

UNIQUENESS OF LIMIT MODELS IN CLASSES WITH AMALGAMATION

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ABSTRACT. We prove:

Main Theorem: *Let \mathcal{K} be an abstract elementary class satisfying the joint embedding and the amalgamation properties. Let μ be a cardinal above the Löwenheim-Skolem number of the class. Suppose \mathcal{K} satisfies the disjoint amalgamation property for limit models of cardinality μ . If \mathcal{K} is μ -Galois-stable, has no μ -Vaughtian Pairs, does not have long splitting chains, and satisfies locality of splitting, then any two (μ, σ_ℓ) -limits over M for $(\ell \in \{1, 2\})$ are isomorphic over M .*

This theorem extends results of Shelah from [Sh 394], [Sh 576], [Sh 600], Kolman and Shelah in [KoSh] and Shelah and Villaveces from [ShVi]. A preliminary version of our uniqueness theorem was used by Grossberg and VanDieren to prove a case of Shelah’s categoricity conjecture for tame abstract elementary classes in [GrVa2].

1. INTRODUCTION

We work in the general context of abstract elementary classes (AECs) with the amalgamation property (AP), the disjoint amalgamation property, the joint embedding property (JEP), and Galois-stability at one fixed cardinality μ above the Löwenheim-Skolem number. We prove the uniqueness of limit models under a unidimensionality-like assumption of no μ -Vaughtian pairs and superstability-like assumptions of the μ -splitting dependence relation.

The basic model theory of abstract elementary classes (definitions, the role of the AP and the JEP, the existence of a “monster model” \mathfrak{C} , Galois types and the foundational development of stability theory in that context) can be checked in the monograph [Gr2] and the books [Ba], [Sh i]. For the

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sake of completeness, we include some of the fundamentals of this context here.

In 1977, Shelah, building on the work of Fraïssé and Jónsson, identified a non-elementary context in which a model theoretic analysis could be carried out. Shelah began to study classes of models, together with a partial ordering of the class, which exhibit many of the properties that the models of a first order theory have with respect to the elementary submodel relation. Such classes were named abstract elementary classes. They include classes of models axiomatizable in $L_{\omega_1, \omega}(\mathbf{Q})$. Both classification theory and stability theory may be carried out to some extent within these classes. One strong advantage is that there are no a priori compactness assumptions. We reproduce the definition here.

Definition 1.1. Let \mathcal{K} be a class of structures all in the same similarity type $L(\mathcal{K})$, and let $\prec_{\mathcal{K}}$ be a partial order on \mathcal{K} . The ordered pair $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ is an *abstract elementary class, AEC for short* iff

A0 (Closure under isomorphism)

- (a) For every $M \in \mathcal{K}$ and every $L(\mathcal{K})$ -structure N if $M \cong N$ then $N \in \mathcal{K}$.
- (b) Let $N_1, N_2 \in \mathcal{K}$ and $M_1, M_2 \in \mathcal{K}$ such that there exist $f_l : N_l \cong M_l$ (for $l = 1, 2$) satisfying $f_1 \subseteq f_2$ then $N_1 \prec_{\mathcal{K}} N_2$ implies that $M_1 \prec_{\mathcal{K}} M_2$.

A1 For all $M, N \in \mathcal{K}$ if $M \prec_{\mathcal{K}} N$ then $M \subseteq N$.

A2 Let M, N, M^* be $L(\mathcal{K})$ -structures in \mathcal{K} . If $M \subseteq N$, $M \prec_{\mathcal{K}} M^*$ and $N \prec_{\mathcal{K}} M^*$, then $M \prec_{\mathcal{K}} N$.

A3 (Downward Löwenheim-Skolem) $LS(\mathcal{K})$ is the minimal infinite cardinal $\geq |L(\mathcal{K})|$ such that for every $M \in \mathcal{K}$ and for every $A \subseteq |M|$ there exists $N \in \mathcal{K}$ such that $N \prec_{\mathcal{K}} M$, $|N| \supseteq A$ and $\|N\| \leq |A| + LS(\mathcal{K})$.

A4 (Tarski-Vaught Chain)

- (a) For every regular cardinal μ and every $N \in \mathcal{K}$ if $\langle M_i \in \mathcal{K} \mid M_i \prec_{\mathcal{K}} N, i < \mu \rangle$ is $\prec_{\mathcal{K}}$ -increasing (i.e. $i < j \implies M_i \prec_{\mathcal{K}} M_j$) then $\bigcup_{i < \mu} M_i \in \mathcal{K}$ and $\bigcup_{i < \mu} M_i \prec_{\mathcal{K}} N$.
- (b) For every regular μ , if $\langle M_i \in \mathcal{K}_\mu \mid i < \mu \rangle$ is $\prec_{\mathcal{K}}$ -increasing then $\bigcup_{i < \mu} M_i \in \mathcal{K}$ and $M_0 \prec_{\mathcal{K}} \bigcup_{i < \mu} M_i$.

For M and $N \in \mathcal{K}$ a monomorphism $f : M \rightarrow N$ is called an \mathcal{K} -embedding iff $f[M] \prec_{\mathcal{K}} N$. Thus, $M \prec_{\mathcal{K}} N$ is equivalent to “ id_M is a \mathcal{K} -embedding from M into N .”

