Chapter 3

Reducing Horizontal Linear Complementarity Problems

3.1 Introduction

This chapter discusses various classes of nonlinear generalizations of the linear programming problem (LP), explains how some of these classes contain others and study the conditions under which some of the more general classes can be reduced to more specific ones. In particular, we define the quadratic programming problem (QP) and the linear complementarity problem (LCP) and explain how to write LPs and QPs as LCPs. Furthermore, we study the horizontal linear complementarity problem (HLCP)—a generalization of the LCP, and show how an HLCP can be reduced to the standard LCP whenever it is possible to do so. The main contribution of this chapter is the polynomial time algorithm we describe to reduce HLCPs to equivalent LCPs. Part of our discussion is based on the paper by Tütüncü and Todd [TT95].

The standard LCP is to find two complementary n-dimensional nonnegative vectors that satisfy a linear equation defined by an $n \times n$ matrix M and an n-dimensional vector q. For the HLCP the objective is the same, however this time the linear equality that the unknown vectors has to satisfy is defined by two $n \times n$ matrices N and M and an n-dimensional vector q; if $N = I$ the problem is the same as the LCP defined by M and q. Given an HLCP defined by matrices N and M and the vector q, our algorithm seeks an equivalent problem defined by matrices N', M' and a vector q' such that $N' = I$. *Reduction* refers to this process.

We note that the reduction algorithm we describe can be seen as a special case of the matroid intersection algorithm for finding a maximum weight common independent set in two matroids [Law76]. Here both matroids share the ground set $\{1, 2, \cdots, 2n\}$; for one, the independent sets are the index sets of linearly independent columns of $[N, M]$, while in the other they are the subsets with at most one element from $\{i, n+i\}$ for each i from 1 to n. In our case, all weights are equal, say to 1, so that we seek a maximum cardinality common independent set. Equivalently, this problem can be represented as finding an invertible matrix $A \in \mathcal{C}(N, M)$, where $\mathcal{C}(N, M)$ is defined as the set of $n \times n$ matrices which can be constructed by juxtaposing columns taken from N or M while maintaining the original relative ordering of the columns. This definition of $\mathcal{C}(N, M)$ is due to Willson [Wil71]. Previously, Anstreicher, Lee, and Rutherford [ALR92] had used the matroid intersection algorithm in the context of the LCP to find a maximum weight complementary basis (e.g., one containing as many elements of a target set as possible), but they did not consider explicitly the case where no complementary basis existed, which we do.

We also study the properties of matrix pairs that are analogous to the properties of a square matrix such as positive semidefiniteness and P- and P_0-properties. Such properties of the matrix M play an important role in the analysis of the solutions of the LCP defined by M and some q. For example, when M is a P-matrix then this LCP has a unique solution for every $q \in \Re^n$. Cottle, Pang, and Stone [CPS92] give an extensive discussion of these issues. We describe some of these results in Section 3.5. Such results motivate us to generalize these properties of the square matrix M to the matrix pair (N, M) in order to analyze the solutions of HLCPs. We do this by extending the particular characterizations of these properties which involve sign relations between corresponding entries of the vectors x and Mx. We then show that a large class of matrix pairs, including P_0-pairs, define reducible HLCPs, and that the reduction preserves the matrix properties. This last observation leads to a characterization of the solutions of HLCPs defined by particular matrix pair classes, analogous to the case of LCPs.

3.2 Generalizations of LP

3.2.1 The Quadratic Programming Problem

The canonical form *quadratic programming* (QP) problem is defined as follows:

$$
\begin{array}{lll}
(\mathcal{QP}) \qquad & \min_x & \frac{1}{2}x^TQx + c^Tx \\
& & Ax \ \geq \ b \\
& & \ \ x \ \geq \ 0,
\end{array}
\tag{3.1}
$$

where $A \in I\!R^{m \times n}$, $b \in I\!R^m$, $c \in I\!R^n$, $Q \in I\!R^{n \times n}$ are given, and $x \in I\!R^n$. Since $x^TQx = \frac{1}{2}x^T(Q + Q^T)x$, Q is assumed to be symmetric without loss of generality.

The dual problem associated with $(\mathcal{QP})$ is:

$$
\begin{array}{llrcrcl}
(\mathcal{QD}) & \max_{y,x} & b^T y & - & \frac{1}{2} x^T Q x & & \\
 & & A^T y & - & Qx & \leq & c \\
 & & y, & & x & \geq & 0,
\end{array}
\tag{3.2}
$$

where $y \in I\!R^m$ and $x \in I\!R^n$.

Quadratic programming problems arise in constrained regression (the unconstrained case is the well known least-squares problem) and also in approximating nonlinear optimization problems. But, probably the best known example of the QP model comes from the exciting theory of portfolio selection.

The seminal works on portfolio selection are the article [Mar52] and the book [Mar59] by Harry Markowitz. His work formalized the diversification principles in portfolio selection and earned him the 1990 Nobel prize for economics. Here we describe his model briefly.

Consider an investor who has a certain amount of money to be invested in a number of different securities. For each security i, $i = 1, \ldots, k$, estimates of its expected return, μ_i, and variance, σ_i^2, are given. Furthermore, for any two securities i and j, their correlation coefficient ρ_{ij} is also assumed to be known. Then, the problem of finding a minimum variance portfolio of the securities 1 to k that yields a target value of expected return (say b) is the *portfolio selection problem* and can be formulated as a QP:

$$
\begin{array}{lrcl}
\min_x & x^T Q x & & \\
 & e^T x & = & 1 \\
 & \mu^T x & = & b \\
 & x & \geq & 0,
\end{array}
\tag{3.3}
$$

where $Q_{ij} = \rho_{ij}\sigma_i\sigma_j$ for $i \neq j$ and $Q_{ii} = \sigma_i^2$, $\mu = (\mu_1, \ldots, \mu_k)$, and e is a k-dimensional vector of ones. Each variable x_i corresponds to the proportion of the total money available that is invested in security i, so they add up to 1. The second constraint sets the expected return to the target value and the objective function corresponds to the total variance of the portfolio. Nonnegativity constraints on x_i are introduced to disallow short sales. Note that the matrix Q has to be positive semidefinite since x^TQx, the variance of the portfolio, must be nonnegative for every portfolio x.

The model (3.3) is rather versatile. For example, if short sales are permitted on some or all of the securities then this can be incorporated into the model simply by removing the nonnegativity constraint on the corresponding variables. Or, if regulations or the investor's considerations limit the amount of investment in a subset of the securities the model can be augmented with a linear constraint to reflect such a limit. In principle, any linear constraint can be added to the model without making it much harder to solve.

