Energy and uniqueness

The aim of this note is to show you a strategy in order to derive a uniqueness result for a PDEs problem by using the *energy* of the problem.

Example: the heat equation.
Let us consider a function u that satisfies the problem:

$$\begin{cases} u_t - Du_{xx} = 0 & \text{in } [0,L]\times(0,\infty)\,, \\ u(0,x) = g(x) & \text{for } x\in[0,L]\,, \\ u(0,t) = u(L,t) = 0 & \text{for } t>0\,. \end{cases} \tag{1}$$

By multiplying the fist equation by u, we get

$$uu_t - Duu_{xx} = 0\,.$$

If we now integrate both sides of the above equality in $[0,L]$, we get

$$\int_0^L uu_t\,\mathrm{d}x - D\int_0^L uu_{xx}\,\mathrm{d}x = 0\,. \tag{2}$$

Now

$$\int_0^L uu_t\,\mathrm{d}x = \frac{1}{2}\int_0^L \frac{\mathrm{d}}{\mathrm{d}t}(u^2)\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_0^L u^2\,\mathrm{d}x\right)\,,$$

while, by integrating by parts the second term of (2), we get

$$\int_0^L uu_{xx}\,\mathrm{d}x = uu_x\Big|_0^L - \int_0^L u_x^2\,\mathrm{d}x = -\int_0^L u_x^2\,\mathrm{d}x\,.$$

where in the last step we used the boundary conditions that u satisfies. Thus, from (2), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_0^L u^2\,\mathrm{d}x\right) = -D\int_0^L u_x^2\,\mathrm{d}x \le 0\,.$$

By calling

$$\mathcal{E}(t) := \frac{1}{2}\int_0^L u^2(x,t)\,\mathrm{d}x\,,$$

the *energy* of the system (relative to u), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \le 0\,,$$

that is, the energy is non-increasing in time. Notice that

$$\mathcal{E}(0) = \frac{1}{2}\int_0^L g^2(x)\,\mathrm{d}x\,.$$

Now, we would like to use the energy of the system in order to prove that (1) admits a unique solution. For, let us now take u, v be two solutions of (1) and consider the function $w := u - v$. Then, by using the **linearity** of the heat equation and the **homogeneity** of the boundary conditions, we obtain that w satisfies

$$\begin{cases} w_t - Dw_{xx} = 0 & \text{in } [0,L]\times(0,\infty)\,, \\ w(0,x) = 0 & \text{for } x\in[0,L]\,, \\ w(0,t) = w(L,t) = 0 & \text{for } t>0\,. \end{cases} \tag{3}$$

Notice that the previous computations are valid for any function satisfying a system like (1) or (3). In particular, they are valid for w.

So, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_0^L w^2(x,t)\,\mathrm{d}x\right) \leq 0\,,$$

and

$$\int_0^L w^2(x,0)\,\mathrm{d}x = 0\,,$$

since the initial data is the null function. Since clearly

$$\int_0^L w^2(x,t)\,\mathrm{d}x \geq 0\,,$$

for every $t \geq 0$, we obtain

$$\int_0^L w^2(x,t)\,\mathrm{d}x = 0\,,$$

for every $t \geq 0$. Since the function we are integrating is non-negative, this means that

$$w(x,t) = 0\,,$$

for all $x \in [0,L]$ and all $t \geq 0$. That is $u = v$ in $[0,L] \times [0,\infty)$. Thus, the problem (1) admits a **unique** solution!

The above strategy to prove uniqueness is called the **energy method** and can be applied for a wide variety of equations and with different boundary conditions.

Notice: we did **not** prove that the problem (1) admits a solution. We only showed that, **if** it has a solution, than this solution is unique! Indeed, our really first hypothesis was to assume that we do have a function u solving the problem.

Bottom line. Suppose we have a problem on a bounded interval I of the form

$$\begin{cases} \mathcal{L}u = f(x,t) & \text{on } I\,, \\ \text{initial conditions} & \text{on } I\,, \\ \text{boundary conditions} & \text{for } t > 0\,, \end{cases}$$

where the operator $\mathcal{L}$ is **linear** (*i.e.*, the PDE is **linear**) and $f : I \times (0,\infty) \to \mathbb{R}$ is a given function. Let u be a function solving the above problem (notice that we **assume** such a solution to exist!). If we

(i) multiply the equation by u in the case of the heat equation and by u_t in the case of the wave equation,
(ii) integrate both sides over the interval,
(iii) use integration by parts and the boundary condition to rewrite some terms,

we usually find a quantity of the form

$$\mathcal{E}(t) := \int_0^L e(x,t)\,\mathrm{d}x\,,$$

that is constant or non-increasing, and whose value at the time $t = 0$ depends only on the initial data of the problem. With these information in hand, it is usually possible to deduce an uniqueness theorem for the PDE in the following way: let u, v be two solution of the problem (*i.e.*, PDE+initial condition(s)+boundary conditions). Then, if we consider the function $w := u - v$, we have that w solves the PDE

$$\mathcal{L}w = 0\,,$$

thanks to the linearity of $\mathcal{L}$. Moreover, it will solve homogeneous initial conditions and homogeneous boundary conditions. Then, by considering the energy $\mathcal{E}$ for w, it is usually possible to deduce that $w \equiv 0$, that is, the problem has a **unique** solution.