For $M_0 \prec_{\mathcal{K}} M_1$ and $N \in \mathcal{K}$, the formula $f : M_1 \xrightarrow{M_0} N$ stands for f is a \mathcal{K} -embedding such that $f \upharpoonright M_0 = \text{id}_{M_0}$.

For a class \mathcal{K} and a cardinal $\mu \geq \text{LS}(\mathcal{K})$ let

$$\mathcal{K}_\mu := \{M \in \mathcal{K} : \|M\| = \mu\}.$$

In practice, abstract elementary classes were not as approachable as one would hope and much work in non-elementary model theory takes place in contexts which additionally satisfy the amalgamation property:

Definition 1.2. Let $\mu \geq \text{LS}(\mathcal{K})$. We say that \mathcal{K} has the μ -*amalgamation property* (μ -AP) iff for any $M_\ell \in \mathcal{K}_\mu$ (for $\ell \in \{0, 1, 2\}$) such that $M_0 \prec_{\mathcal{K}} M_1$ and $M_0 \prec_{\mathcal{K}} M_2$ there are $N \in \mathcal{K}_\mu$ and \mathcal{K} -embeddings $f_\ell : M_\ell \rightarrow N$ such that $f_\ell \upharpoonright M_0 = \text{id}_{M_0}$ for $\ell = 1, 2$.

We say that \mathcal{K} has the *amalgamation property* (AP) iff any triple of models from $\mathcal{K}_{\geq \text{LS}(\mathcal{K})}$ can be amalgamated.

- Remark 1.3.** (1) Using the isomorphism axioms we can see that \mathcal{K} has the λ -AP iff for any $M_\ell \in \mathcal{K}_\lambda$ (for $\ell \in \{0, 1, 2\}$) such that $M_0 \prec_{\mathcal{K}} M_\ell$ (for $\ell \in \{1, 2\}$) there are $N \in \mathcal{K}_\lambda$ and $f : M_1 \xrightarrow{M_0} N$ such that $N \succ_{\mathcal{K}} M_2$.
- (2) Using the axioms of AECs it is not difficult to prove that if \mathcal{K} has the λ -AP for every $\lambda \geq \text{LS}(\mathcal{K})$ then \mathcal{K} has the AP.

The roots of the following fact can be traced back to Jónsson's 1960 paper [Jo]; the present formulation is from [Gr1]:

Fact 1.4. Let $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ be an AEC with no maximal models and suppose that there is $\lambda \geq \kappa > \text{LS}(\mathcal{K})$ such that $\mathcal{K}_{< \lambda}$ has the AP and the JEP. Suppose $M \in \mathcal{K}$. If $\lambda^{< \kappa} = \lambda \geq \|M\|$ then there exists $N \succ M$ of cardinality λ which is κ -model-homogeneous.

Thus if an AEC \mathcal{K} has AP and JEP, then like in first-order stability theory we may assume that there is a large model-homogeneous $\mathfrak{C} \in \mathcal{K}$ that acts like a monster model.

We will refer to the model \mathfrak{C} as the *monster model*. All models considered will be of size less than $\|\mathfrak{C}\|$, and we will find realizations of types we construct inside this monster model. From now on, we assume that the monster model \mathfrak{C} has been fixed. We use the notation $\text{Aut}_M(\mathfrak{C})$ to denote the set of automorphisms of \mathfrak{C} fixing M pointwise.

The notion of type as a set of formulas, even when the class is described in some infinitary logic, does not behave as nicely as in first-order logic. A replacement was introduced by Shelah in [Sh 394]. In order to avoid confusion between this and the classical, syntactic notion, we will use the terminology in [Gr2] and call this alternative notion the *Galois type*.

Since in this paper we deal only with AECs with the AP property, the notion of Galois type has a simpler definition than in the general case.

Definition 1.5 (Galois types). Suppose that \mathcal{K} has the AP.

- (1) Given $M \in \mathcal{K}$ consider the action of $\text{Aut}_M(\mathfrak{C})$ on \mathfrak{C} , for an element $a \in |\mathfrak{C}|$ let $\text{ga-tp}(a/M)$ denote the *Galois type of a over M* which is defined as the orbit of a under $\text{Aut}_M(\mathfrak{C})$.
- (2) For $M \in \mathcal{K}$, we let

$$\text{ga-S}(M) = \{ \text{ga-tp}(a/M) : a \in |\mathfrak{C}| \}.$$

- (3) \mathcal{K} is λ -Galois-stable iff

$$N \in \mathcal{K}_\lambda \implies |\text{ga-S}(N)| \leq \lambda.$$

- (4) Given $p \in \text{ga-S}(M)$ and $N \in \mathcal{K}$ such that $N \succ_{\mathcal{K}} M$, we say that p is *realized* by $a \in N$ iff $\text{ga-tp}(a/M) = p$. Just as in the first-order case we will write $a \models p$ when a is a realization of p .
- (5) For $h \in \text{Aut}(\mathfrak{C})$ and $p = \text{ga-tp}(a/M)$, then the notation $h(p)$ refers to $\text{ga-tp}(h(a)/h(M))$.