3.2.2 The Linear Complementarity Problem

Recall from Chapter 1 that given a matrix $M \in \Re^{n\times n}$ and a vector $q \in \Re^n$, the *linear complementarity problem* (LCP) is defined as the problem of finding vectors x and y satisfying

$$\begin{aligned} y - Mx &= q \\ x \geq 0, y \geq 0, x^Ty &= 0, \end{aligned} \tag{3.4}$$

or possibly showing that no such solution exists. We denote this problem LCP(M, q).

We say that an LCP is *feasible* if all its requirements, except possibly complementarity ($x^Ty = 0$), can be satisfied. If there exist x and y satisfying (3.4), then we say that LCP(M, q) is called *solvable.*

We first note that the LCP contains the LP and the QP as special cases. Indeed, for the quadratic programming problem in (3.1) the Karush-Kuhn-Tucker conditions which are necessary (and sufficient if Q is positive semidefinite) for optimality are satisfied if there exist vectors x, y, u, and v such that

$$\begin{array}{rcccccccl} u & = & Qx & - & A^T y & + & c & \geq & 0 \\ v & = & Ax & & & - & b & \geq & 0 \\ & & u^T x & & & & & = & 0 \\ & & & & v^T y & & & = & 0 \\ & & x & , & y & & & \geq & 0, \end{array} \tag{3.5}$$

which is just LCP(M, q) with

$$M = \begin{bmatrix} Q & -A^T \\ A & 0 \end{bmatrix}, \quad \text{and} \quad q = \begin{bmatrix} c \\ -b \end{bmatrix}$$

with variable vectors $[u^T, v^T]^T$ and $[x^T, y^T]^T$. Moreover, a solution to (3.5) yields a pair (y, x) that satisfies the necessary optimality conditions for the dual problem $(\mathcal{QD})$. Note that when $Q = 0$, (3.1) is just an LP (with inequality constraints as opposed to the standard form LP of Chapter 2).

Conversely, the LCP (M, q) can be represented in the following quadratic programming formulation:

$$\begin{array}{rrcl} \min_x & q^T x + x^T Mx & & \\ & Mx & \geq & -q \\ & x & \geq & 0. \end{array} \tag{3.6}$$

Observing that the objective function of the QP above is always bounded below by zero on the set of feasible solutions, we easily conclude that vectors (x, y) solve the LCP (M, q) if and only if x is a global minimizer of (3.6) with objective value zero. The formulation (3.6) is especially instrumental in establishing existence and uniqueness results for the solutions of LCP (M, q). Note, however, that M cannot be assumed to be symmetric in (3.6).

As seen above, the LCP is a more general and therefore a harder problem than the LP. However, both simplicial and polynomial-time interior-point algorithms have been generalized to certain classes of LCPs. Kojima *et al.* [KMNY91] discuss interior-point algorithms for LCPs.

The LCP is also a well studied extension of the LP and has several applications in optimization, engineering, and finance, among other fields. The excellent book by Cottle, Pang, and Stone [CPS92] provides a thorough discussion of the problem, its applications and solution techniques. Here we will describe a less known but still very useful application of the LCP, namely the American put option pricing problem.

Definition 3.2.1 *An* American put option *on an underlying security, with strike price K and maturity date T, is a security that pays to the holder the stochastic reward $(K - S_\tau)^+$, where S_τ is the price of the underlying security at the time of exercise (τ), when exercised by the holder at any stopping time $\tau \in [0, T]$.*

The American put option is an example of a *derivative security*, or *contingent claim*; these are securities that provide payoffs determined entirely by other securities such as bond and stock prices, market index values, etc.

A solid understanding of an *optimal exercise time* is essential for the pricing of an American put option. Determining an optimal exercise time for general underlying securities is usually a very hard problem. Nevertheless, if the underlying security follows the *geometric Brownian motion* model of Black and Scholes [BS73], and if the market satisfies their several simplifying assumptions such as the lack of transactions costs and arbitrage opportunities, then the American put option pricing problem can be cast as a free boundary problem, or equivalently, as an obstacle problem, as a variational inequality problem, and finally, as a linear complementarity problem.

We should note that all these problems are infinite-dimensional problems defined on Hilbert spaces. In practice, these problems are discretized and the solutions are only approximate. Without further ado, we describe the linear complementarity formulation of the American put option pricing problem.

Theorem 3.2.1 *The American put value P as a function of ξ, the natural logarithm of the stock price, and t, time, is the unique solution to the following infinite-dimensional complementarity problem*

$$\begin{aligned} u-\psi &\geq 0 \\ -\mathcal{L}u-\frac{\partial u}{\partial t} &\geq 0 \\ (-\mathcal{L}u-\frac{\partial u}{\partial t})\wedge(u-\psi) &= 0 \ a.s. \ in\ \Re\times[0,T], \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} \psi &= (K-e^{\xi})^{+}, \\ (f\wedge g)(\xi,t) &= \min\{f(\xi,t),g(\xi,t)\}, \end{aligned}$$

and $\mathcal{L}$ is the operator $\frac{1}{2}\sigma^2\frac{\partial^2}{\partial\xi^2}+(r-\frac{1}{2}\sigma^2)\frac{\partial}{\partial\xi}-r$. In the last expression, r denotes the risk-free interest rate *(which is assumed to be constant) and σ^2 denotes the constant* volatility *of the underlying security.* □

Remark 3.2.1 *A proof of Theorem 3.2.1 can be found, e.g., in [Hut95]. Note that (3.7) is a complementarity formulation for* linear *operators on the variable function u. Furthermore, the discretization of the spatial and time variables in (3.7) through standard finite differences leads to an LCP in Euclidean space, with the vector x a discretization of the function $u - \psi$. This is why we include this example here.*

Remark 3.2.2 *Interestingly, the matrix that defines the discretized LCP is a Z-matrix, i.e., its off-diagonal entries are all nonpositive. Therefore, it can be solved by a single linear program, see [Man76, Man79, CD80]. Consequently, the discretization of (3.7) constitutes a special case of the linear complementarity problem. In general, however, LCPs are not reducible to LPs. Indeed, there exist NP-complete LCPs [KMNY91, pp. 33–34] whereas LPs are solvable in polynomial time.*

Remark 3.2.3 *In [Hut95], Hutton presents favorable numerical results on the performance of solution methods for the American option pricing problem based on the LCP formulation. Alternatively, one can use the so-called* analytic approximation *methods. The reader is referred to the recent working paper by Carr and Faguet [CF95] for details, references, and extensive computational results on such methods.*

3.2.3 The Horizontal Linear Complementarity Problem

A slightly more general form of the LCP is the *horizontal linear complementarity problem* (HLCP), i.e., the problem of finding n-dimensional vectors x and y

satisfying

$$\begin{aligned} Ny - Mx &= q \\ x \geq 0, y \geq 0, x^T y &= 0, \end{aligned} \tag{3.8}$$

given $n \times n$ real matrices N, M and an n-dimensional vector q; this problem is denoted the HLCP(N, M, q). The HLCP was apparently first introduced by Samelson, Thrall, and Wesler [STW58], motivated by a problem in structural engineering. The term *horizontal* reflects the geometric shape of the $n \times 2n$ constraint matrix $[N, -M]$. This is in contrast with the *vertical LCP* that has a constraint matrix with more rows than columns [CPS92, pp. 32-33].

