cap \partial B_S).$$

Using this together with the fact that $\mathcal{H}^{n-1}(G(C \cap \partial B_S)) = O(S^{n-1})$ and the L^∞ bound for W from (5.17), we obtain

$$\delta F_{S,R}(\tilde{u})[W] = O(S^{n-1-\gamma_1^- - 1}) o(S^{-\gamma_1^-}) = o(S^{n-2-2\gamma_1^-}) = o(S^\beta). \quad (5.20)$$

Similarly, using the estimate (5.14) for $Z_{\tilde{u}}$, the estimate (5.17) for W and $\nabla_C W$, and the fact that $\text{spt } W \subset C \cap B_{2S}$, we have

$$\|Z_{\tilde{u}}\|_{L^\infty} Q_{S,R}(W, W) = O(S^{-\gamma_1^- - 1 + n - 2\gamma_1^- - 2}) = o(S^\beta). \quad (5.21)$$

Inserting (5.20) and (5.21) into (5.19) yields (5.16).

To finish the proof of (5.15), it remains to prove the reverse inequality

$$F(\tilde{u}) - F(u) \leq o(S^\beta). \quad (5.22)$$

The proof is nearly identical to the proof of (5.16), so we provide a summary instead of complete details. First, using the area minimality of the graph of $\tilde{u}$, we have the analogue

$$F(\tilde{u}) \leq F_{S,R}(u - W) \leq F_{S,R}(u) - \delta F_{S,R}(u)[W] + C_0 \|Z_u\|_{L^\infty} Q_{S,R}(W, W)$$

of (5.19). The first and second variation estimates then follow exactly as in (5.20) and (5.21), since u satisfies the same decay estimates as $\tilde{u}$. This concludes the proof of (5.22) and thus the proof of (5.15).

Step three (equivalence of area estimate on $A_{S,R}$ with inner Jacobi field data to estimate of second variations). Next, let U and V solve the Jacobi equation $L_C \cdot = 0$ on $C \cap A_{S,R}$ with boundary data equal to $\tilde{u}$ and $\tilde{v}$ respectively on $C \cap (\partial B_S \cup \partial B_R)$. In this step we show

$$F_{S,R}(\tilde{u}) - F_{S,R}(\tilde{v}) = \frac{1}{2} Q_{S,R}(U, U) - \frac{1}{2} Q_{S,R}(V, V) + o(S^\beta). \quad (5.23)$$

By symmetry, since a and $\tilde{a}$ are arbitrary, to show (5.23) it is enough to show the inequality

$$F_{S,R}(\tilde{u}) - F_{S,R}(\tilde{v}) \leq \frac{1}{2} Q_{S,R}(U, U) - \frac{1}{2} Q_{S,R}(V, V) + o(S^\beta); \quad (5.24)$$

the opposite inequality is the exact same.

Set $\Psi = U - V$, which by linearity solves the Jacobi equation

$$L_C \Psi = 0 \text{ on } C \cap A_{S,R}, \quad \Psi|_{C \cap \partial B_S} = (\tilde{a} - a) S^{-\gamma_1^-} \phi_1, \quad \Psi|_{C \cap \partial B_R} = 0. \quad (5.25)$$

Since $\tilde{v} + \Psi = \tilde{u}$ on $C \cap (\partial B_S \cup \partial B_R)$, we can test the area minimality of the graph of $\tilde{u}$ against the graph of $\tilde{v} + \Psi$, so that, using the expansion of the area functional over C from step zero, we obtain

$$\begin{aligned} F_{S,R}(\tilde{u}) - F_{S,R}(\tilde{v}) &\leq F_{S,R}(\tilde{v} + \Psi) - F_{S,R}(\tilde{v}) \\ &= \frac{1}{2} Q_{S,R}(\tilde{v} + \Psi, \tilde{v} + \Psi) - \frac{1}{2} Q_{S,R}(\tilde{v}, \tilde{v}) + \mathcal{R}_{S,R}(\tilde{v} + \Psi, \tilde{v} + \Psi) - \mathcal{R}_{S,R}(\tilde{v}, \tilde{v}). \end{aligned} \quad (5.26)$$

Next, we claim that

$$Q_{S,R}(\tilde{v} + \Psi, \tilde{v} + \Psi) - Q_{S,R}(\tilde{v}, \tilde{v}) = Q_{S,R}(V + \Psi, V + \Psi) - Q_{S,R}(V, V). \quad (5.27)$$

Indeed, we expand the square and integrate by parts, recalling that Ψ solves the Jacobi equation $L_C \Psi = 0$ and vanishes on $C \cap \partial B_R$ to find

$$\begin{aligned} Q_{S,R}(\tilde{v} + \Psi, \tilde{v} + \Psi) - Q_{S,R}(\tilde{v}, \tilde{v}) &= Q_{S,R}(\Psi, \Psi) + 2Q_{S,R}(\tilde{v}, \Psi) \\ &= - \int_{\partial B_S \cap C} \Psi \partial_r \Psi + 2 \left(\int_{\partial B_R \cap C} \tilde{v} \partial_r \Psi - \int_{\partial B_S \cap C} \tilde{v} \partial_r \Psi \right) \end{aligned}$$

Since $\tilde{v} = V$ on $C \cap (\partial B_S \cup \partial B_R)$, the expansion with V in place of $\tilde{v}$ yields the identical right-hand side. As a consequence, inserting (5.27) into (5.26) and recalling that $\Psi + V = U$ gives

$$F_{S,R}(\tilde{u}) - F_{S,R}(\tilde{v}) \leq \frac{1}{2} Q_{S,R}(U, U) - \frac{1}{2} Q_{S,R}(V, V) + \mathcal{R}_{S,R}(\tilde{v} + \Psi) - \mathcal{R}_{S,R}(\tilde{v}) \quad (5.28)$$

So, to finish proving (5.24), it remains to show that

$$\mathcal{R}_{S,R}(\tilde{v} + \Psi) - \mathcal{R}_{S,R}(\tilde{v}) = o(S^\beta). \quad (5.29)$$

With this aim in mind, we begin by computing Ψ using (5.25). Since the boundary data for Ψ involves only the ϕ_1 mode, Ψ is a linear combination of the Jacobi fields corresponding to γ_1^- and γ_1^+ . A direct calculation which we omit that uses the boundary conditions to solve for these coefficients yields the following expression for Ψ in polar coordinates:

$$\Psi(r, \omega) = \frac{S^\beta}{1 - S^\beta/R^\beta} (\tilde{a} - a) (r^{-\gamma_1^+} - R^{-\beta} r^{-\gamma_1^-}) \phi_1(\omega).$$

Since $2S \leq R$, recalling also that $-\gamma_1^- - \beta = -\gamma_1^+$, this implies

$$|\Psi| \leq CS^\beta |\tilde{a} - a| \left(|r^{-\gamma_1^+}| + r^{-\beta} |r^{-\gamma_1^-}| \right) \|\phi_1\|_{L^\infty} \leq CS^\beta r^{-\gamma_1^+}.$$

Similarly, differentiating Ψ leads to the bound $|\nabla_C \Psi| \leq CS^\beta r^{-\gamma_1^+ - 1}$ which, combined with the previous inequality shows

$$Z_\Psi \leq CS^\beta r^{-\gamma_1^+ - 1}. \quad (5.30)$$

Using the fundamental theorem of calculus, (5.6), the triangle inequality, and the inequality $Z_{t\Psi} \leq Z_\Psi$ for $t \in [0, 1]$, we estimate

$$\begin{aligned} |\mathcal{R}_{S,R}(\tilde{v} + \Psi) - \mathcal{R}_{S,R}(\tilde{v})| &\leq \int_0^1 |\delta \mathcal{R}_{S,R}(\tilde{v} + t\Psi)[\Psi]| dt \leq \int_0^1 \int_{C \cap A_{S,R}} (Z_{\tilde{v}+t\Psi})^2 Z_\Psi d\mathcal{H}^n dt \\ &\leq \int_{C \cap A_{S,R}} (Z_{\tilde{v}} + Z_\Psi)^2 Z_\Psi d\mathcal{H}^n = \int_{C \cap A_{S,R}} Z_{\tilde{v}}^2 Z_\Psi + 2Z_{\tilde{v}} Z_\Psi^2 + Z_\Psi^3 d\mathcal{H}^n. \end{aligned}$$

Recalling the bounds $Z_{\tilde{v}} = O(r^{-\gamma_1^- - 1})$ (from (5.14)) and $Z_\Psi \leq CS^\beta r^{-\gamma_1^+ - 1}$ (from (5.30)), and the identities $\gamma_1^+ + \gamma_1^- = n - 2$ and $\gamma_1^+ - \gamma_1^- = \beta$, we estimate

$$\begin{aligned} |\mathcal{R}_{S,R}(\tilde{v} + \Psi) - \mathcal{R}_{S,R}(\tilde{v})| &\lesssim \int_S^R S^\beta r^{n-1} [r^{-2\gamma_1^- - 2 - \gamma_1^+ - 1} + S^\beta r^{-\gamma_1^- - 1 - 2\gamma_1^+ - 2} + S^{2\beta} r^{-3\gamma_1^+ - 3}] dr \\ &\leq S^\beta \int_S^R r^{-2 - \gamma_1^-} dr = o(S^\beta), \end{aligned}$$

This concludes the remainder bound and thus the proofs of (5.24) and (5.23).

Step four (computation of second variation on $A_{S,R}$ with inner Jacobi data). In this step we prove that

$$Q_{S,R}(U, U) - Q_{S,R}(V, V) = \left[\gamma_1^- (\tilde{a}^2 - a^2) + \frac{\beta(a - \tilde{a})^2}{1 - (S/R)^\beta} \right] S^\beta + o(S^\beta). \quad (5.31)$$

Integrating by parts and using the fact that U and V solve the Jacobi equation in $A_{S,R}$ and agree on ∂B_R , we first notice that

$$\begin{aligned} Q_{S,R}(U, U) - Q_{S,R}(V, V) &= \int_{C \cap \partial B_R} U(\partial_r U - \partial_\nu V) - \int_{C \cap \partial B_S} U \partial_r U d\mathcal{H}^{n-1} + \int_{C \cap \partial B_S} V \partial_r V d\mathcal{H}^{n-1} \\ &= \int_{C \cap \partial B_R} U(\partial_r U - \partial_\nu V) d\mathcal{H}^{n-1} - \tilde{a} \int_{C \cap \partial B_S} S^{-\gamma_1^-} \phi_1 \partial_r U d\mathcal{H}^{n-1} \\ &\quad + a \int_{C \cap \partial B_S} S^{-\gamma_1^-} \phi_1 \partial_r V d\mathcal{H}^{n-1} \end{aligned}$$