For a more detailed discussion of Galois types, their extensions, restrictions, equivalent forms and generalizations, the reader may consult [Gr2].

While the amalgamation property is useful for dealing with Galois types, in this paper we require a stronger version of AP for one of the steps in the proof of the uniqueness of limit models. Specifically, we use μ -disjoint amalgamation over limit models to prove that relatively full towers are limit models (Theorem 4).

Definition 1.6. Let \mathcal{K} be an abstract elementary class. \mathcal{K} has the μ -disjoint amalgamation property (μ -DAP) iff for every $M_\ell \in \mathcal{K}_\mu$ (for $\ell = 0, 1, 2$) such that $M_0 \prec_{\mathcal{K}} M_\ell$ (for $\ell = 1, 2$) there are $N \in \mathcal{K}_\mu$ which is a \mathcal{K} -extension of M_2 and a \mathcal{K} -embedding $f : M_1 \xrightarrow{M_0} N$ such that $f[M_1] \cap M_2 = M_0$.

We say that a class has the *disjoint amalgamation property* iff it has the μ -disjoint amalgamation property for every $\mu \geq \text{LS}(\mathcal{K})$. We write DAP for

short. In this paper we only require that disjoint amalgamation hold for the subclass of all limit models of \mathcal{K}_μ .

The next notion to consider is that of a saturated model. In homogeneous abstract elementary classes (see, for example, [GrLe]) where one may study classes of models omitting given sets of types, the existence of a saturated model presents some problems. One solution is to consider models which realize as many types as possible. Such models are called Galois-saturated. More formally, a model M of size $\kappa > \text{LS}(\mathcal{K})$ is *Galois-saturated* if it realizes all Galois types over submodels $N \prec_{\mathcal{K}} M$ of cardinality $< \kappa$. When stability theory has been ported to contexts more general than first order logic, many situations have appeared when Galois-saturated models do not fulfill the main roles that saturated models play in elementary classes.

The main concept of this paper is Shelah's *limit model* which (among other things) serves as a substitute for the role of saturation in stability theory (see [Gr2],[ShVi],[Sh i], etc.) or at least serves as a stepping stone to prove the properties of Galois-saturated models. For example, under the assumption of categoricity with reasonable stability conditions, the existence of Galois-saturated models in singular cardinals is not straightforward and is proved by first considering limit models [Sh 394]. In some contexts limit models have been successfully used as “tools” towards finding Galois-saturated models ([KoSh] and [Sh 472]). Furthermore, the notion of limit model refines the notion of saturation; more detailed information is given on the particular way one model is embedded inside another.

Limit models appear in [KoSh] and in [Sh 576] under the name (μ, α) -saturated models. In [Sh 600], Shelah calls this notion *brimmed*. Later papers, beginning with Shelah-Villaveces [ShVi], adopt the name *limit models*. We use the more recent terminology. Before defining limit models, we must introduce their building blocks, universal extensions.

Definition 1.7. (1) Let κ be a cardinal $\geq \text{LS}(\mathcal{K})$. We say $M^* \succ_{\mathcal{K}} N$ is κ -universal over N iff for every $N' \in \mathcal{K}_\kappa$ with $N \prec_{\mathcal{K}} N'$ there exists a \mathcal{K} -embedding $g : N' \xrightarrow[N]{} M^*$ such that the following diagram commutes:

$$\begin{array}{ccc}
N' & & \\
\text{id} \uparrow & \searrow g & \\
N & \xrightarrow{\text{id}} & M^*
\end{array}$$

- (2) We say M^* is *universal over N* or M^* is a *universal extension of N* iff M^* is $\|N\|$ -universal over N .

Definition 1.8. [Limit models] Consider $\mu \geq \text{LS}(\mathcal{K})$ and $\alpha < \mu^+$ a limit ordinal and $N \in \mathcal{K}_\mu$. We say that M is (μ, α) -*limit model over N* iff there exists an increasing and continuous chain $\langle M_i \in \mathcal{K}_\mu \mid i < \alpha \rangle$ such that $M_0 = N$; $M = \bigcup_{i < \alpha} M_i$; M_i is a proper \mathcal{K} -submodel of M_{i+1} ; and M_{i+1} is universal over M_i for all $i < \alpha$.

From Theorem 1 we get that for $\alpha \leq \mu^+$ there always exists a (μ, α) -limit model provided \mathcal{K} has the AP, has no maximal models and is μ -Galois-stable. This theorem was stated without proof as Claim 1.16 in [Sh 600], for a proof see [GrVa1] or [Gr1].