Obviously, HLCP(I, M, q) is just LCP(M, q). Furthermore, when N is invertible, the first equation of (3.8) can be replaced by $y - N^{-1}Mx = N^{-1}q$, and the problem is just $LCP(N^{-1}M, N^{-1}q)$. However, N is often singular for the HLCPs appearing in practice, justifying a separate study.

As for the LCP, an HLCP is called *feasible* if all its requirements, except possibly complementarity, can be satisfied; if there exist x and y satisfying (3.8), then HLCP (N, M, q) is called *solvable*.

HLCPs arise for instance in electrical engineering. Vandenberghe *et al.* [DMVV92, VDMV89] provide examples of resistive piecewise linear circuits whose analyses lead to HLCPs, as well as further generalizations of the LCP. This is not too surprising as Eaves and Lemke [EL81] show that the HLCP is equivalent to a piecewise linear system

$$f(x) = y \tag{3.9}$$

where $f : \Re^n \rightarrow \Re^n$ is piecewise linear. Also, HLCPs provide a unifying framework for the study of infeasible-interior-point methods to solve some optimization problems; see e.g. [Zha94].

One of our main goals in this chapter is to develop an algorithm to reduce HLCPs to standard LCPs whenever such a reduction is possible. The next section describes such an algorithm.

3.3 Reduction Algorithm

Although there has been a recent surge of interest in the study of horizontal LCPs [Gow94,Zha94,MP95,SG95], the standard LCP is a much better studied problem and polynomial time algorithms have been devised for certain matrix classes. This gives us the motivation to design an algorithm to reduce HLCPs to standard LCPs whenever it is possible to do so.

As observed above, when N is invertible the HLCP (N, M, q) can be readily written as an equivalent LCP. Such reductions are possible even when N is singular as long as there is another HLCP (N', M', q') equivalent to the original HLCP with the property that N' is invertible. We now make the notion of equivalence more precise.

For a pair of $n \times n$ matrices N and M, we say that $\{N', M'\}$ is a *twisted column rearrangement* of $\{N, M\}$ if for each index i, either $N'_i = N_i$ and $M'_i = M_i$ or $N'_i = -M_i$ and $M'_i = -N_i$. We write $(\tilde{N}, \tilde{M}) \sim (N, M)$ if $\tilde{N}$ and $\tilde{M}$ are obtained by possibly applying the same permutation to the columns of N' and M' for a twisted column rearrangement $\{N', M'\}$ of $\{N, M\}$. Thus

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \right) \sim \left(\begin{bmatrix} 0 & -3 & -1 \\ 1 & -6 & -4 \\ 0 & -9 & -7 \end{bmatrix}, \begin{bmatrix} 2 & 0 & -1 \\ 5 & 0 & 0 \\ 8 & -1 & 0 \end{bmatrix} \right).$$

If $(\tilde{N}, \tilde{M}) \sim (N, M)$ and (y, x) is a pair of n-vectors, we write $(\tilde{y}, \tilde{x})$ for the vectors obtained by performing the same permutation and exchange operations on the components of y and x, without switching signs. Thus $Ny - Mx = \tilde{N}\tilde{y} - \tilde{M}\tilde{x}$. Similarly, given $(\tilde{y}, \tilde{x})$, (y, x) denotes the result of applying the exchange operation (without sign switches) and the inverse permutation to $(\tilde{y}, \tilde{x})$.

We also note that premultiplication of the matrices N, M, and the vector q with a nonsingular square matrix does not alter the solutions to the HLCP. We add this operation to the ones described above to define equivalence of HLCPs. We say that the HLCPs (N, M, q) and (N', M', q') are *equivalent* if there is a nonsingular $n \times n$ matrix U and some $(\tilde{N}, \tilde{M}) \sim (N, M)$ such that $N' = U\tilde{N}$, $M' = U\tilde{M}$, and $q' = Uq$. It is easy to see that this is an equivalence relation, and (y, x) solves the HLCP (N, M, q) (satisfies $Ny - Mx = 0$, respectively) if and only if $(\tilde{y}, \tilde{x})$ solves the HLCP (N', M', q') (respectively, satisfies $N'\tilde{y} - M'\tilde{x} = 0$). We say the HLCP (N, M, q) is *reducible* to a standard LCP if there exists an HLCP (N', M', q') in its equivalence class with $N' = I$; clearly the latter is just the LCP (M', q').

The rest of this section will be devoted to the description of an algorithm that repeatedly applies the operations described above to a given HLCP in order to reduce it to an LCP. The algorithm fails to achieve this goal only when there is no HLCP (N', M', q') with $N' = I$ in the equivalence class of the original problem. As mentioned in the introduction, this algorithm turns out to be a specialization of the matroid intersection algorithm.

Before presenting a high-level description of our algorithm, we give an informal discussion of the main ideas. Given an HLCP (N, M, q), the algorithm starts by performing row operations on the augmented matrix $\left[\begin{array}{c|c|c} N & M & q \end{array}\right]$ to obtain $(N^{(1)}, M^{(1)}, q^{(1)})$ where $N^{(1)}$ is in row reduced echelon form. In a particular iteration of the algorithm, say the pth, we have the HLCP $(N^{(p)}, M^{(p)}, q^{(p)})$ which is equivalent to HLCP (N, M, q). We keep $N^{(p)}$ in row reduced echelon form. The algorithm terminates with the desired reduction if $k := \text{rank}(N^{(p)})$ reaches n, but if $k < n$ the algorithm tries to find a twisted column rearrangement $(\overline{N}^{(p)}, \overline{M}^{(p)})$ of $(N^{(p)}, M^{(p)})$ such that rank$(\overline{N}^{(p))}$ is larger than k. For example, if a column of $M^{(p)}$, say column l with $l > k$, has a nonzero entry below its k^{th} entry, we have that the l^{th} column of $M^{(p)}$ is linearly independent of the first k columns of $N^{(p)}$, whereas the l^{th} column of $N^{(p)}$ is not; then exchanging these two columns (with sign switches) would result in an equivalent problem with the rank of the new N matrix being $k + 1$. More generally, one can exchange several columns at once to increase the rank as much as possible.