Now, we may expand V and U in polar coordinates as

$$V(r, \omega) = \sum_{j=1}^{\infty} (a_j \phi_j(\omega) r^{-\gamma_j^-} + b_j \phi_j(\xi) r^{-\gamma_j^+}), \quad U(r, \omega) = \sum_{j=1}^{\infty} (\tilde{a}_j \phi_j(\omega) r^{-\gamma_j^-} + \tilde{b}_j \phi_j(\xi) r^{-\gamma_j^+}) \quad (5.32)$$

where $\{\phi_j\}_j$ are a basis of eigenfunctions (with ϕ_1 as before) of $\Delta_{C \cap \partial B_1} + |A_{C \cap \partial B_1}|^2$ with $\int_{C \cap \partial B_1} \phi_j^2 = 1$ and eigenvalues $\lambda_j \rightarrow \infty$ as $j \rightarrow \infty$, and $\gamma_j^\pm = \frac{n-2}{2} \pm \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_j}$ (cf. Section 2), and since the boundary data of V and U lie in the span of ϕ_1 alone, the coefficients $a_j, b_j, \tilde{a}_j, \tilde{b}_j$ satisfy

$$a_j = b_j = \tilde{a}_j = \tilde{b}_j = 0 \quad \text{for } j \geq 2,$$

while

$$\begin{cases} a_1 R^{-\gamma_1^-} + b_1 R^{-\gamma_1^+} = a R^{-\gamma_1^-} + \sigma \\ a_1 S^{-\gamma_1^-} + b_1 S^{-\gamma_1^+} = a S^{-\gamma_1^-}. \end{cases}$$

and

$$\begin{cases} \tilde{a}_1 R^{-\gamma_1^-} + \tilde{b}_1 R^{-\gamma_1^+} = a R^{-\gamma_1^-} + \sigma \\ \tilde{a}_1 S^{-\gamma_1^-} + \tilde{b}_1 S^{-\gamma_1^+} = \tilde{a} S^{-\gamma_1^-}. \end{cases}$$

for a constant σ which is $o(R^{-\gamma_1^-})$. This yields

$$\begin{aligned} a_1 &= a + \sigma \frac{R^{\gamma_1^+}}{R^\beta - S^\beta}, \quad b_1 = -\sigma \frac{R^{\gamma_1^-} R^\beta S^\beta}{R^\beta - S^\beta} \\ \tilde{a}_1 &= \frac{a R^\beta - \tilde{a} S^\beta}{R^\beta - S^\beta} + \sigma \frac{R^{\gamma_1^+}}{R^\beta - S^\beta}, \quad \tilde{b}_1 = \frac{(\tilde{a} - a - \sigma R^{\gamma_1^-}) R^\beta S^\beta}{R^\beta - S^\beta}. \end{aligned}$$

In other words,

$$V(r, \omega) = \left[\left(a + \sigma \frac{R^{\gamma_1^+}}{R^\beta - S^\beta} \right) r^{-\gamma_1^-} - \sigma \frac{R^{\gamma_1^-} R^\beta S^\beta}{R^\beta - S^\beta} r^{-\gamma_1^+} \right] \phi_1(\omega)$$

$$U(r, \omega) = \left[\left(\frac{aR^\beta - \tilde{a}S^\beta}{R^\beta - S^\beta} + \sigma \frac{R^{\gamma_1^+}}{R^\beta - S^\beta} \right) r^{-\gamma_1^-} + \frac{(\tilde{a} - a - \sigma R^{\gamma_1^-}) R^\beta S^\beta}{R^\beta - S^\beta} r^{-\gamma_1^+} \right] \phi_1(\omega).$$

Let us now insert this into the above computation for the difference between $Q_{S,R}(U, U)$ and $Q_{S,R}(U, V)$. Noticing that U and V share the same σ terms, we observe that the corresponding terms in $\partial_r U - \partial_r V$ cancel, and so we are left with

$$\begin{aligned} & \int_{C \cap \partial B_R} V(\partial_r U - \partial_r V) d\mathcal{H}^{n-1} \\ &= (aR^{-\gamma_1^-} + \sigma) \left(-\gamma_1^- \frac{aR^\beta - \tilde{a}S^\beta}{R^\beta - S^\beta} R^{\beta+\gamma_1^-} - \gamma_1^+ (\tilde{a} - a) \frac{R^\beta S^\beta}{R^\beta - S^\beta} R^{\gamma_1^-} + \gamma_1^- a R^{\beta+\gamma_1^-} \right) \\ &= \beta(aR^{-\gamma_1^-} + \sigma) \frac{(a - \tilde{a})S^\beta R^{\beta+\gamma_1^-}}{R^\beta - S^\beta}. \end{aligned}$$

Similarly,

$$a \int_{C \cap \partial B_S} S^{-\gamma_1^-} \phi_1 \partial_r V = a \left[-\gamma_1^- \left(a + \sigma \frac{R^{\gamma_1^+}}{R^\beta - S^\beta} \right) S^\beta + \sigma \gamma_1^+ \frac{R^{\gamma_1^-} R^\beta S^\beta}{R^\beta - S^\beta} \right]$$

and

$$-\tilde{a} \int_{C \cap \partial B_S} S^{-\gamma_1^-} \phi_1 \partial_r U = \tilde{a} \left[\gamma_1^- \left(\frac{aR^\beta - \tilde{a}S^\beta}{R^\beta - S^\beta} + \sigma \frac{R^{\gamma_1^+}}{R^\beta - S^\beta} \right) S^\beta + \gamma_1^+ \frac{(\tilde{a} - a - \sigma R^{\gamma_1^-}) R^\beta S^\beta}{R^\beta - S^\beta} \right]$$

We therefore arrive at

$$\begin{aligned} & Q_{S,R}(U, U) - Q_{S,R}(V, V) \\ &= \left\{ \beta(aR^{-\gamma_1^-} + \sigma) \frac{(a - \tilde{a})S^\beta R^{\beta+\gamma_1^-}}{R^\beta - S^\beta} + a \left[-\gamma_1^- \left(a + \sigma \frac{R^{\gamma_1^+}}{R^\beta - S^\beta} \right) S^\beta + \sigma \gamma_1^+ \frac{R^{\gamma_1^-} R^\beta S^\beta}{R^\beta - S^\beta} \right] \right. \\ & \quad \left. + \tilde{a} \left[\gamma_1^- \left(\frac{aR^\beta - \tilde{a}S^\beta}{R^\beta - S^\beta} + \sigma \frac{R^{\gamma_1^+}}{R^\beta - S^\beta} \right) S^\beta + \gamma_1^+ \frac{(\tilde{a} - a - \sigma R^{\gamma_1^-}) R^\beta S^\beta}{R^\beta - S^\beta} \right] \right\} \\ &= \gamma_1^- (\tilde{a}^2 - a^2) S^\beta + \beta(a - \tilde{a})^2 \frac{S^\beta}{1 - (S/R)^\beta} + 2\beta\sigma \frac{R^{\gamma_1^-} S^\beta}{1 - (S/R)^\beta} (a - \tilde{a}) \end{aligned}$$

Keeping in mind that $\sigma = o(R^{-\gamma_1^-})$, this exactly simplifies to (5.31).

Conclusion. Combining (5.10), (5.15), (5.23), (5.31), and the fact that $u_k = q\phi_1$ on $C \cap \partial B_{R_k}$, we compute

$$\begin{aligned} \mathcal{H}^n(M) - \mathcal{H}^n(\text{graph}_C^{\rho_0, R}(q\phi_1)) &= \mathcal{H}^n(M_1) - \mathcal{H}^n(\text{graph}_C^{\rho_0, S}(q\phi_1)) + F_{S,R}(u) - F_{S,R}(v) \\ &= \frac{1}{2}(a^2 - \tilde{a}^2) \gamma_1^- S^\beta + F_{S,R}(\tilde{u}) - F_{S,R}(\tilde{v}) + o(S^\beta) \\ &= \frac{1}{2}(a^2 - \tilde{a}^2) \gamma_1^- S^\beta + \frac{1}{2}(Q_{S,R}(U, U) - Q_{S,R}(V, V)) + o(S_k^\beta) \\ &= \frac{1}{2}(a^2 - \tilde{a}^2) \gamma_1^- S^\beta + \frac{1}{2} \left[\gamma_1^- (\tilde{a}^2 - a^2) + \frac{\beta(a - \tilde{a})^2}{1 - (S/R)^\beta} \right] S^\beta + o(S^\beta) \\ &= \frac{1}{2} \frac{\beta(a - \tilde{a})^2}{1 - (S/R)^\beta} S^\beta + o(S^\beta) \rightarrow \infty \end{aligned}$$

as $S \rightarrow \infty$ since $a \neq \tilde{a}$.

In the non-strictly stable case, as explained at the beginning of the proof, by the same reasoning we instead obtain

$$\mathcal{H}^n(M) - \mathcal{H}^n(\text{graph}_C^{\rho_0, R}(q\phi_1)) = \frac{1}{2} \frac{\beta(a - \tilde{a})^2}{(1 - \log S / \log R)} \log S + o(\log S) \rightarrow \infty.$$

□

Remark 5.3. Note that if C is not strictly minimizing then the asymptotic behavior of T_λ is instead given by (2.4), so our final lower bound on the difference of areas would have γ_1^+ in place of γ_1^- , resulting in a lower bound on the difference of areas that will instead decay to zero as $S \rightarrow \infty$.

Proof of Theorem 1.8. The proof is divided into two steps. In the first, we construct locally minimizing (1,2)-clusters decaying to the cone at the same rate as T_λ for every $|\lambda|$ sufficiently small, and with volumes uniformly bounded away from zero. In the second step, we argue that rescaling each of these so that the volume of the first chamber is one yields infinitely many distinct local minimizers (up to rigid motions).

Step one: existence of infinite (not necessarily distinct) continuum of local minimizers with prescribed decay. Let λ_0 be as in section 4 and $\lambda \in [-\lambda_0, \lambda_0]$. Let $q = ar^{-\gamma_1^-} + o(r^{-\gamma_1^-})$ and $R_0 > 0$ be such that T_λ satisfies (5.2). We claim that for any such λ , there exists a locally minimizing (1,2)-cluster $\mathcal{X}_\lambda$ as in Theorem 1.3 with $\partial\mathcal{X}_\lambda(2) \cap \partial\mathcal{X}_\lambda(3) \cap B_\rho^c \subset \text{graph}_C v$ where $v = q\phi_1 + o(r^{-\gamma_1^-})$. Suppose for a contradiction that this is not the case. Repeating the argument of the proof of Theorem 1.3, we excise at most $\lceil 1/\bar{v} \rceil - 1$ lens-like concentrations to arrive at the family of variational problems $\tau_{R_k, \lambda}(v)$ with $v \geq \bar{v}$ for which $\lambda_* < \infty$ (as defined in (4.6)) and whose corresponding minimizers $\mathcal{X}_{\lambda, k}^v$ converge to the claimed minimizing (1,2)-cluster $\mathcal{X}_\lambda$ of Theorem 1.3. There is a radius ρ_0 such that $\mathcal{X}_{\lambda, k}(1), \mathcal{X}_\lambda(1) \subset B_{\rho_0}$.