Theorem 1 (Existence). *Let \mathcal{K} be an AEC without maximal models and suppose it is Galois-stable in μ . If \mathcal{K} has the amalgamation property then for every $N \in \mathcal{K}_\mu$ there exists $M^* \succeq_{\mathcal{K}} N$, universal over N of cardinality μ .*

The following theorem partially clarifies the analogy with saturated models:

Theorem 2. *Let T be a stable, complete, first-order theory and let \mathcal{K} be the elementary class of models of T with the usual notion of elementary submodel. If M is a (μ, δ) -limit model for δ a limit ordinal with $\text{cf}(\delta) \geq \kappa(T)$, then M is saturated.*

Proof. Use an argument similar to the proof of [Sh e, Theorem III 3.11]. \dashv

Thus in elementary classes superstability implies that limit models are saturated, in particular are unique. This raises the following natural question for AECs:

Question 1.9 (Uniqueness problem). *Let \mathcal{K} be an AEC, $\mu \geq \text{LS}(\mathcal{K})$, $M \in \mathcal{K}_\mu$ and σ_1, σ_2 limit ordinals $< \mu^+$, and suppose that for $\ell = 1, 2$, N_ℓ is a (μ, σ_ℓ) -limit model over M . What “reasonable” assumptions on \mathcal{K} will imply that $\exists f : N_1 \cong_M N_2$?*

Question 1.9 is non-trivial only for the case where $\text{cf}(\sigma_1) \neq \text{cf}(\sigma_2)$. Using a back and forth argument one can show that when $\text{cf}(\sigma_1) = \text{cf}(\sigma_2)$, we get uniqueness without any assumptions on \mathcal{K} . More precisely:

Fact 1.10. *Let $\mu \geq \text{LS}(\mathcal{K})$ and $\sigma < \mu^+$. If M_1 and M_2 are (μ, σ) -limits over M , then there exists an isomorphism $g : M_1 \rightarrow M_2$ such that $g \upharpoonright M = \text{id}_M$. Moreover if M_1 is a (μ, σ) -limit over M_0 , if N_1 is a (μ, σ) -limit over N_0 and if $g : M_0 \cong N_0$, then there exists a \mathcal{K} -embedding, \hat{g} , extending g such that $\hat{g} : M_1 \cong N_1$.*

Fact 1.11. *Let μ be a cardinal and σ a limit ordinal with $\sigma < \mu^+$. If M is a (μ, σ) -limit model, then M is a $(\mu, \text{cf}(\sigma))$ -limit model.*

The main result of this paper provides an answer to Question 1.9.

Theorem 3 (Main Theorem). *Let \mathcal{K} be an AEC without maximal models, and $\mu > \text{LS}(\mathcal{K})$. Suppose \mathcal{K} satisfies the AP and JEP and the subclass of limit models of \mathcal{K} satisfies μ -DAP. If \mathcal{K} is μ -Galois-stable, does not have long splitting chains, has no μ -Vaughtian pairs and satisfies locality of splitting¹, then any two (μ, σ_ℓ) -limits over M for $(\ell \in \{1, 2\})$ are isomorphic over M .*

Remark 1.12. After reading preprints of this paper, Fred Drueck in his Ph.D. thesis [Dr] pointed out that the disjoint amalgamation property is not necessary to carry out the arguments here. In particular, it is not needed in Theorem 4. We leave the assumption in this paper for historical accuracy.

The last section of this paper (see pages 23 and ff.) describes different approaches to Question 1.9.

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2. THE SETTING

In what follows, \mathcal{K} is assumed to be an AEC, and μ is a cardinal $\geq \text{LS}(\mathcal{K})$. In this section we summarize all of the assumptions that will be made on the class \mathcal{K} , and in the subsequent sections we introduce two of the main components of the proof of the uniqueness of limit models: strong types and towers.

¹See Assumption 2.8 for the precise description of long splitting chains and locality.

We will prove the uniqueness of limit models in μ -Galois stable AECs that are essentially unidimensional and are equipped with a moderately well-behaved dependence relation. We will use μ -splitting as the dependence relation, but any dependence relation which is local and has existence, uniqueness and extension properties suffices.

Definition 2.1. A type $p \in \text{ga-S}(M)$ μ -splits over $N \in \mathcal{K}_{\leq \mu}$ if and only if N is a $\prec_{\mathcal{K}}$ -submodel of M and there exist $N_1, N_2 \in \mathcal{K}_{\mu}$ and a \mathcal{K} -mapping h such that $N \prec_{\mathcal{K}} N_l \prec_{\mathcal{K}} M$ for $l = 1, 2$ and $h : N_1 \rightarrow N_2$ with $h \upharpoonright N = \text{id}_N$ and $p \upharpoonright N_2 \neq h(p \upharpoonright N_1)$.

The existence property for non- μ -splitting types follows from Galois stability in μ :

Fact 2.2 (Existence - Claim 3.3 of [Sh 394]). *Assume \mathcal{K} has AP and is Galois-stable in μ . For every $M \in \mathcal{K}_{\geq \mu}$ and $p \in \text{ga-S}(M)$, there exists $N \in \mathcal{K}_{\mu}$ such that p does not μ -split over N .*

The uniqueness and extension properties of non- μ -splitting types hold for types over limit models:

Fact 2.3 (Uniqueness - Theorem I.4.15 of [Va1]). *Let $N \prec_{\mathcal{K}} M \prec_{\mathcal{K}} M'$ be models in \mathcal{K}_{μ} such that M' is universal over M and M is universal over N . If $p \in \text{ga-S}(M)$ does not μ -split over N , then there is a unique $p' \in \text{ga-S}(M')$ such that p' extends p and p' does not μ -split over N .*

A variation of this fact is later used in an induction construction in the proof of Theorem 6. We state it explicitly here:

Fact 2.4 (Theorem I.4.10 of [Va1]). *Let M, N, M^* be models in \mathcal{K}_{μ} . Suppose that M is universal over N and that M^* is an extension of M . If a type $p = \text{ga-tp}(a/M)$ does not μ -split over N then there exists an automorphism g of \mathfrak{C} fixing M such that $\text{ga-tp}(g(a)/M^*)$ does not μ -split over N and $\text{ga-tp}(g(a)/M) = p$.*

The other concepts that show up in the assumptions of the main theorem of this paper are minimal types and μ -Vaughtian Pairs.