A key observation that we make is that rank increasing column exchanges correspond to paths with special properties in a directed graph induced by the matrices $M^{(p)}$ and $N^{(p)}$. For example, the single column change we described above corresponds to a path consisting of a single arc in this graph. Furthermore, we show that if this graph does not contain such paths, then the problem is irreducible. Based on these observations, the problem of finding rank increasing twisted column rearrangements in each iteration reduces to a path finding problem (say a shortest path) on the induced graph, which can be done in polynomial time. This also indicates that our algorithm has polynomial complexity, and therefore whenever an HLCP is reducible to an LCP which can be solved in polynomial time, that HLCP can be solved in polynomial time as well. The algorithm is presented in Figure 3.1.

Step 1. Initialization: Form the augmented matrix $\left[\begin{array}{c|c|c} N & M & q \end{array}\right]$, and row reduce N and apply the same row operations to M and q. Also permute the columns of N (and the corresponding columns of M) to get

$$\left[\begin{array}{c|c|c} N^{(1)} & M^{(1)} & q^{(1)} \end{array}\right] = \left[\begin{array}{cc|cc|c} I & * & * & * & * \\ \\ 0 & 0 & * & * & * \end{array}\right].$$

Set p=1.

Step 2. Set $k := \mathrm{rank}(N^{(p)})$.

<u>If</u> $(k = n)$ <u>then</u>

<u>return</u>(`done`)

<u>else</u>

form the directed graph $G^{(p)} := (V, E^{(p)})$ where
$V := \{1, \cdots, n\}$ and $E^{(p)} := E_N^{(p)} \cup E_M^{(p)}$
$:= \{(i,j) : N_{ij}^{(p)} \neq 0\} \cup \{(i,j) : M_{ij}^{(p)} \neq 0\}$.

<u>End if</u>

Step 3. Now, if there exists one, find a shortest path of length at least one (set all arc lengths to 1) starting at a vertex $v_0 \in \{k+1, \cdots, n\}$ and ending at $v_l \in \{k+1, \cdots, n\}$, allowing the possibility of $v_0 = v_l$.

Step 4. <u>If</u> (there is no such path) <u>then</u>

<u>return</u>(`failure`)

<u>else</u>

For all $i = 1, \cdots, l$ replace $M_{v_i}^{(p)}$ with $-N_{v_i}^{(p)}$ and $N_{v_i}^{(p)}$ with $-M_{v_i}^{(p)}$ whenever $(v_{i-1}, v_i) \in E_M^{(p)} \setminus E_N^{(p)}$. Let the resulting matrices be $\overline{N}^{(p)}$,$\overline{M}^{(p)}$, and row reduce $[\overline{N}^{(p)}, \overline{M}^{(p)}, q^{(p)}]$ (possibly with column permutations) to get $[N^{(p+1)}, M^{(p+1)}, q^{(p+1)}]$, as in Step 1. Set $p := p + 1$.

<u>End if</u>

Step 5. Go to Step 2.

Figure 3.1: Reduction algorithm

3.4 Analysis of the Algorithm

In this section we will show that the reduction algorithm achieves its intended goals. We start with showing that the algorithm terminates in at most n iterations, where n is the size of the input matrices. This result follows immediately from the following lemma.

Lemma 3.4.1 *If in iteration p, the algorithm does not terminate in Step 2 or 4, then* $\mathrm{rank}(N^{(p+1)}) > \mathrm{rank}(N^{(p)})$.

Proof:

It suffices to prove that $\mathrm{rank}(\overline{N}^{(p)}) > \mathrm{rank}(N^{(p)})$; since $N^{(p+1)}$ is row equivalent to a column permutation of $\overline{N}^{(p)}$, $\mathrm{rank}(\overline{N}^{(p)}) = \mathrm{rank}(N^{(p+1)})$.

Let $(v_0, v_1, \ldots, v_l)$ be the path found in Step 4, and define $\{j_1, \ldots, j_s\} := \{1, \ldots, k\} \setminus \{v_0, \ldots, v_l\}$. Recall that $k = \mathrm{rank}(N^{(p)})$. Now define the index sets I and J as follows:

$$I \;:=\; \{j_1, \ldots, j_s, v_{l-1}, v_{l-2}, \ldots, v_0\}$$

and

$$J \;:=\; \{j_1, \ldots, j_s, v_l, v_{l-1}, \ldots, v_1\}.$$

Consider $\overline{N}^{(p)}_{IJ}$, namely the square submatrix of $\overline{N}^{(p)}$ obtained by taking the rows indexed by I, and columns indexed by J, keeping the written order. Since $N^{(p)}$ is in row reduced echelon form, its first k columns coincide with those of the identity matrix. Therefore, the columns indexed by $j_1, \ldots, j_s$ in $\overline{N}^{(p)}_{IJ}$ have only one nonzero, which is a 1 and appears on the diagonal. Furthermore, for $i = 1, \ldots, l$, $\overline{N}^{(p)}_{v_{i-1}v_i}$ is nonzero for $i = 1, \ldots, l$, because of the construction. Therefore $\overline{N}^{(p)}_{IJ}$ has nonzero entries on its diagonal. On the other hand, $\overline{N}^{(p)}_{v_i v_j}$ must be zero for $i < j - 1$, because otherwise $v_0 \to v_1 \to \cdots \to v_i \to v_j \to \cdots \to v_l$ would be a shorter path, contradicting that $(v_0, v_1, \ldots, v_l)$ is a shortest path. This establishes that $\overline{N}^{(p)}_{IJ}$ is upper triangular with a nonzero diagonal, whence it has full rank. Consequently, $\mathrm{rank}(\overline{N}^{(p)}) \geq \mathrm{rank}(\overline{N}^{(p)}_{IJ}) = |I| = |J|$.

But, $I = \{1, \ldots, k\} \cup \{v_0, \ldots, v_{l-1}\}$ and since $v_0 \in \{k+1, \ldots, n\}$, $|I| \geq k+1$. Thus $\mathrm{rank}(N^{(p+1)}) = \mathrm{rank}(\overline{N}^{(p)}) > k = \mathrm{rank}(N^{(p)})$. □

The lemma above indicates that, if the reduction algorithm does not terminate in Step 2 or 4 of an iteration, then it finds an HLCP in the equivalence class of the original problem with an N matrix that has strictly larger rank than that of the previous iteration. Consequently, after at most $n - \mathrm{rank}(N^{(1)})$ iterations the reduction algorithm will terminate, either in Step 2 or in Step 4. Clearly, if the algorithm terminates in Step 2, we do reduce the HLCP to a standard LCP, since an $n \times n$ row reduced echelon matrix of rank n is the identity.