The exterior interfaces $M_k := \partial\mathcal{X}_{\lambda, k}(2) \cap \partial\mathcal{X}_{\lambda, k}(3)$, and hence additionally the limiting exterior interface $M_\infty = \partial\mathcal{X}_\lambda(2) \cap \partial\mathcal{X}_\lambda(3)$, are trapped between the leaves $T_{\pm 2\lambda_*}$ of the foliation. In particular, $M_\infty \subset \text{graph}_C u_\infty$ on $B_{\rho_0}^c$. Now, the decay $|u_\infty| \lesssim r^{-\gamma_1^-}$ together with [33, Lemma 2.10 and Claim 3.3], both of which hold for exterior minimal surfaces with no modifications to the proof, guarantee that there is a constant $\tilde{a} \in \mathbb{R}$ with $\tilde{a} \neq a$ such that

$$u_\infty(r, \omega) = \tilde{a}\phi_1(\omega)r^{-\gamma_1^-} + o(r^{-\gamma_1^-}). \quad (5.33)$$

Here we are using that $C_{k, l}$ is an area-minimizing quadratic cone in order to apply the results of [33], together with the contradiction assumption to say that $\tilde{a} \neq a$.

Therefore, by the local convergence of M_k to $\text{graph}_C u_\infty$ on $B_{\rho_0}^c$ (which can be improved to smooth convergence by De Giorgi's/Allard's epsilon regularity), there are $S_k \rightarrow \infty$ such that $S_k/R_k \rightarrow 0$ and so that $M_k \subset \text{graph}_C^{\rho_0, S_k} u_k$ and (5.3) holds for $u = u_k$ with $S = S_k$ for k sufficiently large. Note that any area-minimizing quadratic cone is strictly minimizing and strictly stable, thanks to [16, Theorem 3.2, Remark 3.3] and [33, Proposition 2.7] (see also [13, Section 2.5]). Moreover, the hypothesis (5.2) holds for all area-minimizing quadratic cones, since the works [6, 11, 12, 18] guarantee that the leaves of the foliation inherit the $SO(\mathbb{S}^{k+1}) \times SO(\mathbb{S}^{l+1})$ symmetry of the cone asymptotically, which in particular implies the hypothesis, since ϕ_1 is constant. Now, for each such k let $M_{k,1}$ to be $\text{graph}_C^{\rho_0, S_k} u_k$ and let $M_{k,2}$ be an area-minimizing integral current with $\partial[M_{k,2}] = [\text{graph}_{C \cap \partial B_{7R_k/2}} q\phi_1] + [\text{graph}_{C \cap \partial B_{S_k}} u_k]$. In light of the above discussion, we may apply Lemma 5.1 for k sufficiently large to the surface $\tilde{M}_k := M_{k,1} \sqcup M_{k,2}$ with $R = 7R_k/2$, to obtain

$$\begin{aligned} \tilde{\Lambda}_{T_\lambda}(v) &= \lim_{k \rightarrow \infty} (\mathcal{E}_{R_k}(\mathcal{X}_{\lambda, k}) - \mathcal{H}^n(T_\lambda \cap B_{4R_k})) \\ &\geq \lim_{k \rightarrow \infty} (\mathcal{H}^n(\tilde{M}_k) - \mathcal{H}^n(\text{graph}_C^{\rho_0, 7R_k/2}(q\phi_1))) - c_{\rho_0} \\ &= +\infty \end{aligned}$$

for some ρ_0 -dependent constant c_0 (independent of k), where we have also used that

$$(M_{k,2} \setminus \text{graph}_C^{\rho_0, 7R_k/2}(q\phi_1)) \cap B_{4R_k} = (T_\lambda \setminus \text{graph}_C^{\rho_0, 7R_k/2}(q\phi_1)) \cap B_{4R_k}.$$

This gives the desired contradiction.

Step two: existence of infinitely many distinct local minimizers with unit volume first chambers. For $|\lambda| < \lambda_0$, let $\mathcal{X}_\lambda$ denote the locally minimizing (1,2)-cluster with exterior interface decaying to C at the same leading order rate as T_λ . We define an equivalence relation $\sim$ on these clusters by saying $\mathcal{X}_\lambda \sim \mathcal{X}_{\lambda'}$ if there is $r > 0$ such that, up to rigid motions, $\mathcal{X}_\lambda = r\mathcal{X}_{\lambda'}$. To finish the proof of the theorem, it suffices to show that there are infinitely many equivalence classes. Suppose for contradiction that this were not the case. Then there must exist non-zero $\lambda_j \rightarrow 0$, $r_j > 0$, and λ such that $r_j\mathcal{X}_{\lambda_j} = \mathcal{X}_\lambda$ for each j . By scaling,

$$r_j = (|\mathcal{X}_\lambda(1)|/|\mathcal{X}_{\lambda_j}(1)|)^{1/(n+1)}.$$

Since $|\mathcal{X}_\lambda(1)| \in [\bar{v}, 1]$ for all $|\lambda| < \lambda_0$ by Remark 4.4 with $\bar{v} > 0$, we have

$$0 < \limsup_{j \rightarrow \infty} r_j \leq \limsup_{j \rightarrow \infty} r_j < \infty. \quad (5.34)$$

On the other hand, let a_j and a denote the coefficients in front of $\phi_1 r^{-\gamma_1^-}$ in the leading order term of the graphical expansions of $\partial\mathcal{X}_j(2) \cap \partial\mathcal{X}_j(3)$ and $\partial\mathcal{X}(2) \cap \partial\mathcal{X}(3)$ respectively. Since the leading order decay of $\mathcal{X}_{\lambda_j}$ must match the leading order decay of $\mathcal{X}_{\lambda_i}$, we have

$$r_j a_j = a. \quad (5.35)$$

If $a = 0$, then (5.35) implies that $r_j = 0$, contradicting (5.34), and if $a > 0$, then (5.35) implies that $\limsup_{j \rightarrow \infty} r_j = \infty$, again contradicting (5.34). Thus our original assumption that there are finitely many equivalence classes cannot hold, and the proof is complete. $\square$

6. PROOF OF THEOREM 1.6

6.1. $\Lambda_{C_{k,k}}$ **estimate in all even dimensions** $n+1 = 2k+2$. To simplify notation slightly, we will henceforth set $m := n+1 \geq 8$, with m an even integer. Note that in our preceding work [8], joint with Lia Bronsard, the ambient dimension is n in place of $n+1$. We recall from [8] that

$$\Lambda_{\text{plane}}(m) = \frac{2(m-1)\omega_{m-1} \int_{1/2}^1 (1-t^2)^{\frac{m-3}{2}} dt - \omega_{m-1} \left(\frac{\sqrt{3}}{2}\right)^{m-1}}{\left(2\omega_{m-1} \int_{1/2}^1 (1-t^2)^{\frac{m-1}{2}} dt\right)^{\frac{m-1}{m}}} \quad (6.1)$$

With the substitution $t = \cos \theta$, we may rewrite the right-hand side of (6.1) as $\omega_{m-1}^{1/m} \frac{(m-1)P_{m-2} - (\frac{\sqrt{3}}{2})^{m-1}}{P_m^{(m-1)/m}}$ where here and in the sequel we let

$$P_m := 2 \int_0^{\pi/3} \sin^m \theta d\theta. \quad (6.2)$$

Moreover, integrating by parts shows that we have the recursion $mP_m = (m-1)P_{m-2} - (\frac{\sqrt{3}}{2})^{m-1}$ and therefore

$$\Lambda_{\text{plane}}(m) = m\omega_{m-1}^{1/m} P_m^{1/m}. \quad (6.3)$$

Next, in [8] (see also the proof of the validity of (1.6) for $C_{3,3} \times \mathbb{R}$ from Theorem 1.3 below) we constructed a competitor whose normalized energy (i.e. $D_m = M(k, k)$ for $2k+2 = m$ with $M(k, l)$ as in (6.37)) is

$$D_m := \omega_{m/2}^{2/m} \frac{\frac{m^2}{2} r \int_1^{r-h} u^{(m-2)/2} (r^2 - (u+h)^2)^{(m-4)/4} du - \frac{m^2 \sqrt{2}}{4(m-1)}}{\left(1 + m \int_1^{r-h} u^{(m-2)/2} (r^2 - (u+h)^2)^{m/4} du\right)^{(m-1)/m}}.$$

Here

$$h := 1 + \sqrt{3}, \quad r := \sqrt{6} + \sqrt{2} = \sqrt{2} h.$$

We simplify this expression with the change of variables $u = hx$ to obtain

$$D_m = \omega_{m/2}^{2/m} \left(\frac{m^2 \sqrt{2}}{2}\right) \frac{h^{m-1} \int_a^b x^{(m-2)/2} Q(x)^{(m-4)/4} dx - \frac{1}{2(m-1)}}{\left(1 + mh^m \int_a^b x^{(m-2)/2} Q(x)^{m/4} dx\right)^{(m-1)/m}}$$

where we have introduced the shorthand notation

$$a := \frac{1}{h} = \frac{\sqrt{3}-1}{2}, \quad b := \sqrt{2}-1, \quad Q(x) := 1 - 2x - x^2. \quad (6.4)$$

As we did for $\Lambda_{\text{plane}}(m)$, we rewrite the numerator as the integral of a positive function. To this end, define the auxiliary functions

$$W_m(x) := x^{(m-2)/2} Q(x)^{(m-4)/4}, \quad G_m(x) := 1 - \frac{mh}{4(m-1)} (3x + 2x^2 - 1). \quad (6.5)$$

Direct computation verifies that $(x^{m/2} Q(x)^{m/4})' = \frac{m}{2} W_m(x)(1 - 3x - 2x^2)$, and $Q(a) = a^2$, $Q(b) = 0$, so that

$$a^m = \frac{m}{2} \int_a^b W_m(x)(3x + 2x^2 - 1) dx.$$

Recalling that $a = 1/h$, the numerator of D_m becomes

$$M_m := h^{m-1} \int_a^b W_m(x) G_m(x) dx, \quad (6.6)$$

Finally, we introduce the shorthand

$$L_m := h^m \int_a^b x^{(m-2)/2} Q(x)^{m/4} dx \quad (6.7)$$

for the integral in the denominator of D_m . With this notation, D_m takes the consolidated form

$$D_m = \omega_{m/2}^{2/m} \left(\frac{m^2 \sqrt{2}}{2} \right) \frac{M_m}{(1 + mL_m)^{(m-1)/m}}. \quad (6.8)$$

Our goal is to show that $D_m < \Lambda_{\text{plane}}(m)$ in every even dimension m . Recall that in [8], we verified this computationally up to $m = n + 1 = 2700$. Here we prove:

Theorem 6.1. *For every even $m \geq 2000$, $\Lambda_{\text{plane}}(m) > D_m$.*

Since $2001 < 2700$, the validity of $\Lambda_{C_{k,k}} < \Lambda_{\text{plane}}(n + 1)$ for all even $n + 1 = 2k + 2$ follows immediately from Theorem 6.1 and the main results of [8].