Definition 2.5. (1) For M a model of cardinality μ , $p \in \text{ga-S}(M)$ is *minimal* if it is non-algebraic and for each N extending M of cardinality μ there is a unique non-algebraic extension of p to N .

- (2) For M a limit model of cardinality μ a μ -Vaughtian Pair is a pair of models M' and N' of cardinality μ if $M \preceq_{\mathcal{K}} M' \prec_{\mathcal{K}} N'$ and if there exists $p \in \text{ga-S}(M)$ a minimal type so that N' contains no new realizations of p , in other words, $p(M') = p(N')$.

Fact 2.6 (Existence of minimal types - reference [Sh 394]). *Let $\mu > \text{LS}(\mathcal{K})$. If \mathcal{K} is Galois-stable in μ , then for every $M \in \mathcal{K}_{\mu}$ and every $q \in \text{ga-S}(M)$, there are $N \in \mathcal{K}_{\mu}$ and $p \in \text{ga-S}(N)$ such that $M \preceq_{\mathcal{K}} N$, $q \leq p$ and p is minimal.*

Fact 2.7 (Claim $(*)_8$ of Theorem 9.7 of [Sh 394]). *If \mathcal{K} is categorical in some successor cardinal $\lambda^+ > \text{LS}(\mathcal{K})^+$, then for every μ satisfying $\text{LS}(\mathcal{K}) \leq \mu \leq \lambda$, there are no μ -Vaughtian Pairs.*

It is worth mentioning that our “no μ -Vaughtian pairs” assumption is much weaker in general than assuming categoricity (as in earlier version of the proof): even in First Order, theories such as the theory of Real Closed Fields are quite far from being categorical but also have no Vaughtian pairs. Of course, under ω -stability, no Vaughtian pairs and categoricity are equivalent (in First Order). But our stability assumptions are of “superstable” nature - under these, categoricity is quite stronger than no μ -Vaughtian pairs.

Here are the assumptions of the paper:

Assumption 2.8. \mathcal{K} is an AEC with the μ -DAP² over limit models and JEP, and \mathcal{K} satisfies the following properties:

- (1) All models are submodels of a fixed monster model \mathfrak{C} .³
- (2) \mathcal{K} is stable in μ .
- (3) There are no μ -Vaughtian Pairs.
- (4) μ -splitting in \mathcal{K} satisfies the following locality (sometimes called continuity) and “no long splitting chains” properties.

For all infinite α , for every sequence $\langle M_i \mid i < \alpha \rangle$ of limit models of cardinality μ and for every $p \in \text{ga-S}(M_{\alpha})$, where $M_{\alpha} = \bigcup_{i < \alpha} M_i$, we have that

- (a) If for every $i < \alpha$, the type $p \upharpoonright M_i$ does not μ -split over M_0 , then p does not μ -split over M_0 .
- (b) There exists $i < \alpha$ such that p does not μ -split over M_i .

²See Remark 1.12 which indicates only the amalgamation property is necessary.

³Notice that this already implies the full AP.

In the context of an AEC with the full amalgamation property and JEP, categoricity in a cardinal $\lambda > \mu$ implies all parts of Assumption 2.8. For a proof of Assumption 2.8.2 from categoricity, see Claim 1.7 of [Sh 394] or [Ba]. The observation that assumption 2.8(4a) follows from categoricity is a consequence of Observation 6.2 and Main Lemma 9.4 of [Sh 394]. Lemma 6.3 of [Sh 394] is the statement that assumption 2.8(4b) follows from categoricity when the cofinality of the categoricity cardinal is larger than μ .

Assumption 2.8 also holds in contexts without the assumption of categoricity. First let us consider μ -DAP. The μ -DAP over limit models holds for free in first order classes of the form $(Mod(T), \prec)$ for complete T . As the referee has pointed out, in AECs, μ -DAP does not generally hold over arbitrary models. Consider the class \mathcal{K} of structures with two sorts U and V and a binary relation $<$ on U such that for each model M , U^M is well-ordered by $<^M$ with order type at most ω , V^M is empty when U^M is finite and if non-empty, V^M is infinite. By defining $\prec_{\mathcal{K}}$ by $M \prec_{\mathcal{K}} N$ iff U^M is an initial segment of U^N and $V^M \subset V^N$, we get an AEC with $LS(\mathcal{K})$ equal to \aleph_0 . \mathcal{K} satisfies the AP and JEP and is \aleph_1 -categorical. It fails to have the \aleph_0 -DAP yet has \aleph_0 -DAP over limit models.