It remains to be proven that if the algorithm terminates in Step 4, then there is no HLCP (N', M', q') in the equivalence class of the original problem HCLP (N, M, q) with $N' = I$. Recall that $\mathcal{C}(N, M)$ denotes the set of $n \times n$ matrices that are constructed by juxtaposing columns taken from N and M.

Lemma 3.4.2 *If the algorithm terminates in Step 4 of the p^{th} iteration with* $k := \mathrm{rank}(N^{(p)})$, *then* $\mathrm{rank}(A) \leq k$ *for all* $A \in \mathcal{C}(N^{(p)}, -M^{(p)})$.

Proof:

Assume that the algorithm terminates in Step 4. Then there are no paths in $G^{(p)}$ starting and ending in vertices from the set $T := \{k+1, \cdots, n\}$. Let us say that a vertex v is *reachable* from a set U if there exists a path in $G^{(p)}$ from a vertex in U

to v. Let S be the set of vertices in $G^{(p)}$ that are not in T but are reachable from T and let $S^c := \{1, \cdots, k\} \setminus S$. Clearly, S does not contain any member of T, i.e., $S \subset \{1, \cdots, k\}$. Permute the rows and columns of $N^{(p)}$ and $M^{(p)}$ to get $\tilde{N}^{(p)}$, $\tilde{M}^{(p)}$ which have row and column indices S, S^c, and T, in this order. This permutation does not alter the rank of the matrices, so $\operatorname{rank}(\tilde{N}^{(p)}) = \operatorname{rank}(N^{(p)}) = k$. This indexing partitions $\tilde{N}^{(p)}, \tilde{M}^{(p)}$ into blocks, say $\tilde{N}_{ij}$, $\tilde{M}_{ij}$, $i, j = 1, 2, 3$:

$$\begin{bmatrix} \tilde{N}^{(p)} & \tilde{M}^{(p)} \end{bmatrix} = \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} & \tilde{N}_{13} & \tilde{M}_{11} & \tilde{M}_{12} & \tilde{M}_{13} \\ \tilde{N}_{21} & \tilde{N}_{22} & \tilde{N}_{23} & \tilde{M}_{21} & \tilde{M}_{22} & \tilde{M}_{23} \\ \tilde{N}_{31} & \tilde{N}_{32} & \tilde{N}_{33} & \tilde{M}_{31} & \tilde{M}_{32} & \tilde{M}_{33} \end{bmatrix}. \tag{3.10}$$

One can easily see that $\tilde{N}_{11} = I$,$\tilde{N}_{21} = 0$, $\tilde{N}_{31} = 0$, $\tilde{N}_{12} = 0$, $\tilde{N}_{22} = I$, $\tilde{N}_{32} = 0$, and $\tilde{N}_{33} = 0$. We also claim that $\tilde{M}_{33} = 0$, $\tilde{N}_{13} = 0 = \tilde{M}_{13}$, $\tilde{M}_{12} = 0$, and $\tilde{M}_{32} = 0$.

$\tilde{M}_{33} = 0$, because if there were a nonzero entry in $\tilde{M}_{33}$, then there would be a path (of length 1) in $G^{(p)}$ consisting of just the arc corresponding to this nonzero entry, and therefore, the algorithm would not have stopped at Step 4.

Also, all entries in $\tilde{N}_{13}$ and $\tilde{M}_{13}$ are zero, because otherwise there would be an arc from a vertex in S to a vertex in T, thus at least one vertex in T would be reachable from the set of vertices T (via S), and thus the algorithm would have found this path and would not have stopped.

Finally, $\tilde{M}_{12}$ and $\tilde{M}_{32}$ cannot have any nonzero entries, because otherwise at least one vertex in S^c would be reachable from T, contradicting the definition of S^c. Therefore, $[\tilde{N}^{(p)}, \tilde{M}^{(p)}]$ has the following form:

$$\left[\begin{array}{cc} \tilde{N}^{(p)} & \tilde{M}^{(p)} \end{array}\right] = \left[\begin{array}{cccccc} I & 0 & 0 & * & 0 & 0 \\ 0 & I & * & * & * & * \\ 0 & 0 & 0 & * & 0 & 0 \end{array}\right]. \qquad (3.11)$$

So, for any $A \in \mathcal{C}(N^{(p)}, -M^{(p)})$ the columns indexed by $S^c \cup T$ of A have rank at most $k - |S|$, and thus rank(A) is at most $k - |S| + |S| = k$. □

If the algorithm terminates in Step 4, then we have $k < n$, so the lemma above shows that $\det(A) = 0$ for all $A \in \mathcal{C}(N, -M)$. But on the other hand, by our definition of equivalence, if HLCP (N', M', q') is equivalent to HLCP (N, M, q) then N' must be of the form $N' = QPA$, where Q is an invertible matrix, P is a permutation matrix and $A \in \mathcal{C}(N, -M)$. Therefore, rank (N') = rank $(A) \leq k < n$, for any N' such that HLCP (N', M', q') is equivalent to HLCP (N, M, q). Consequently, there is no HLCP (N', M', q') in the equivalence class of the problem HCLP (N, M, q) with $N' = I$. This shows that the algorithm fails to attain a reduction to a standard LCP only when such reduction is not possible.

Theorem 3.4.1 *The following are equivalent:*

(i) For all $A \in \mathcal{C}(N, -M)$ *we have* $\det(A) = 0$.

(ii) There exists an $(\tilde{N}, \tilde{M}) \sim (N, M)$ *and an invertible matrix* U *such that* $[U\tilde{N}, U\tilde{M}]$ *has the form in (5).*

(iii) For all diagonal D *we have* $\det(MD + N) = 0$.