The desired inequality is equivalent to $\log(\Lambda_{\text{plane}}(m)/D_m) > 0$. We begin with formal asymptotics of this quantity as $m \rightarrow \infty$. From (6.3) and (6.8), we have

$$\log \frac{\Lambda_{\text{plane}}(m)}{D_m} = -\log m + \frac{1}{m} \log \frac{\omega_{m-1}}{\omega_{m/2}^2} + \frac{1}{m} \log P_m - \log M_m + \frac{m-1}{m} \log(1 + mL_m) - \log \frac{\sqrt{2}}{2}. \quad (6.9)$$

Recall that P_m, M_m , and L_m were defined in (6.2), (6.6), and (6.7) respectively. Stirling's formula shows that the first term on the right-hand side of (6.9) behaves as

$$\frac{1}{m} \log \frac{\omega_{m-1}}{\omega_{m/2}^2} = -\log \sqrt{2} + \frac{\log m}{m} - \frac{1}{m} \log(2\sqrt{2}) + O(m^{-2}). \quad (6.10)$$

Moreover, using Laplace's method (in the endpoint case; see Chapter 4.3 of Asymptotic Expansions of Integrals by de Bruijn), one shows that the functions P_m, M_m , and L_m have asymptotics

$$P_m = \frac{2\sqrt{3}}{m} \left(\frac{\sqrt{3}}{2} \right)^m (1 + O(m^{-1})), \quad (6.11)$$

$$M_m = \frac{3}{2m} \left[1 - \frac{7 - \frac{4\sqrt{3}}{3}}{m} + O(m^{-2}) \right], \quad (6.12)$$

$$1 + mL_m = \sqrt{3} \left[1 - \frac{6 - \frac{4}{\sqrt{3}}}{m} + O(m^{-2}) \right]. \quad (6.13)$$

Below, we will verify these estimates and obtain quantitative errors on the remainders, in order to obtain the explicit threshold on m given by Theorem 6.1 above which we have $\Lambda_{\text{plane}}(m) > D_m$. Let us first see that the asymptotics (6.10)-(6.13) yield $\Lambda_{\text{plane}}(m) > D_m$ for a sufficiently large (not quantified) dimension m . Substituting the asymptotics into the right-hand side of (6.9), we see that the terms with coefficient $\frac{\log m}{m}$ cancel (see the proof of Theorem 6.1 below for more details) and the leading contribution is of order $\frac{1}{m}$:

$$\log \frac{\Lambda_{\text{plane}}(m)}{D_m} = \frac{1 - \log \sqrt{2}}{m} + O(m^{-2}).$$

As this coefficient is positive, the desired inequality holds asymptotically as $m \rightarrow \infty$. The asymptotics above guide the estimates proven in the following lemma, which will allow us to show the estimate holds for m above an explicit threshold.

Lemma 6.2. *The following estimates hold:*

$$\frac{1}{m} \log \frac{\omega_{m-1}}{\omega_{m/2}^2} \geq -\log \sqrt{2} + \frac{\log m}{m} - \frac{\log(2\sqrt{2})}{m} \quad \text{for all } m \geq 1, \quad (6.14)$$

$$P_m \geq \left(\frac{\sqrt{3}}{2} \right)^m \frac{2\sqrt{3}}{m} \left(1 - \frac{13}{m} \right) \quad \text{for all } m \geq 1, \quad (6.15)$$

$$M_m \leq \frac{3}{2m} \left(1 - \frac{4.1}{m} \right) \quad \text{for all } m \geq 2000, \quad (6.16)$$

$$1 + mL_m \geq \sqrt{3} \left[1 - \frac{6.03 - \frac{4}{\sqrt{3}}}{m} \right] \quad \text{for all } m \geq 2000. \quad (6.17)$$

Before proving the lemma, let us see how it yields Theorem 6.1.

Proof of Theorem 6.1. Let $m \geq 2000$. Starting from (6.9) and substituting (6.14), we find

$$\log \frac{\Lambda_{\text{plane}}(m)}{D_m} \geq -\log m + \frac{\log m}{m} - \frac{\log(2\sqrt{2})}{m} + \frac{1}{m} \log P_m - \log M_m + \frac{m-1}{m} \log(1 + mL_m). \quad (6.18)$$

Now, we take the logarithm of (6.15). Noting that for $m \geq 26$ (i.e. $13/m \leq 1/2$) we have $\log(1 - \frac{13}{m}) \geq -26/m$ since $\log(1 - u) \geq -2u$ for $0 \leq u \leq 1/2$, we arrive at the bound

$$\frac{1}{m} \log P_m \geq \log \frac{\sqrt{3}}{2} - \frac{\log m}{m} + \frac{\log(2\sqrt{3})}{m} - \frac{26}{m^2}.$$

Similarly, we take the logarithm of (6.16) and use concavity of the logarithm, which in particular gives $\log(1 - x) \leq -x$, to obtain

$$-\log M_m \geq \log m - \log \frac{3}{2} + \frac{4.1}{m}.$$

Summing these terms and combining with (6.18) yields

$$\log \frac{\Lambda_{\text{plane}}(m)}{D_m} \geq -\log \sqrt{3} + \frac{4.1 + \log(\sqrt{3/2})}{m} + \frac{m-1}{m} \log(1 + mL_m) - \frac{26}{m^2}. \quad (6.19)$$

Finally, taking the logarithm of (6.17) and noticing that we have the bounds

$$\begin{cases} \log(1 - t) \geq -t - t^2 & \text{for } t \in (0, 1/2); \\ \frac{6.03 - \frac{4}{\sqrt{3}}}{m} \leq \frac{1}{2} & \text{for } m \geq 8; \\ (6.03 - \frac{4}{\sqrt{3}})^2 < 14, \end{cases}$$

we obtain

$$\begin{aligned} \frac{m-1}{m} \log(1 + mL_m) &\geq \log \sqrt{3} - \frac{1}{m} \log \sqrt{3} + \log \left(1 - \frac{6.03 - \frac{4}{\sqrt{3}}}{m} \right) \\ &\geq \log \sqrt{3} + \frac{1}{m} \left(-6.03 + \frac{4}{\sqrt{3}} - \log \sqrt{3} \right) - \frac{14}{m^2}. \end{aligned}$$

Combining this with (6.19) yields

$$\log \frac{\Lambda_{\text{plane}}(m)}{D_m} \geq \frac{1}{m} \left[4.1 - \log(\sqrt{2}) + \frac{4}{\sqrt{3}} - 6.03 \right] - \frac{40}{m^2}. \quad (6.20)$$

The coefficient $C_0 := 4.1 - \log(\sqrt{2}) + \frac{4}{\sqrt{3}} - 6.03$ of $1/m$ is positive, indeed an explicit calculation yields

$$4.1 - 6.03 + \frac{4}{\sqrt{3}} - \log \sqrt{2} > 0.032.$$

Thus

$$\log \frac{\Lambda_{\text{plane}}(m)}{D_m} \geq \frac{C_0}{m} - \frac{40}{m^2} > \frac{1}{m} \left(0.032 - \frac{40}{m} \right).$$

When $m \geq 1251$, we have $0.032 - \frac{40}{m} > 0$, thus completing the proof. $\square$

Now we prove the lemma.

Proof of Lemma 6.2. Proof of (6.14): Recall that $\omega_m = \frac{\pi^{m/2}}{\Gamma(1+m/2)}$. Using this and the identity $\Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma(z) \Gamma(z + \frac{1}{2})$ with $z = (m+1)/4$, we have

$$\frac{\omega_{m-1}}{\omega_{m/2}^2} = \frac{1}{2^{(m-1)/2}} \frac{\Gamma(\frac{m}{4} + 1)^2}{\Gamma(\frac{m}{4} + \frac{1}{4}) \Gamma(\frac{m}{4} + \frac{3}{4})}.$$

Next, applying Gautschi's inequality, which states that $\frac{\Gamma(x+1)}{\Gamma(x+s)} \geq x^{1-s}$ for all $x > 0$ and $0 < s < 1$, with the choices $x = m/4$ and $s = 1/4, 3/4$ yields

$$\frac{\Gamma(\frac{m}{4} + 1)^2}{\Gamma(\frac{m}{4} + \frac{1}{4}) \Gamma(\frac{m}{4} + \frac{3}{4})} \geq \frac{m}{4}.$$

Therefore $\frac{\omega_{m-1}}{\omega_{m/2}^2} \geq \frac{m}{2^{(m+3)/2}}$. Taking logarithms and dividing by m gives (6.14).

Proof of (6.15): Since the integrand of P_m is maximized at the right endpoint $\pi/3$ where $\sin(\pi/3) = \sqrt{3}/2$, it is convenient to change variables $x = \pi/3 - \theta$ and factor out this contribution, expressing P_m as

$$P_m = 2 \left(\frac{\sqrt{3}}{2} \right)^m \int_0^{\pi/3} \left(\frac{2}{\sqrt{3}} \sin \left(\frac{\pi}{3} - x \right) \right)^m dx = 2 \left(\frac{\sqrt{3}}{2} \right)^m \int_0^{\pi/3} \exp\{m\phi(x)\} dx \quad (6.21)$$

where

$$\phi(x) := \log \frac{2 \sin(\pi/3 - x)}{\sqrt{3}}.$$

With the aim of bounding the integral below, we note that $\phi'(0) = -\frac{1}{\sqrt{3}}$ and $\phi''(x) = -\csc^2(\pi/3 - x)$, so in particular on the interval $[0, \pi/6]$ we have $\phi''(x) \geq -4$ and thus $\phi(x) \geq -\frac{x}{\sqrt{3}} - 2x^2$ by Taylor's Theorem. Thus

$$\int_0^{\pi/3} \exp\{m\phi(x)\} dx \geq \int_0^{\pi/6} \exp \left\{ -\frac{mx}{\sqrt{3}} - 2mx^2 \right\} dx.$$

Now we make the change of variables $y = mx$ and further truncate the interval of integration: noting that $m\pi/6 \geq \sqrt{3} \log m$ for all $m \geq 6$, and recalling that $e^{-z} \geq 1 - z$, we have

$$\begin{aligned} \int_0^{\pi/6} \exp \left\{ -\frac{mx}{\sqrt{3}} - 2mx^2 \right\} dx &\geq \frac{1}{m} \int_0^{\sqrt{3} \log m} \exp \left\{ -\frac{y}{\sqrt{3}} - \frac{2y^2}{m} \right\} dy \\ &\geq \frac{1}{m} \int_0^{\sqrt{3} \log m} \exp \left\{ -\frac{y}{\sqrt{3}} \right\} \left(1 - \frac{2y^2}{m} \right) dy \\ &\geq \frac{1}{m} \int_0^{\sqrt{3} \log m} \exp \left\{ -\frac{y}{\sqrt{3}} \right\} dy - \frac{2}{m^2} \int_0^\infty \exp \left\{ -\frac{y}{\sqrt{3}} \right\} y^2 dy \\ &= \sqrt{3} \left(\frac{1}{m} - \frac{1}{m^2} \right) - \frac{12\sqrt{3}}{m^2}. \end{aligned}$$

In conclusion, we obtain

$$\int_0^{\pi/3} \exp\{m\phi(x)\} dx \geq \frac{\sqrt{3}}{m} \left(1 - \frac{13}{m} \right) \quad (6.22)$$

for $m \geq 6$. On the other hand, for $1 \leq m \leq 5$, $1 - \frac{13}{m}$ is negative, so (6.22) holds for all integers $m \geq 1$. Together with (6.21), this completes the proof.