However, there are AECs in which μ -DAP does hold. DAP holds in homogeneous classes (see [Sh 3] or [Po]), in excellent classes (see [Sh 87b]) and is an axiom in the definition of finitary classes (see [HyKe]). It also holds for cats consisting of existentially closed models of positive Robinson theories ([Za]). In each of these contexts dependence relations satisfying Assumption 2.8 have been developed. Finally, the locality and existence of non- μ -splitting extensions are akin to consequences of superstability in first order logic.

Fact 2.9 (“No long splitting chains” follows from stability in FO). *Suppose that T is first order complete. If T is stable then Assumption 2.8(4b) holds for α such that $\text{cf}(\alpha) \geq |T|^+$.*

Proof. Let $\langle M_i | i \leq \alpha \rangle$ be an increasing sequence of saturated models and $p \in S(M_\alpha)$ be such that $\forall i < \alpha$, p μ -splits over M_i . Let $\varphi_i(\bar{x}, \bar{y})$ be a formula witnessing the splitting of $p \upharpoonright M_{i+1}$ over M_i . As $\text{cf}(\alpha) \geq |T|^+$, there exists $S \subset \alpha$ infinite such that $i, j \in S \Rightarrow \varphi_i = \varphi_j$.

Without loss of generality, suppose that $\langle M_n | n \leq \omega \rangle$ is an increasing sequence of saturated models, and $p \in S_\varphi(M_\omega)$ is such that $\bar{a}_i, \bar{b}_i \in M_{i+1}$ witness that $p \upharpoonright M_{i+1}$ splits over M_i . Then $p(x_1, \bar{y}_1, \bar{z}_1, x_2, \bar{y}_2, \bar{z}_2)$ and $\{\bar{d}_i | i <$

ω witness that p has the order property, where $\bar{d}_i = \bar{a}_i \hat{\ } \bar{b}_i \hat{\ } c_i$, $c_i \in M_{i+2}$ and

$$c_i \models p \upharpoonright \{\bar{a}_k, \bar{b}_k \mid k \leq i\} \cup \{d_k \mid k < i\}.$$

Now use [Gr1, Lemma VII, 2.12]. ⊣

3. STRONG TYPES

Under the assumption of μ -stability, we can define *strong types* as in [ShVi]. These strong types will allow us to achieve a better control of extensions of towers of models than what we obtain using just Galois types.

Definition 3.1 (Definition 3.2.1 of [ShVi]). For M a (μ, θ) -limit model, let

$$\mathfrak{St}(M) := \left\{ (p, N) \left| \begin{array}{l} N \prec_{\mathcal{K}} M; \\ N \text{ is a } (\mu, \theta)\text{-limit model}; \\ M \text{ is universal over } N; \\ p \in \text{ga-S}(M) \text{ is non-algebraic} \\ \text{and } p \text{ does not } \mu\text{-split over } N. \end{array} \right. \right\}$$

Elements of $\mathfrak{St}(M)$ are called *strong types*. Two strong types $(p_1, N_1) \in \mathfrak{St}(M_1)$ and $(p_2, N_2) \in \mathfrak{St}(M_2)$ are *parallel* iff for every M' of cardinality μ extending M_1 and M_2 there exists $q \in \text{ga-S}(M')$ such that q extends both p_1 and p_2 and q does not μ -split over N_1 nor over N_2 .

Remark 3.2. Under the assumption of the existence of universal extensions, it is equivalent to say two strong types $(p_1, N_1) \in \mathfrak{St}(M_1)$ and $(p_2, N_2) \in \mathfrak{St}(M_2)$ are parallel iff for some M' of cardinality μ universal over some common extension of M_1 and M_2 there exists $q \in \text{ga-S}(M')$ such that q extends both p_1 and p_2 and q does not μ -split over N_1 and N_2 .

Lemma 3.3 (Monotonicity of parallel types). *Suppose $M_0, M_1 \in \mathcal{K}_\mu$ and $M_0 \prec_{\mathcal{K}} M_1$ and $(p, N) \in \mathfrak{St}(M_1)$. If M_0 is universal over N , then we have $(p \upharpoonright M_0, N)$ is parallel to (p, N) .*

Proof. Straightforward using the uniqueness of non- μ -splitting extensions. ⊣

Notation 3.4. Let $M, M' \in \mathcal{K}_\mu$ and suppose that $M \prec_{\mathcal{K}} M'$. For $(p, N) \in \mathfrak{St}(M')$, if M is universal over N , we define the restriction $(p, N) \upharpoonright M \in \mathfrak{St}(M)$ to be $(p \upharpoonright M, N)$. If we write $(p, N) \upharpoonright M$, we mean that p does not μ -split over N and M is universal over N . We denote by \sim the parallelism relation between strong types in $\mathfrak{St}(M)$, for fixed M .