Proof:

Clearly, the discussion above indicates the equivalence of the first two conditions. To see the equivalence of (i) and (iii), first note that (i) holds if and only if $\det(A) =$

0 for all $A \in \mathcal{C}(N, M)$. Consider $D =: \mathrm{diag}(d_1, d_2, \cdots n)$. Then, as shown in Willson [Wil71], $\det(MD+N)$ is a polynomial in the variables $\{d_1, d_2, \cdots n\}$ whose coefficients are the determinants of the matrices in $\mathcal{C}(N, M)$. Moreover each of these determinants is the coefficient of exactly one term in the polynomial. More precisely, each term in the polynomial is of the form $\Delta \cdot \prod_{i=1}^{n} d_i^{k_i}$, where each $k_i \in \{0, 1\}$, $\Delta := \det(A)$ and A is the $n \times n$ matrix with columns $A_i := M_i$ when k_i is 1 and $A_i := N_i$ otherwise. Then, $\det(MD + N) = 0$ for all diagonal D if and only if the corresponding polynomial vanishes for all values of the variables $\{d_1, d_2, \cdots, d_n\}$, which is true if and only if all the coefficients of the polynomial are zero. This establishes the equivalence of the conditions (i) and (iii). $\square$

Remark 3.4.1 *Theorem 3.4.1 gives various characterizations of irreducible HLCPs. Simply by negating these conditions, we can characterize the reducible ones. The equivalence of items (i) and (ii) can also be seen via Edmonds' max-min theorem for matroid intersections: the maximum cardinality of a common independent set equals the minimum rank of a covering (see Lawler [Law76], Theorem 8.5.2). Here the covering is $E_1 \cup E_2$, where E_1 contains the indices of the second and third blocks of columns of $\tilde{N}^{(p)}$ and $\tilde{M}^{(p)}$ in (4) and (5), with matrix rank equal to the order of $\tilde{N}_{22}$, while E_2 contains the indices of the first blocks of columns of these matrices, containing at most the order of $\tilde{N}_{11}$ elements with no pair of elements of the form $\{i, n+i\}$ for some i.*

Remark 3.4.2 *The first condition in Theorem 3.4.1 gives a finite method to check for reducibility. However, the number of matrices in $\mathcal{C}(N, -M)$ grows exponentially with the matrix size. In contrast, the reduction algorithm presented in the previous section checks for reducibility in polynomial time.*

3.5 Matrix Properties, the LCP and the HLCP

In this section we focus on several interesting matrix properties such as the positive (semi)definiteness, P- and P_0-properties. These properties become especially important in the analysis of feasibility and solvability of LCPs. We first define some of these properties, and then generalize them to the case of matrix pairs. Then we study how these properties relate to the reducibility and solvability of the corresponding HLCPs.

Definition 3.5.1 *A matrix $M \in \Re^{n\times n}$ is called positive definite (positive semidefinite) if for every nonzero $x \in \Re^n$ we have that $x^T(Mx) \geq 0$.*

Definition 3.5.2 *A matrix $M \in \Re^{n\times n}$ is called a P-matrix (a P_0-matrix) if all of its principal minors are positive (nonnegative).*

The following characterization of P and P_0-matrices is due to Fiedler and Pták [FP66].

Proposition 3.5.1 *Let $M \in \Re^{n\times n}$.*

(i) M is a P_0-matrix if and only if for every nonzero $x \in \Re^n$, there exists an index i such that $x_i \neq 0$ and $x_i(Mx)_i \geq 0$.

(ii) M is a P-matrix if and only if for every nonzero $x \in \Re^n$, there exists an index i such that $x_i(Mx)_i > 0$.

□

In the study of LCPs, sign properties of the corresponding entries of vectors x and Mx, such as those in Proposition 3.5.1, become important. Given $M \in \Re^{n\times n}$

and $x \in \Re^n$ we let $y := Mx$ and define

$$z(x) := x \circ y = x \circ Mx. \tag{3.12}$$

Here $\circ$ denotes the Hadamard product, i.e., $z_i(x) = x_i y_i$ for all i. Observe that $M \in \Re^{n\times n}$ is positive definite (positive semidefinite) if for every nonzero $x \in \Re^n$, $e^T z(x) > 0$ ($e^T z(x) \geq 0$) where e is an n-dimensional vector of ones. Similarly, $M \in \Re^{n\times n}$ is a P-matrix if and only if for every nonzero $x \in \Re^n$ we have an index i such that $z_i(x) > 0$.

The following definitions are also inspired by the sign properties of the vector $z(x)$:

Definition 3.5.3 *A matrix $M \in \Re^{n\times n}$ is called a* column sufficient *matrix if for every x, $z(x) \leq 0$ implies $z(x) = 0$. The matrix M is called* row sufficient *if M^T is column sufficient. When both M and M^T are column sufficient M is said to be* sufficient.

Definition 3.5.4 *A matrix $M \in \Re^{n\times n}$ is called a $P_*(\kappa)$-*matrix *if for every $x \in \Re^n$ the following holds:*

$$(1 + 4\kappa) \sum_i [z_i(x)]^+ - \sum_i [z_i(x)]^- \geq 0. \tag{3.13}$$

Above, $[x]^+$ and $[x]^-$ denote the positive and negative parts of the real number x, i.e., $[x]^+ = \max\{x, 0\}$ and $[x]^- = \max\{-x, 0\}$.

Definition 3.5.5 *A matrix $M \in \Re^{n\times n}$ is called a P_*-*matrix *if it is a $P_*(\kappa)$-matrix for some $\kappa \geq 0$.*

The classes of column and row sufficient matrices were introduced by Cottle, Pang and Venkateswaran [CPV81]. The classes of $P_*(\kappa)$ and P_* matrices were introduced by Kojima et al. [KMNY91]. Recently, Väliaho showed that the class of P_*-matrices coincides with the class of sufficient matrices [Väl95].

We now summarize how the above matrix properties are related to the solutions of corresponding LCPs.

Theorem 3.5.1 *Let $M \in \Re^{n\times n}$.*

1. *If M is positive semidefinite then the LCP (M, q) is solvable whenever it is feasible. [Cot64a,Cot64b]*
2. *LCP (M, q) has a unique solution for all $q \in \Re^n$ if and only if M is a P-matrix [STW58].*
3. *M is column sufficient if and only if the LCP (M, q) has a (possibly empty) convex solution set for every $q \in \Re^n$ [CPV81].*
4. *M is row sufficient if and only if for each $q \in \Re^n$, if (x, u) is a Karush-Kuhn-Tucker pair of the QP (3.6), then $(x, Mx + q)$ solves the LCP (M, q) [CPV81].*
5. *If M is a $P_*(\kappa)$-matrix, then by an interior-point algorithm one can either obtain a solution $(y, x) \geq 0$ to the LCP (M, q) such that $x^T y < \epsilon$ and $\|y - Mx - q\| < \epsilon$ or show that the problem is not solvable, in $\mathcal{O}((1+\kappa)^2 n \ln \frac{1}{\epsilon})$ iterations [PS94].* □

It is worth noting that the class of P_0-matrices include all the other matrix classes we defined above. Furthermore, the class of column sufficient matrices include the the class of P_*-matrices (which is just the class of sufficient matrices, cf. [Väl95]), which in turn include the classes of P-matrices and positive semidefinite matrices.