Proof of (6.16): We proceed in several steps.

Step 1: Define the auxiliary functions

$$H_m(x) := xG_m(x), \quad \Phi(x) := \frac{1}{2} \log x + \frac{1}{4} \log Q(x), \quad (6.23)$$

with G_m as in (6.5) and Q as in (6.4). Then, observing that $W_m(x) = x e^{(m-4)\Phi(x)}$, we can write

$$M_m = h^{m-1} \int_a^b H_m(x) e^{(m-4)\Phi(x)} dx,$$

where we recall that $a = \frac{\sqrt{3}-1}{2} = \frac{1}{h}$ and $b = \sqrt{2} - 1$, as in (6.4). Since we additionally have $e^{\Phi(a)} = \frac{1}{h}$ and $-\Phi'(a) = \frac{h^2}{2} = 2 + \sqrt{3}$, we change variables $x = a + t$ and write

$$M_m = h^3 \int_0^{b-a} H_m(a+t) e^{(m-4)\Phi'(a)t} e^{(m-4)R(t)} dt,$$

where

$$R(t) := \Phi(a+t) - \Phi(a) - \Phi'(a)t$$

denotes the remainder of $\Phi(a+t)$ from its linear approximation at $\Phi(a)$. We record for later use that

$$H_m(a) = a \frac{3m-4}{4(m-1)}, \quad H'_m(a) = -\frac{1+am}{m-1}, \quad m \geq 2, \quad (6.24)$$

$$\Phi''(a) = -8 \left(4\sqrt{3} + 7 \right) \in [-112, -111]. \quad (6.25)$$

Letting $c := -\Phi'(a)$ (which is positive), our main claim is that

$$M_m \leq h^3 \left[\frac{H_m(a)}{(m-4)c} + \frac{1}{(m-4)^2} \left(\frac{H'_m(a)}{c^2} + \frac{\Phi''(a)H_m(a)}{c^3} \right) \right] + \frac{2h+655}{(m-4)^3} \quad (6.26)$$

provided that $m \geq 2000$.

Step 2: We split M_m into pieces and bound each piece separately. To begin, let

$$Y := \frac{2 \log(m-4)}{c}.$$

and split $M_m = M_m^{\text{loc}} + M_m^{\text{tail}}$ where

$$M_m^{\text{tail}} := h^3 \int_{Y/(m-4)}^{b-a} H_m(a+t) e^{-(m-4)ct} e^{(m-4)R(t)} dt.$$

Note that the lower endpoint $\frac{Y}{m-4}$ of the integral is less than $b-a$ provided that $m \geq 46$. Otherwise, we simply set $M_m^{\text{tail}} = 0$.

Now, observe that Φ is concave so $R(t) \leq 0$, and $|H_m| \leq 1$ on $[a, b]$. So, since $c = \frac{h^2}{2}$,

$$M_m^{\text{tail}} \leq h^3 \int_{Y/(m-4)}^{\infty} e^{-(m-4)ct} dt = \frac{h^3}{(m-4)c} e^{-cY} = \frac{h^3}{(m-4)c} (m-4)^{-2} = \frac{2h}{(m-4)^3}. \quad (6.27)$$

Step 3: Toward estimating M_m^{loc} , first note that $H_m(x) = x - \frac{mh}{4(m-1)}(3x^2 + 2x^3 - x)$ is concave on $[a, b]$ since $H_m''(x) = -\frac{mh}{4(m-1)}(6 + 12x)$, and hence

$$H_m(a+t) \leq H_m(a) + H'_m(a)t.$$

As a consequence, letting $s = (m-4)t$,

$$\begin{aligned} M_m^{\text{loc}} &\leq h^3 \int_0^{Y/(m-4)} \{H_m(a) + H'_m(a)t\} e^{-(m-4)ct} e^{(m-4)R(t)} dt \\ &= \frac{h^3}{m-4} \int_0^Y \left\{ H_m(a) + H'_m(a) \frac{s}{m-4} \right\} e^{-cs} e^{(m-4)R(s/(m-4))} ds \end{aligned} \quad (6.28)$$

We examine the error term $R(t)$. First of all, for $m \geq 707$, we have $Y/(m-4) \leq 0.005$. Direct computation verifies that

$$\Phi'''(x) = \frac{1}{x^3} - \frac{(x+1)(x^2+2x+7)}{(1-2x-x^2)^3}$$

is a negative and decreasing function on $[a, a+0.005]$ and thus $|\Phi'''| \leq |\Phi'''(a+0.005)| \leq 6200$. As a consequence, since $6200/6 \leq 1100$, Taylor's Theorem gives

$$R(t) = \frac{\Phi''(a)}{2} t^2 + \rho_{\Phi}(t), \quad |\rho_{\Phi}(t)| \leq \max_{s \in [0, 0.005]} |\Phi'''(s)| \frac{t^3}{6} \leq 1100 t^3$$

for $t \in [0, 0.005]$. In particular, recalling the bound (6.25), for $s \in [0, Y]$ we have

$$|(m-4)R(s/(m-4))| \leq \frac{|\Phi''(a)|}{2} \frac{(2 \log(m-4)/c)^2}{m-4} + 1100 \frac{(2 \log(m-4)/c)^3}{(m-4)^2} \leq \frac{1}{2},$$

when $m \geq 2000$. Consequently, since another application of Taylor's Theorem yields $e^t \leq 1 + t + t^2$ for $|t| < 1/2$, we have

$$e^{(m-4)R(s/(m-4))} \leq 1 + \frac{\Phi''(a)}{2} \frac{s^2}{m-4} + 1100 \frac{s^3}{(m-2)^2} + \left(\frac{|\Phi''(a)|}{2} \frac{s^2}{m-4} + 1100 \frac{s^3}{(m-4)^2} \right)^2.$$

Inserting this bound into (6.28) and using that $H_m(a) + H'_m(a)t > 0$ on $[a, a+0.005]$ for all $m \geq 6$, this yields

$$M_m^{\text{loc}} \leq M_m^{\text{main}} + M_m^{\text{err}},$$

where

$$M_m^{\text{main}} = \frac{h^3}{m-4} \left\{ H_m(a) \int_0^Y e^{-cs} ds + \frac{H'_m(a)}{m-4} \int_0^Y e^{-cs} s ds + \frac{H_m(a)\Phi''(a)}{2(m-4)} \int_0^Y e^{-cs} s^2 ds \right\} \quad (6.29)$$

and

$$M_m^{\text{err}} = \frac{h^3}{(m-4)^3} \frac{|H'_m(a)||\Phi''(a)|}{2} \int_0^\infty e^{-cs} s^3 ds + \frac{h^3}{m-4} H_m(a) \int_0^\infty e^{-cs} \left(\frac{1}{2} |\Phi''(a)| \frac{s^2}{m-4} + 1100 \frac{s^3}{(m-4)^2} \right)^2 ds. \quad (6.30)$$

Here, we have used in addition that $H'_m(a) < 0$, so in particular

$$\frac{h^3}{(m-4)^2} H'_m(a) \int_0^\infty e^{-cs} s \left(\frac{1}{2} |\Phi''(a)| \frac{s^2}{m-4} + 1100 \frac{s^3}{(m-4)^2} \right)^2 ds < 0.$$

Step 4: Let us first bound (6.30). Using

$$\int_0^\infty y^k e^{-cy} dy = \frac{k!}{c^{k+1}}, \quad (6.31)$$

and expanding the square $(\frac{1}{2} |\Phi''(a)| \frac{s^2}{m-4} + 1100 \frac{s^3}{(m-4)^2})^2 = \frac{|\Phi''(a)|^2}{4(m-4)^2} s^4 + \frac{(1100)^2}{(m-4)^4} s^6 + 1100 \frac{|\Phi''(a)|}{(m-4)^3} s^5$, we get

$$M_m^{\text{err}} \leq \frac{h^3}{m-4} \left[\frac{|H'_m(a)||\Phi''(a)|}{2(m-4)^2} \frac{6}{c^4} + H_m(a) \left(\frac{|\Phi''(a)|^2}{4(m-4)^2} \frac{24}{c^5} + \frac{(1100)^2}{(m-4)^4} \frac{720}{c^7} + \frac{1100|\Phi''(a)|}{(m-4)^3} \frac{120}{c^6} \right) \right].$$

Recalling (6.24) and (6.25), in particular for $m \geq 4$ we have the crude estimates

$$0 < H_m(a) \leq \frac{3a}{4}, \quad |H'_m(a)| \leq 1, \quad -\Phi''(a) \leq 112, \quad (6.32)$$

which in turn give

$$M_m^{\text{err}} \leq \frac{h^3}{m-4} \left\{ \frac{112}{2(m-4)^2} \frac{6}{c^4} + \frac{3a}{4} \left\{ \frac{(112)^2}{4(m-4)^2} \frac{24}{c^5} + \frac{(1100)^2}{(m-4)^4} \frac{720}{c^7} + 112 \cdot \frac{1100}{(m-4)^3} \frac{120}{c^6} \right\} \right\} \leq \frac{650}{(m-4)^3} \quad (6.33)$$

for $m \geq 2000$.

Step 5: We may next evaluate M_m^{main} directly. Keeping in mind that

$$\begin{aligned} \int_0^Y e^{-cy} dy &= \frac{1 - e^{-cY}}{c} \leq \frac{1}{c}, \\ \int_0^Y y e^{-cy} dy &= \frac{1}{c^2} - e^{-cY} \left(\frac{Y}{c} + \frac{1}{c^2} \right), \\ \int_0^Y y^2 e^{-cy} dy &= \frac{2}{c^3} - e^{-cY} \left(\frac{Y^2}{c} + \frac{2Y}{c^2} + \frac{2}{c^3} \right), \end{aligned} \quad (6.34)$$

we arrive at

$$\begin{aligned} M_m^{\text{main}} &\leq h^3 \left[\frac{H_m(a)}{(m-4)c} + \frac{1}{(m-4)^2} \left(\frac{H'_m(a)}{c^2} + \frac{H_m(a)\Phi''(a)}{c^3} \right) \right] \\ &\quad + \frac{h^3 e^{-cY}}{(m-4)^2} \left[-H'_m(a) \left(\frac{Y}{c} + \frac{1}{c^2} \right) - \frac{1}{2} H_m(a) \Phi''(a) \left(\frac{Y^2}{c} + \frac{2Y}{c^2} + \frac{2}{c^3} \right) \right]. \end{aligned} \quad (6.35)$$

Note that $H_m(a) > 0$, while $H'_m(a) < 0$ and $\Phi''(a) < 0$, so each term in (6.35) is positive. Recall that $Y = \frac{2 \log(m-4)}{c}$, so $e^{-cY} = (m-4)^{-2}$. Again using the crude bounds in (6.32), the term in (6.35) is bounded above by $C_0 \cdot (m-2)^{-4}$ for

$$C_0 := h^3 \left[\left(\frac{Y}{c} + \frac{1}{c^2} \right) + 112 \cdot \frac{3a}{8} \left(\frac{Y^2}{c} + \frac{2Y}{c^2} + \frac{2}{c^3} \right) \right] \leq 5(m-4)$$

with the final inequality holding for $m-4 \geq 156$.