Notice that \sim is an equivalence relation on $\mathfrak{St}(M)$ (see [Va1]). Stability in μ implies that there are few strong types over any model of cardinality μ :

Fact 3.5 (Claim 3.2.2 (3) of [ShVi]). *If \mathcal{K} is Galois-stable in μ , then for any $M \in \mathcal{K}_\mu$, $|\mathfrak{St}(M)/\sim| \leq \mu$.*

4. TOWERS

To each (μ, θ) -limit model M we can naturally associate a $\prec_{\mathcal{K}}$ -increasing chain $\bar{M} = \langle M_i \in \mathcal{K}_\mu \mid i < \theta \rangle$ witnessing that M is a (μ, θ) -limit model (that is, $\bigcup_{i < \theta} M_i = M$ and M_{i+1} is universal over M_i). Furthermore, by Facts 1.10 and 1.11 we can require that this chain satisfies additional requirements such as M_{i+1} is a limit model over M_i . In this section we will be considering a related chain of models which we will refer to as a tower (see Definition 4.1). But first, we will describe how towers will be used to prove the main theorem of this paper.

To prove the uniqueness of limit models we will construct a model which is simultaneously a (μ, θ_1) -limit model over some fixed model M and a (μ, θ_2) -limit model over M . Notice that, by Fact 1.10, it is enough to construct a model M^* that is simultaneously a (μ, ω) -limit model and a (μ, θ) -limit model for arbitrary ordinal $\theta < \mu^+$. By Fact 1.11 we may assume that θ is a limit ordinal $< \mu^+$ such that $\theta = \mu \cdot \theta$.

So, we actually construct an array of models with $\omega + 1$ rows and the number of columns of this array will have the same cofinality as θ . See the big picture of the construction on page 22. We intend to carry out the construction **down** and **to the right** in that picture. In the array, the bottom right hand corner (M^*) will be a (μ, ω) -limit model witnessed by a chain of models as described in the first paragraph of this section. This chain will appear in the last column of the array. We will see that M^* is a (μ, θ) -limit model by examining the last (the ω^{th}) row of the array. This last row will be an $\prec_{\mathcal{K}}$ -increasing sequence of models, \bar{M}^* whose length will have the same cofinality as θ . However we will not be able to guarantee that M_{i+1}^* is universal over M_i^* in this last row. Thus we need another method to conclude that M^* is a (μ, θ) -limit model. This involves attaching more information to our sequence \bar{M}^* . We call this accessorized sequence of models a tower (see Definition 4.1 below). Each row in our construction of the array of models will be such a tower.

Under the assumption of Galois-superstability, given any sequence $\langle a_i \mid i < \theta \rangle$ of elements with $a_i \in M_{i+1} \setminus M_i$, we can identify some $N_i \prec_{\mathcal{K}} M_i$ such

that $\text{ga-tp}(a_i/M_i)$ does not μ -split over N_i . Furthermore, by Assumption 2.8, we may choose this N_i such that M_i is a limit model over N_i . We abbreviate this situation by the triple $(\bar{M}, \bar{a}, \bar{N})$.

Definition 4.1 (Towers). Let $(I, <)$ be a well ordering of cardinality $< \mu^+$. For cleaner notation, we will identify I with θ , its order-type, and we will denote the successor of i in the ordering I by $i + 1$ when it is clear. Then, we define a *tower* to be a triple $(\bar{M}, \bar{a}, \bar{N})$ where $\bar{M} = \langle M_i \mid i < \theta \rangle$ is a $\prec_{\mathcal{K}}$ -increasing sequence of limit models of cardinality μ ; $\bar{a} = \langle a_i \mid i + 1 < \theta \rangle$ and $\bar{N} = \langle N_i \mid i + 1 < \theta \rangle$ satisfy $a_i \in M_{i+1} \setminus M_i$; $\text{ga-tp}(a_i/M_i)$ does not μ -split over N_i ; and M_i is a (μ, σ) -limit model over N_i .

Notation 4.2. We denote by $\mathcal{K}_{\mu, I}^*$ the set of towers of the form $(\bar{M}, \bar{a}, \bar{N})$ where the sequences \bar{M} , \bar{a} and \bar{N} are indexed by I . Occasionally, I will be an ordinal θ with the usual ordering, and we write $\mathcal{K}_{\mu, \theta}^*$ for this set of towers. At times, we will be considering towers based on different well orderings I and I' simultaneously. In these contexts if $i \in I \cap I'$, the notation $i + 1$ is not necessarily well-defined so we will use the notation $\text{succ}_I(i)$ for the successor of i in the ordering I . Finally when I is a sub-order of I' for any $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I'}^*$ we write $(\bar{M}, \bar{a}, \bar{N}) \upharpoonright I$ for the tower in $\mathcal{K}_{\mu, I}^*$ given by the subsequences $\langle M_i \mid i \in I \rangle$, $\langle N_i \mid i \in I \rangle$ and $\langle a_i \mid i \in I \rangle$.

In addition to having control over the last row of the array, we also need to be able to guarantee that the last column of the tower witnesses that M^* is a (μ, ω) -limit model. This will be done by prescribing the following ordering on rows of the array:

Definition 4.3. For towers $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I}^*$ and $(\bar{M}', \bar{a}', \bar{N}') \in \mathcal{K}_{\mu, I'}^*$ with $I \subseteq I'$, we write $(\bar{M}, \bar{a}, \bar{N}) < (\bar{M}', \bar{a}', \bar{N}')$ if and only if for every $i \in I$, $a_i = a'_i$, $N_i = N'_i$ and M'_i is a proper universal extension of M_i .