Indeed, $P_*(0)$-matrices are just the positive semidefinite matrices. These results are proved in [KMNY91].

One of the main goals of this section is to extend the matrix classes defined above to the case of matrix pairs in order to study the solutions of the HLCP (N, M, q). First, we extend the notation introduced in (3.12) to two arbitrary n-dimensional vectors x and y and let

$$z(x, y) := x \circ y. \tag{3.14}$$

Next, we state the following simple result.

Proposition 3.5.2 *Consider equivalent HLCPs (N, M, q) and (N', M', q'). Let (y, x) be such that $Ny - Mx = 0$ and let $(\tilde{y}, \tilde{x})$ be the vectors obtained from (y, x) by the permutation and exchange operations (without the sign switches) used in transforming the HLCP (N, M, q) to HLCP (N', M', q'). Then $N'\tilde{y} - M'\tilde{x} = 0$ and*

$$z(\tilde{y}, \tilde{x}) \quad = \quad P\, z(y, x) \text{ for a permutation matrix } P.$$

Proof:

Straightforward. $\square$

We also present a complementary result that will be useful in the definition and study of row-sufficient pairs.

Proposition 3.5.3 *Consider equivalent HLCPs (N, M, q) and (N', M', q'). Let $[y^T, x^T]^T$ be in the range space of $[N, M]^T$ and let $\tilde{y}$ and $\tilde{x}$ be the vectors obtained from y and x by the permutation and exchange operations (with the sign switches) used in transforming the HLCP (N, M, q) to HLCP (N', M', q'). Then, $[\tilde{y}^T, \tilde{x}^T]^T$*

is in the range space of $[N', M']^T$ *and*

$$z(\tilde{y}, \tilde{x}) \;=\; P\, z(y, x) \text{ for a permutation matrix } P.$$

Proof:

Since the HLCPs (N, M, q) and (N', M', q') are equivalent, there exist an invertible matrix U and some $(\tilde{N}, \tilde{M}) \sim (N, M)$ such that $N' = U\tilde{N}$, $M' = U\tilde{M}$, and $q' = Uq$. Also, note that the range spaces of $[N', M']^T$ and $[\tilde{N}, \tilde{M}]^T$ coincide. But then, by construction $[\tilde{y}^T, \tilde{x}^T]^T$ is in the range space of $[\tilde{N}, \tilde{M}]^T$ if and only if $[y^T, x^T]^T$ is in the range space of $[N, M]^T$, and for all $i = 1, \ldots, n$

$$\tilde{y}_i \tilde{x}_i \;=\; y_{\sigma_i} x_{\sigma_i},$$

for a permutation $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ of $\{1, 2, \ldots, n\}$. $\square$

Now we can introduce the following standard generalization of the positive semidefiniteness property.

Definition 3.5.6 (N, M) *is called a* monotone pair *if* $Ny - Mx = 0$ *implies* $e^T z(x, y) \geq 0$.

Clearly, (I, M) is a monotone pair if and only if M is positive semidefinite. Also, if $(\tilde{N}, \tilde{M}) \sim (N, M)$ then (N, M) is a monotone pair if and only if $(\tilde{N}, \tilde{M})$ is a monotone pair. Hence, if the reduction algorithm of Section 3.3 reduces the HLCP (N, M, q) to the LCP (M', q') for a given monotone pair (N, M), then M' is positive semidefinite. This follows immediately from Proposition 3.5.2.

One important observation that we make about the HLCPs defined by monotone pairs is that they are always reducible. This observation is made independently and simultaneously by Güler [Gül95] and Sznajder and Gowda [SG95]. See also [Gow95] where Gowda describes a method to reduce a monotone HLCP to an LCP based on the algorithm we presented in Section 3.3. Therefore we will omit the proof here and will prove a more general result at the end of this section.

We now continue with the generalizations of the aforementioned matrix properties to matrix pairs.

Definition 3.5.7 *A matrix pair (N, M) is called a* column sufficient pair *if for every (y, x) satisfying $Ny - Mx = 0$, $z(y, x) \leq 0$ implies $z(y, x) = 0$. The pair (N, M) is called a* row sufficient pair *if $z(y, x) \leq 0$ implies $z(y, x) = 0$ for all $[y^T, x^T]^T$ in the range space of $[N, M]^T$. When the pair (N, M) is both row and column sufficient, (N, M) is said to be a* sufficient pair.

The definitions of column and row sufficient pairs were introduced by Mangasarian and Pang [MP95].

Definition 3.5.8 *A matrix pair (N, M) is called a* $P_*(\kappa)$-pair *if $Ny - Mx = 0$ implies the following:*

$$(1 + 4\kappa) \sum_i [z_i(y, x)]^+ - \sum_i [z_i(y, x)]^- \geq 0. \tag{3.15}$$

This definition also appears in [ALP95].

Definition 3.5.9 *A matrix pair (N, M) is called a* P_*-pair *if it is a $P_*(\kappa)$-pair for some $\kappa \geq 0$.*

Finally, we want to generalize the P- and P_0- properties along these lines. Although it is straightforward to generalize the P-property using Fiedler and Ptàk's characterization, the P_0-property as given in Proposition 3.5.1 lacks symmetry in terms of the entries of the vectors x and Mx and it is not appropriate to define

P_0-pairs with such an asymmetry. However, we have the following result that establishes a symmetric equivalent of the P_0-property:

Proposition 3.5.4 *Let $M \in \Re^{n\times n}$. Then the following are equivalent:*

(i) M is a P_0-matrix.

(ii) For every nonzero $x \in \Re^n$, there exists an index i such that $x_i \neq 0$ or $(Mx)_i \neq 0$, and $x_i(Mx)_i \geq 0$.

Proof:

Clearly (i) implies (ii). We prove that (ii) implies (i) by contradiction. Assume that M is not a P_0-matrix, so that there exists a nonzero vector x such that for all indices i, either we have $x_i = 0$ or $x_i(Mx)_i < 0$. Consider the index sets $I_j(x)$, $j = 1, \ldots, 4$, defined as follows:

$$\begin{aligned} I_1(x) &:= \{i : x_i = 0 = (Mx)_i\} \\ I_2(x) &:= \{i : x_i = 0, (Mx)_i \neq 0\} \\ I_3(x) &:= \{i : x_i \neq 0, x_i(Mx)_i \geq 0\} \\ I_4(x) &:= \{i : x_i(Mx)_i < 0\} \end{aligned}$$

By our assumption we must have that $I_3(x) = \emptyset$. We prove that there exists an $\overline{x} \neq 0$ such that $I_2(\overline{x}) \cup I_3(\overline{x}) = \emptyset$, showing that M fails to satisfy (ii).