Hence, for $m \geq 200$, recalling that $c = -\Phi'(a) = 2 + \sqrt{3}$, we obtain

$$M_m^{\text{main}} \leq h^3 \left[\frac{H_m(a)}{(m-4)c} + \frac{1}{(m-4)^2} \left(\frac{H'_m(a)}{c^2} + \frac{H_m(a)\Phi''(a)}{c^3} \right) \right] + \frac{5}{(m-4)^3}. \quad (6.36)$$

Combining this with (6.27) and (6.33) gives (6.26).

Step 6: Finally, let us see how the estimate (6.26) implies the desired bound (6.16) on M_m . Note that for any $m \geq 5$

$$A := h^3 \left[\frac{H_m(a)}{(m-4)c} + \frac{1}{(m-4)^2} \left(\frac{H'_m(a)}{c^2} + \frac{H_m(a)\Phi''(a)}{c^3} \right) \right] = \frac{H_m(a)h^3}{(m-4)c} \left[1 - \frac{1}{m-4} \left(\frac{4(ma+1)}{a(3m-4)c} + \frac{-\Phi''(a)}{c^2} \right) \right]$$

$$\leq \frac{H_m(a)h^3}{(m-4)c} \left[1 - \frac{1}{m} \left(\frac{4(ma+1)}{(3m-4)c} + \frac{-\Phi''(a)}{c^2} \right) \right].$$

In the final inequality we used $a \leq 1$. Recalling that $c = \frac{h^2}{2}$, we may then directly compute

$$\frac{H_m(a)h^3}{(m-4)c} = \frac{2hH_m(a)}{m} \left(\frac{m}{m-4} \right) = \frac{3}{2m} \left(1 + \frac{4}{m-4} \right) \left(1 - \frac{1}{3(m-1)} \right) \leq \frac{3}{2m} \left(1 + \frac{4}{m-4} - \frac{1}{3(m-1)} \right),$$

so that, after first using this to crudely bound

$$-\frac{H_m(a)h^3}{(m-4)c} \cdot \frac{1}{m} \left(\frac{4(ma+1)}{(3m-4)c} - \frac{\Phi''(a)}{c^2} \right) \leq -\frac{3}{2m} \cdot \frac{1}{m} \left(\frac{1}{c} \frac{4(ma+1)}{3m-4} - \frac{\Phi''(a)}{c^2} \right),$$

which always holds for $m \geq 5$, we obtain

$$\begin{aligned} A &\leq \frac{3}{2m} \left[1 - \frac{1}{m} \left(\frac{4(ma+1)}{(3m-4)c} - \frac{\Phi''(a)}{c^2} \right) + \frac{4}{m-4} - \frac{1}{3(m-1)} \right] \\ &= \frac{3}{2m} \left[1 - \frac{1}{m} \left(\frac{4(ma+1)}{(3m-4)c} - \frac{\Phi''(a)}{c^2} - 4\frac{m}{m-4} + \frac{m}{3(m-1)} \right) \right]. \end{aligned}$$

Recalling the additional error term from (6.26), we observe that in order to conclude (6.16), it remains to check that

$$\left(\frac{4(ma+1)}{(3m-4)c} - \frac{\Phi''(a)}{c^2} - 4\frac{m}{m-4} + \frac{m}{3(m-1)} \right) - \frac{2m^2}{3} \cdot \frac{2h+655}{(m-4)^3} \geq 4.1$$

for large enough m . This indeed holds for $m \geq 1300$.

Proof of (6.17):

We use the same notation of the proof of (6.16). Letting Φ be as in (6.23), we note that

$$L_m = h^m \int_a^b e^{m\Phi(x)} x^{-1} dx,$$

Recall that $-\Phi'(a) = \frac{h^2}{2} = 2 + \sqrt{3}$, so using the fact that $e^{\Phi(a)} = \frac{1}{h}$, the change of variables $x = a + t$ gives

$$L_m = \int_0^{b-a} e^{m\Phi'(a)t} e^{mR(t)} (a+t)^{-1} dt,$$

where

$$R(t) := \Phi(a+t) - \Phi(a) - \Phi'(a)t.$$

As before, let $c := -\Phi'(a)$, and let

$$Z := \frac{2 \log m}{c}.$$

Since e^t and $(a+t)^{-1}$ are convex and $a^{-1} - a^{-2}t \geq 0$ for $t \in [0, Z/m]$ and any $m \geq 1$, we have

$$L_m \geq \int_0^{Z/m} e^{-mct} (1 + mR(t)) [a^{-1} - a^{-2}t] dt.$$

As in Step 3, for $m \geq 703$, we have $Z/m \leq 0.005$, and for $t \in [0, 0.005]$ we have

$$R(t) \geq \frac{1}{2} \Phi''(a)t^2 - 1100t^3.$$

Combining this with the change of variables $t = \frac{s}{m}$, and recalling that $a^{-1} = h$ and $\Phi''(a) < 0$, we obtain

$$\begin{aligned} L_m &\geq \frac{1}{m} \int_0^Z e^{-cs} (1 + mR(s/m)) \left(h - h^2 \frac{s}{m} \right) ds \\ &\geq \frac{1}{m} \int_0^Z e^{-cs} \left(1 + \frac{\Phi''(a)}{2} \frac{s^2}{m} - \frac{1100s^3}{m^2} \right) \left(h - h^2 \frac{s}{m} \right) ds \\ &\geq \frac{h}{m} \int_0^Z e^{-cs} ds + \frac{h}{m^2} \frac{\Phi''(a)}{2} \int_0^Z e^{-cs} s^2 ds - \frac{h^2}{m^2} \int_0^Z e^{-cs} s ds - \frac{1100h}{m^3} \int_0^Z e^{-cs} s^3 ds \\ &\geq \frac{h}{m} \int_0^Z e^{-cs} ds + \frac{h}{m^2} \frac{\Phi''(a)}{2} \int_0^\infty e^{-cs} s^2 ds - \frac{h^2}{m^2} \int_0^\infty e^{-cs} s ds - \frac{1100h}{m^3} \int_0^\infty e^{-cs} s^3 ds. \end{aligned}$$

We can evaluate each of these four integrals explicitly. Keeping in mind (6.34) and recalling that the Gaussian moments satisfy (6.31), we have

$$L_m \geq \frac{h}{mc} - \frac{he^{-cZ}}{mc} + \frac{1}{m^2} \left(\frac{\Phi''(a)h}{c^3} - \frac{h^2}{c^2} \right) - \frac{6600h}{c^4 m^3}.$$

Since $\frac{h}{c} = \frac{2}{h} = \sqrt{3} - 1$ and $e^{-cZ} \leq 1$, multiplying by m and adding 1 yields

$$\begin{aligned} 1 + mL_m &\geq \sqrt{3} + \frac{1}{m} \left(\frac{\Phi''(a)h}{c^3} - \frac{h^2}{c^2} \right) - \frac{1}{m^2} \left(\frac{6600h}{c^4} + \frac{h}{c} \right) \\ &\geq \sqrt{3} \left(1 - \frac{1}{m} \left[\frac{1}{\sqrt{3}} \left(\frac{(-\Phi''(a))h}{c^3} + \frac{h^2}{c^2} \right) + \frac{1}{\sqrt{3}m} \left(\frac{6600h}{c^4} + \frac{h}{c} \right) \right] \right). \end{aligned}$$

Recalling that $\Phi''(a) = -56 - 32\sqrt{3}$, as well as the values of c , a and h , a direct calculation gives

$$\left[\frac{1}{\sqrt{3}} \left(\frac{(-\Phi''(a))h}{c^3} + \frac{h^2}{c^2} \right) + \frac{1}{\sqrt{3}m} \left(\frac{6600h}{c^4} + \frac{h}{c} \right) \right] \leq 6.03 - \frac{4}{\sqrt{3}}$$

provided $m \geq 2000$. This completes the proof. $\square$

6.2. Proof that $C_{3,3} \times \mathbb{R}$ satisfies (1.6). We will proceed to construct a competitor produced from that constructed in [8] for $C_{3,3}$, extended suitably in the transverse direction to $C_{3,3}$ (i.e. the direction of the spine of the cone). In order to do this, let us first recall the procedure used to construct the $(SO(4) \times SO(4))$ -invariant competitor E for the set $K = \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x|^2 = |y|^2\}$ whose boundary is $C_{3,3}$ (see [8, Section 7.2]).

In this case, we define the “slice” $\hat{E}$ of E in a two-dimensional quadrant $Q = \{(u, v) : u > 0, v > 0\}$, with $u = |(x_1, \dots, x_4)|$ and $v = |(x_5, \dots, x_8)|$. Then

$$\partial K \cap Q = \{(u, v) : u > 0, v > 0, u = v\},$$

In Q , define $\hat{E} \cap \{u < v\}$ to be the circular arc of radius $r > 0$ and center $(0, -h)$ forming angle $\pi/3$ with the line $\{u = v\}$ and angle $\pi/2$ with the v -axis $\{u = 0\}$. Such an arc can be parameterized as the graph over the v -axis of the function $f : [1, -h + r] \rightarrow [0, 1]$ given by $f(v) = \sqrt{r^2 - (v + h)^2}$. Let $\theta = \frac{\pi}{4}$ denote the angle that the line $\{u = v\}$ makes with the v -axis $\{u = 0\}$. Since $f(1) = 1 = \sqrt{r^2 - (1 + h)^2}$ and $f'(1) = -\tan(2\pi/3 - \pi/4) = -(1 + h)$, this yields

$$h = \tan(2\pi/3 - \pi/4) - 1, \quad r = \sec(2\pi/3 - \pi/4).$$

The set $\hat{E} \cap \{u > v\}$ is defined analogously. Namely, we can also parameterize the arc of a circle of radius $\rho > 0$ and center $(-h, 0)$ that forms angle $\pi/3$ with the line $\{u = v\}$ and angle $\pi/2$ with the u -axis $\{v = 0\}$, via the function $u \mapsto f(u)$ over $[1, -h + r]$.

By the coarea formula, we then obtain

$$\begin{aligned} |E| &= 16\omega_4^2 \int_{\hat{E}} u^3 v^3 du dv = \omega_4^2 \left(1 + 8 \int_1^{\rho-d} v^3 (r^2 - (v + h)^2)^2 dv \right), \\ \mathcal{H}^7(\partial E) &= 16\omega_4^2 \int_{\partial \hat{E}} u^3 v^3 d\mathcal{H}^1 = 32\omega_4^2 r \int_1^{r-h} v^3 (r^2 - (v + h)^2) dv. \end{aligned}$$

Letting $\mathcal{X}$ be the $(1, 2)$ -cluster with $\mathcal{X}(1) = E/|E|^{1/8}$, so that it has the same volume as $\mathcal{X}_{\text{lens}}(1)$, with $\mathcal{X}(2) = K \setminus \mathcal{X}(1)$ and $\mathcal{X}(3) = (\mathbb{R}^8 \setminus K) \setminus \mathcal{X}(1)$. For this cluster, we then consider the perimeter cost $M(3, 3) = P(\mathcal{X}(1)) - \mathcal{H}^7(C_{3,3} \cap \mathcal{X}(1))$ of gluing $\mathcal{X}(1)$ into $C_{3,3}$, which can be expressed as

$$M(3, 3) = \frac{1}{|E|^{\frac{7}{8}}} \left(\mathcal{H}^7(\partial E) - \frac{16\omega_4^2 \sqrt{2}}{7} \right). \quad (6.37)$$

We refer the reader to [8] for more details, noting that here we have $\lambda = 1$, since $k = l = 3$.

Let us now repeat this for the cone $C_{3,3} \times \mathbb{R}$, via an extension of the above competitor E slice-wise in the transverse direction of the spine. More precisely, define a set $F \subset \mathbb{R}^9$ via

$$F = \{(w, t) \in \mathbb{R}^8 \times \mathbb{R} : w \in H(t)E\},$$

where $H : [-L, L] \rightarrow (0, \infty)$ is a Lipschitz function, to be determined. By the coarea formula, this yields

$$|F| = |E| \int_{-L}^L H(t)^8 dt. \quad (6.38)$$

Meanwhile, letting $\Psi : \mathbb{R}^9 \rightarrow \mathbb{R}^9$ denote the map $\Psi(x, t) := (H(t)x, t)$, we observe that $F = \Psi(E \times [-L, L])$, and $\partial F = \Psi(\partial E \times (-L, L))$, and

$$\mathcal{H}^8(\partial F) = \int \int_{\partial E \times (-L, L)} \mathbf{J}^{\partial E \times (-L, L)} \Psi d\mathcal{H}^7 dt$$

where $\mathbf{J}^{\partial E \times (-L, L)} \Psi$ is the tangential Jacobian of Ψ over $\partial E \times (-L, L)$ (well-defined $\mathcal{H}^8$ -a.e.).

Let us first compute the differential of Ψ at any point (x, t) in the $\mathcal{H}^8$ -full measure set where the tangential Jacobian is well-defined. Since we may decompose $T_{(x, t)}(\partial E \times (-L, L))$ as $T_x \partial E \otimes T_t(-L, L) \cong T_x \partial E \otimes \mathbb{R}$, for any $\tau \in T_x \partial E$, we have

$$d\Psi_{x, t}(\tau, 0) = (H(t)\tau, 0) \quad \text{and} \quad d\Psi_{x, t}(0, 1) = (H'(t)x, 1).$$

To simplify computations, we observe that one may compute $\mathbf{J}^{\partial E \times (-L, L)} \Psi$ as the volume of the 8-dimensional parallelepiped spanned by the vectors $\{d\Psi_{x, t}(\tau_i, 0)\}_{i=1}^n$, $d\Psi_{x, t}(0, 1)$, for an orthonormal frame $\{\tau_i\}_{i=1}^n$ of $T_{x, t} \partial E$. This is simply the product of the area of the 7-dimensional parallelepiped spanned by $\{H(t)\tau_i\}_{i=1}^7$ and the length of the projection of the vector $(H'(t)x, 1)$ onto the orthogonal complement of the former span, namely,

$$H(t)^7 \cdot |\mathbf{p}_{d\Psi_{x, t}(\cdot, 0)}^\perp(H'(t)x, 1)|,$$

where $\mathbf{p}_{d\Psi_{x, t}(\cdot, 0)}^\perp$ is the projection onto the orthogonal complement of the span of $\{H(t)\tau_i\}_{i=1}^7$. Notice that the latter orthogonal complement is spanned by the orthonormal vectors $(\nu_{\partial E}(x), 0)$ and $(0, 1)$, where $\nu_{\partial E}(x)$ is the outward unit normal to $\partial^* E$ (defined $\mathcal{H}^7$ -a.e.) we thus have

$$|\mathbf{p}_{d\Psi_{x, t}(\cdot, 0)}^\perp(H'(t)x, 1)| = \sqrt{(H'(t))^2(x \cdot \nu_{\partial E}(x))^2 + 1}.$$

We thus arrive at

$$\mathcal{H}^8(\partial F) = \int_{-L}^L H(t)^7 \int_{\partial E} \sqrt{(H'(t))^2(x \cdot \nu_{\partial E}(x))^2 + 1} d\mathcal{H}^7(x) dt.$$

The coarea formula then yields

$$\mathcal{H}^8(\partial F) = 16\omega_4^2 \int_{-L}^L H(t)^7 \int_{\partial \hat{E}} u^3 v^3 \sqrt{(H'(t))^2((u, v) \cdot \nu_{\partial \hat{E}}(u, v))^2 + 1} d\mathcal{H}_{(u, v)}^1 dt.$$

Now, recalling the parameterization of $\partial \hat{E}$ via the two circle-arcs, we have

$$\nu_{\partial \hat{E}} = \begin{cases} \frac{(-f'(v), 1)}{\sqrt{1+(f'(v))^2}} & \text{when } u = f(v), \text{ and} \\ \frac{(-f'(u), 1)}{\sqrt{1+(f'(u))^2}} & \text{when } v = f(u), \end{cases}$$

with

$$f'(v) = \frac{-(v+h)}{\sqrt{r^2 - (v+h)^2}} \quad \text{and} \quad \sqrt{1+(f'(v))^2} = \frac{r}{\sqrt{r^2 - (v+h)^2}}.$$

Thus, when $u = f(v)$ we have

$$(u, v) \cdot \nu_{\partial \hat{E}}(u, v) = (f(v), v) \cdot \nu_{\partial \hat{E}}(f(v), v) = \frac{v(v+h)}{r} + \frac{f(v)\sqrt{r^2 - (v+h)^2}}{r} = \frac{r^2 - h(v+h)}{r}.$$

Likewise,

$$(u, f(u)) \cdot \nu_{\partial \hat{E}}(u, f(u)) = \frac{r^2 - h(u+h)}{r}.$$

In conclusion,

$$\mathcal{H}^8(\partial F) = 32\omega_4^2 \int_{-L}^L H(t)^7 r \int_1^{r-h} v^3 (r^2 - (v+h)^2) \sqrt{(H'(t))^2 \frac{(r^2 - h(v+h))^2}{r^2} + 1} dv dt.$$

Just like we did for $C_{3,3}$, we may then let $\tilde{\mathcal{X}}$ be the $(1, 2)$ -cluster with $\tilde{\mathcal{X}}(1) = F/|F|^{1/9}$, with $\tilde{\mathcal{X}}(2) = (K \times \mathbb{R}) \setminus \tilde{\mathcal{X}}(1)$ and $\mathcal{X}(3) = (\mathbb{R}^9 \setminus (K \times \mathbb{R})) \setminus \tilde{\mathcal{X}}(1)$. The normalized cost of gluing this competitor into $C_{3,3} \times \mathbb{R}$ is then

$$\tilde{M}(3, 3) := \frac{1}{|F|^{\frac{8}{9}}} \left(\mathcal{H}^8(\partial F) - \frac{16\omega_4^2 \sqrt{2}}{7} \int_{-L}^L H(t)^7 dt \right).$$

Setting

$$H(t) = \left(1 - \frac{|t|}{3}\right)_+$$

with $L = 3$, via an explicit calculation using Mathematica we obtain

$$\tilde{M}(3, 3) \approx 7.79468.$$

Recalling from [8] that

$$\Lambda_{\text{plane}}(9) \approx 7.93735360,$$

we conclude that (1.6) indeed holds for $C = C_{3,3} \times \mathbb{R}$. $\square$

Remark 6.3. We note that there is some flexibility with the choice of length scale $L = 3$ for the above choice of H . Moreover, note that one can perform an analogous computation to the one above for general cylindrical cones over quadratic cones, i.e. $C_{k,l} \times \mathbb{R}^m$. However, we do not pursue this any further, since we believe that merely having one example of a cone with a non-isolated singularity for which (1.6) holds is of interest, and there does not appear to be a single choice of H that universally works in higher dimensions, nor a systematic way to look for an admissible choice of H in higher dimensions.

APPENDIX A. EXPANSION OF THE AREA AND GRAPHICALITY FOR THE PLATEAU PROBLEM

A.1. Expansion of the area functional. We begin by using the area formula to write

$$\mathcal{H}^n(\text{graph}_C^{s,r} u) = \int_{A_{s,r}} J_C u \, d\mathcal{H}^n,$$

where $J_C u$ is the tangential Jacobian of h on C , given by

$$J_C u = \sqrt{\det[dH^t dH]},$$

where $H(x) = x + u(x)\nu_C(x)$. Letting $\{e_i\}_{i=1}^n$ denote an orthonormal frame for TC , we have

$$dH(e_i) = e_i + \partial_i u \nu_C + u \nabla_{e_i} \nu_C = e_i + \partial_i u \nu_C - u A(e_i),$$

where $A(e_i) := -\sum_{j=1}^n (e_j \cdot \nabla_{e_i} \nu_C) e_j$, so that $A(e_i) e_j \equiv A(e_i, e_j) = -(e_j \cdot \nabla_{e_i} \nu_C)$ is the ij -coordinate of the second fundamental form A of C (previously denoted by A_C but here we drop the dependence on C to simplify notation) with respect to the bases $\{e_i\}$, where we identify $(TC)^\perp$ with $\mathbb{R}$. This yields

$$dH^t dH = (I - uA)^2 + \nabla_C u \otimes \nabla_C u,$$

where we identify A with its matrix representation. Thus, factoring out $I - uA$, which is possible for ε_0 sufficiently small, we obtain

$$\sqrt{\det[dH^t dH]} = \det(I - uA) \sqrt{1 + |(I - uA)^{-1} \nabla_C u|^2}.$$

Recalling that

$$\det(I - B) = 1 - \text{tr}(B) + \frac{1}{2}(\text{tr}(B)^2 - \text{tr}(B^2)) - \frac{1}{6}(\text{tr}(B)^3 - 3 \text{tr}(B) \text{tr}(B)^2 + 2 \text{tr}(B^3)) + O(|B|^4),$$

and exploiting the minimality of C together with a Taylor expansion of the square root and the observation that

$$|(I - uA)^{-1} \nabla_C u|^2 = |\nabla_C u|^2 + 2uA(\nabla_C u, \nabla_C u) + O(u^2 |A|^2 |\nabla_C u|^2),$$

we thus have

$$\begin{aligned} \sqrt{\det[dH^t dH]} &= (1 - \frac{1}{2}|A|^2 u^2 - \frac{1}{3} \text{tr}(A^3) u^3 + O(|A|^4 u^4)) \sqrt{1 + |(I - uA)^{-1} \nabla_C u|^2} \\ &= (1 - \frac{1}{2}|A|^2 u^2 - \frac{1}{3} \text{tr}(A^3) u^3 + O(|A|^4 u^4)) (1 + \frac{1}{2} |\nabla_C u|^2 + uA(\nabla_C u, \nabla_C u) + O(u^2 |A|^2 |\nabla_C u|^2)) \\ &=: 1 + \frac{1}{2} (|\nabla_C u|^2 - |A|^2 u^2) + \mathcal{E}(u, \nabla_C u, A). \end{aligned}$$

In particular, for ε_0 sufficiently small this yields (5.6) with

$$|\mathcal{R}_{s,r}(u)| \leq \int_{C \cap A_{s,r}} |\mathcal{E}(u, \nabla_C u, A)| \, d\mathcal{H}^n \leq C_0 \int_{C \cap A_{s,r}} (|A|^3 u^3 + u|A| |\nabla_C u|^2) \, d\mathcal{H}^n.$$

Recalling that $|A| \lesssim |x|^{-1}$ on $C \setminus \{0\}$, the claimed bound (5.7) on $\mathcal{R}_{s,r}$ follows.