If $I_2(x) = \emptyset$, then take $\overline{x} = x$ and we are done. Otherwise pick any $i \in I_2$. Choose ϵ small enough in absolute value such that $\|M(\epsilon e_i)\|_\infty < \delta$, where e_i is the i^{th} unit vector and $\delta := \min_{(Mx)_j \neq 0}\{|(Mx)_j|\}$, and with the opposite sign to $(Mx)_i$.

Now consider $x^1 := x + \epsilon e_i$. We observe that $I_3(x^1) = \emptyset$, and that $I_4(x^1) = I_4(x) \cup \{i\}$, which indicates that $|I_1(x^1) \cup I_2(x^1)| = |I_1(x) \cup I_2(x)| - 1$. Therefore, after a finitely many, say k, repetitions of this procedure, $I_2(x^k)$ will be empty, with $I_3(x^k)$ remaining empty. Then letting $\overline{x} := x^k$ shows that (ii) fails. Hence, (ii) implies (i), and therefore (i) and (ii) are equivalent. $\square$

Now we can introduce the following definitions:

Definition 3.5.10 *A matrix pair (N, M) is called a* P_0-pair *(respectively, a* P-pair*) if $Ny - Mx = 0$ and $(y, x) \neq 0$ implies that there exists an index i such that $x_i \neq 0$ or $y_i \neq 0$, and $z_i(y, x) \geq 0$ (respectively, such that $z_i(y, x) > 0$).*

It is easy to observe that M is a P_0- (P-, column sufficient, row sufficient, P_*-) matrix if and only if (I, M) is a P_0-pair (P, column sufficient, row sufficient, P_*-)pair. This makes it clear that the definitions above are rather natural extensions of the properties of square matrices to square matrix pairs. Furthermore, if $(\tilde{N}, \tilde{M}) \sim (N, M)$, then (N, M) is a P_0- (P-, column sufficient, P_*-)pair if and only if $(\tilde{N}, \tilde{M})$ is a P_0- (P-, column sufficient, P_*-)pair. This statement follows immediately from Proposition 3.5.2. Using Proposition 3.5.3 we easily see that the statement also holds for row sufficient pairs, i.e., if $(\tilde{N}, \tilde{M}) \sim (N, M)$ then (N, M) is a row sufficient pair if and only if $(\tilde{N}, \tilde{M})$ is row sufficient.

Now, we show that HLCPs defined by P_0-pairs are reducible:

Theorem 3.5.2 *If (N, M) is a P_0-pair, then the HLCP (N, M, q) is reducible.*

Proof:

(N, M) is a P_0-pair if and only if there is no vector pair (y, x) satisfying $Ny - Mx = 0$ and such that for all i either $y_i = x_i = 0$ or $y_i x_i < 0$. For such (y, x) there must exist a positive diagonal matrix D such that $x = -Dy$. Thus (N, M) is a P_0-pair if and only if the system $Ny - Mx = 0$, $Dy + x = 0$, has no nonzero solution for all diagonal $D > 0$. This is true if and only if

$$\det \begin{bmatrix} N & -M \\ D & I \end{bmatrix} \neq 0$$

for all diagonal $D > 0$. But this determinant is just $\det(MD + N)$, so that (N, M) is a P_0-pair if and only if $\det(MD + N) \neq 0$ for all positive diagonal matrices D. Now, using Theorem 3.4.1(iii) we conclude that the HLCP (N, M, q) is reducible when (N, M) is a P_0-pair. $\square$

Remark 3.5.1 *In the proof of Theorem 3.5.2, we have shown that* (N, M) *is a* P_0*-pair if and only if* $\det(MD + N) \neq 0$ *for all positive diagonal matrices* D*. This is precisely one of the equivalent conditions given by Willson to define* W_0*-pairs [Wil71]. Therefore, the* P_0*-property for the matrix pairs is actually equivalent to the* W_0*-property. Consequently, the second one of the equivalent conditions given in [Wil71] verifies the result of Theorem 3.5.2.*

Remark 3.5.2 *As in the case of single matrices, classes of matrix pairs exhibit interesting inclusion properties. Indeed, all the inclusion relations we described for classes of matrices extend identically to the corresponding matrix pairs. Consequently,* P_0*-pairs include the other matrix pair classes we defined above. Hence, HLCPs defined by* P*-pairs, column sufficient pairs, row sufficient pairs, and* P_**-pairs are also reducible as a result of Theorem 3.5.2.*

As we observed above, one nice feature of the reduction algorithm presented in the previous section is that it maintains the properties of the matrix pairs while generating equivalent problems. Moreover, if the HLCP (N, M, q) is reduced to the LCP (M', q'), then (N, M) is a P_0- (P-, column sufficient, row sufficient, P_*-)pair if and only if M' is a P_0- (P-, column sufficient, row sufficient, P_*-)matrix. These results follow immediately from Propositions 3.5.2 and 3.5.3. Therefore, analogs of the results of Theorem 3.5.1 hold when the matrix M is replaced with the matrix pair (N, M) and the LCP (M, q) is replaced with the HLCP (N, M, q). (One direction of the "if and only if" statements requires a little more proof; this is provided in the cited references.) We state them for the sake of completeness:

Theorem 3.5.3 *Let $N, M \in \Re^{n\times n}$.*

1. *If (N, M) is a monotone pair then the HLCP (N, M, q) is solvable whenever it is feasible.*
2. *HLCP (N, M, q) has a unique solution for all $q \in \Re^n$ if and only if (N, M) is a P-pair [STW58].*
3. *(N, M) is a column sufficient pair if and only if the HLCP (N, M, q) has a (possibly empty) convex solution set for every $q \in \Re^n$ [MP95].*
4. *(N, M) is a row sufficient pair if and only if for every vector $q \in \Re^n$ for which the HLCP (N, M, q) is feasible, every Karush-Kuhn-Tucker vector of the bilinear program*

$$\begin{array}{llcl} \min_{x,y} & x^T y & & \\ & Ny \quad -Mx & \geq & -q \\ & y, \qquad\quad x & \geq & 0, \end{array} \tag{3.16}$$

   solves the HLCP (N, M, q) [MP95].

5. *If (N, M) is a $P_*(\kappa)$-pair, then by an interior-point algorithm one can either obtain a solution $(y, x) \geq 0$ to the HLCP (N, M, q) such that $x^T y < \epsilon$ and*

$\|Ny - Mx - q\| < \epsilon$ or show that the problem is not solvable in $\mathcal{O}((1+\kappa)^2 n \ln \frac{1}{\epsilon})$ iterations. $\square$