To see (5.8) and (5.9), we simply observe that

$$\begin{aligned} \delta \mathcal{R}_{s,r}(u)[\psi] &= \frac{d}{dt} \Big|_{t=0} \int_{C \cap A_{s,r}} \mathcal{E}(u + t\psi, \nabla_C u + t\nabla_C \psi, A) \, d\mathcal{H}^n \\ &= \int_{C \cap A_{s,r}} D_M \mathcal{E}(u, M, A) \Big|_{M=\nabla_C u} \cdot \nabla_C \psi + D_v \mathcal{E}(v, \nabla_C u, A) \Big|_{v=u} \cdot \psi \, d\mathcal{H}^n, \end{aligned}$$

and thus, letting

$$\mathcal{G}_1(u, \nabla_C u, A) := D_M \mathcal{E}(u, M, A) \Big|_{M=\nabla_C u}, \quad \mathcal{G}_2(u, \nabla_C u, A) := D_v \mathcal{E}(v, \nabla_C u, A) \Big|_{v=u},$$

we further have

$$\begin{aligned}
& \delta^2 \mathcal{R}_{s,r}(u)[\psi_1, \psi_2] \\
&= \frac{d}{dt} \Big|_{t=0} \int_{C \cap A_{s,r}} \mathcal{G}_1(u + t\psi_2, \nabla_C u + t\nabla_C \psi_2, A) \cdot \nabla_C \psi_1 + \mathcal{G}_2(u + t\psi_2, \nabla_C u + t\nabla_C \psi_2, A) \cdot \psi_2 d\mathcal{H}^n \\
&= \int_{C \cap A_{s,r}} D_M^2 \mathcal{E}(u, M, A) \Big|_{M=\nabla_C u} [\nabla_C \psi_1, \nabla_C \psi_2] d\mathcal{H}^n \\
&\quad + \int_{C \cap A_{s,r}} +D_v D_M \mathcal{E}(v, M, A) \Big|_{v=u, M=\nabla_C u} \cdot (\psi_2 \nabla_C \psi_1 + \psi_1 \nabla_C \psi_2) + D_v^2 \mathcal{E}(v, \nabla_C u, A) \Big|_{v=u} \cdot \psi_1 \psi_2 d\mathcal{H}^n.
\end{aligned}$$

Combining these with the fact that $|A| \lesssim |x|^{-1}$ on $C \setminus \{0\}$ and applying Cauchy-Schwarz in the case of (5.9) yields the desired claim.

A.2. Graphicality over C for the Plateau problem. In this section we consider the Plateau problem with data that is perturbative over C .

Lemma A.1. *Let $C \subset \mathbb{R}^{n+1}$ be an area minimizing hypercone with an isolated singularity and let $\lambda > 0$. Let G_λ denote the open set containing C with $\partial G_\lambda = T_\lambda \cup T_{-\lambda}$. There exists $\delta > 0$, $0 < C_1 < 1$, $C_2 > 0$, and $R_0 > 0$ with the following property: if $C_1 r \geq s \geq R_0$, $h \in C^\infty(C \cap \partial B_s)$ satisfies*

$$\| |h|/s + |\nabla_{C \cap \partial B_s} h| + s |\nabla_{C \cap \partial B_s}^2 h| \|_{L^\infty} \leq \delta s^{-\gamma_1^- - 1} \quad \text{and} \quad (\text{A.1})$$

$$\text{graph}_C h \subset\subset G_\lambda, \quad (\text{A.2})$$

$|\lambda'| < \lambda$, $v : C \rightarrow \mathbb{R}$ is such that $T_{\lambda'} = \text{graph}_C v$ on $B_{R_0}^c$, and S is an area-minimizing integral current with boundary data $\llbracket \text{graph}_{C \cap \partial B_s} h \rrbracket + \llbracket \text{graph}_{C \cap \partial B_r} v \rrbracket$, then there is $u : C \cap A_{s,r} \rightarrow \mathbb{R}$ such that $\text{spt } S = \llbracket \text{graph}_C^{s,r} u \rrbracket$ and

$$|Z_u(x)| \leq C_2 |x|^{-\gamma_1^- - 1} \quad \forall x \in C \cap A_{s,r}, \quad (\text{A.3})$$

where we recall that $Z_u(x) = |\nabla_C u(x)| + |u(x)|/|x|$ as in the proof of Lemma 5.1.

Proof. Fix C and $\lambda > 0$.

We show that there are $\delta > 0$, $R_0 > 0$, $C_1 > 0$, and $C_2 > 0$ such that given any boundary data as above and solution to the Plateau problem S in the class of integral currents, $\text{spt } S$ is a graph on the desired annulus satisfying the estimate (A.3). To prove this, it suffices to show that for any set of sequences $\delta_j \rightarrow 0$, $C_1^j \rightarrow 0$, $R_0^j \rightarrow \infty$, r_j and s_j with $C_1^j r_j \geq s_j \geq R_0^j$, $\lambda_j' < \lambda$ with v_j such that $T_{\lambda_j'} = \text{graph}_C v_j$ on $B_{R_0}^c$, h_j satisfying (A.1) with $\delta_j \rightarrow 0$, and area-minimizing integral currents S_j with boundary data $\llbracket \text{graph}_{C \cap \partial B_{s_j}} h_j \rrbracket + \llbracket \text{graph}_{C \cap \partial B_{r_j}} v_j \rrbracket$, there are u_j such that the desired graphicality and estimate holds for large j . Note that by our assumption on the boundary data, the same barrier argument as in Lemma 4.1 implies that $\text{spt } S_j \subset G_\lambda$.

First, by a straightforward area comparison between the leaf and a competitor constructed from S_j , we find $\mathbb{M}(S_j)/r_j^n \leq \mathcal{H}^n(T_{\lambda'} \cap B_{r_j})/r_j^n + o(1)$. Since $\mathcal{H}^n(T_{\lambda'} \cap B_{r_j})/r_j^n \rightarrow \mathcal{H}^n(C \cap B_1)$, $\partial S_j/r_j \rightarrow \partial(\llbracket C \rrbracket \llcorner B_1)$, and $\text{spt } S_j/r_j \subset G_\lambda/r_j$, it follows that S_j/r_j converges to $\llbracket C \rrbracket \llcorner B_1$ as currents. By combining this convergence with Allard's boundary regularity theorem [4], we deduce the existence of ε_0 such that S_j/r_j is regular (with multiplicity one) on $B_{\varepsilon_0}(x)$ for every $x \in (T_{\lambda'}/r_j) \cap \partial B_1$ and large j . Moreover, the convergence of S_j/r_j to C implies that the graphicality of S_j/r_j over boundary tangent planes entailed by Allard's boundary regularity may be converted into graphicality over C for large enough j . Rescaling by r_j , we conclude that $S_j \llcorner B_{(1-\varepsilon_0/2)r_j}^c$ is a multiplicity one graph over C for large j .

Second, we claim that for sufficiently large j and each $x \in C \cap B_{r_j-3} \setminus B_{s_j+3}$, $S_j \cap B_1(x)$ is a multiplicity one graph over C that vanishes as $j \rightarrow \infty$. We begin by decomposing $S_j \llcorner B_2(x) = \sum_{i=1}^{I^{j,x}} \partial \llbracket E_i^{j,x} \rrbracket$ for each such x , where $E_i^{j,x}$ is a perimeter minimizer on $B_2(x)$. By the fact that $\text{spt } S_j \subset G_\lambda$ and G_λ decays to C at infinity (and so decays uniformly to a disk on $B_2(y)$ as $|y| \rightarrow \infty$ along C), we may apply De Giorgi's ε -regularity theorem for perimeter minimizers to conclude that $\partial E_i^{j,x} \cap B_1(x)$ is a graph (of a function with L^∞ norm as small as one desires for j large enough) over C for all j large and $x \in C \cap B_{r_j-3} \setminus B_{s_j+3}$. Next, note that for j large enough so that this holds for all such x , the number $I^{j,x}$ of sheets on $B_1(x)$ is continuous in x . Now by the previous paragraph $I^{j,x} = 1$ near $\partial B_{r_j} \cap C$. Also, $C \cap B_{r_j-3} \setminus B_{s_j+3}$ is connected due to the connectedness of $C \cap \partial B_1$ (which is a consequence of the maximum principle). Since a continuous, discrete-valued function on a connected set is constant, we deduce that $I^{j,x} = 1$ for all j large and $x \in C \cap B_{r_j-3} \setminus B_{s_j+3}$. This concludes the proof of the claim in this second step.

To finish proving the graphicality, it remains to show that $\text{spt } S_j$ is graphical near ∂B_{s_j} . Note that we cannot deduce it directly from the compactness along scales r_j , since it could be that $s_j/r_j \rightarrow 0$. However, we may instead consider the rescalings S_j/s_j and notice that thanks to the argument in the previous paragraph, S_j/s_j converges

to $\llbracket C \rrbracket \llcorner B_1^c$ as currents. An application of Allard's boundary regularity theorem using this convergence as we did near ∂B_{r_j} finishes the proof of graphicality.

It remains to estimate $Z_{u_j}(x)$. The estimate for u_j follows immediately from $\text{spt } S_j \subset G_\lambda$. Also, note that a consequence of the graphicality argument is that $\|u_j\|_{C_1(C \cap A_{s_j, r_j})} \rightarrow 0$ by the convergence of the rescaled surfaces to C . Since u_j and $\nabla_C u_j$ vanish as $j \rightarrow \infty$, the minimal surface operator on C applied to u_j is uniformly elliptic, and the desired estimates follow from flattening coordinates and invoking interior and boundary Schauder estimates. $\square$

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