Remark 4.4. The ordering $<$ on towers is identical to the ordering $<_{\mu}^c$ defined in [ShVi]. The superscript was used by Shelah and Villaveces to distinguish this ordering from others. We only use one ordering on towers, so we omit the superscripts and subscripts here.

Once we have established an ordering on towers, we can define a specific tower which will be called a *union of an increasing sequence of towers*. Suppose that $\langle (\bar{M}, \bar{a}, \bar{N})^{\gamma} \in \mathcal{K}_{\mu, I_{\gamma}}^* \mid \gamma < \beta \rangle$ is an increasing sequence of towers such that the index set I_{γ} of $(\bar{M}, \bar{a}, \bar{N})^{\gamma}$ is a sub-ordering of the

index set $I_{\gamma'}$ for $(\bar{M}, \bar{a}, \bar{N})^{\gamma'}$ whenever $\gamma < \gamma'$. Let $I_\beta := \bigcup_{\gamma < \beta} I_\gamma$. Then denote by $(\bar{M}, \bar{a}, \bar{N})^\beta \in \mathcal{K}_{\mu, I_\beta}^*$ the “union” of the sequence of towers where

$$\begin{aligned} a_i^\beta &= a_i^{\min\{\gamma | i \in I_\gamma\}}, \\ N_i^\beta &= N_i^{\min\{\gamma | i \in I_\gamma\}} \text{ and} \\ \bar{M}^\beta &= \langle M_i^\beta \mid i \in \bigcup_{\gamma < \beta} I_\gamma \rangle \text{ with } M_i^\beta = \bigcup_{\gamma < \beta} \bigcup_{I_\gamma \ni i} M_i^\gamma. \end{aligned}$$

By Assumption 2.8.4a, $(\bar{M}, \bar{a}, \bar{N})^\beta$ is indeed a tower.

Notice that we do not assume an individual tower to be continuous. Nor do we assume that inside of a tower M_{i+1} is universal over M_i . If one considers the approach of defining an array of models row by row, then generally (even in the first order case) even if all rows are continuous and satisfy the universality property mentioned in this paragraph, it is not necessarily true that the union of these rows will be a tower in which every model is universal over its predecessors.

For a tower $(\bar{M}, \bar{a}, \bar{N})$, it was shown in [ShVi], that even if M_{i+1} is not universal over M_i , one can conclude that $\bigcup_{i < \theta} M_i$ is a (μ, θ) -limit model provided that all types over each of the M_i are realized by a sufficient number of a_j s in the tower. Unfortunately constructing such a tower meeting these along with all of our other requirements is beyond reach. However, in [Va1], VanDieren showed that slightly less was needed (see Definition 4.5). In [Va1], the amalgamation property is not assumed resulting in noise that can be avoided in our context. Thus because we have at our disposal the AP, we provide a complete, undistracted proof here.

Definition 4.5 (Relatively Full Towers). Suppose that I is a well-ordered set such that there exists a cofinal sequence $\langle i_\alpha \mid \alpha < \theta \rangle$ of I of order type θ such that there are $\mu \cdot \omega$ many elements between i_α and $i_{\alpha+1}$.

Let $(\bar{M}, \bar{a}, \bar{N})$ be a tower indexed by I such that each M_i is a (μ, σ) -limit model. For each i , let $\langle M_i^\gamma \mid \gamma < \sigma \rangle$ witness that M_i is a (μ, σ) -limit model. The tower $(\bar{M}, \bar{a}, \bar{N})$ is *full relative to* $(M_i^\gamma)_{\gamma < \sigma, i \in I}$ iff for every $\gamma < \sigma$ and every $(p, M_i^\gamma) \in \mathfrak{St}(M_i)$ with $i_\alpha \leq i < i_{\alpha+1}$, there exists $j \in I$ with $i \leq j < i_{\alpha+1}$ such that $(\text{ga-tp}(a_j/M_j), N_j)$ and (p, M_i^γ) are parallel.

Relative fullness of towers can be seen as a (weak) form of “eventual Galois saturation.” Along a full tower, all strong Galois types over members of sequences - sequences which witness the fact that along the tower the models are limits - end up being realized (modulo parallelism) by an element a_j of

the tower. As we see in our proof, this property is much more flexible than regular Galois-saturation - it could be regarded as a “dynamic” and robust version.

Although relatively full towers are used here as a technical device for the proof, the crucial property is that “eventual” or “dynamic” relative Galois saturation. These objects have variously been used by Shelah in various places, Shelah-Villaveces [ShVi], VanDieren [Va1], and other authors. It is reasonable to say that the notion of relatively full towers has potential for other uses outside of these works.

Theorem 4 (Relatively full towers provide limit models). *Let θ be a limit ordinal $< \mu^+$ satisfying $\theta = \mu \cdot \theta$. Suppose that I is a well-ordered set as in Definition 4.5.*

Let $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I}^$ be a tower made up of (μ, σ) -limit models, for some fixed $\sigma < \mu^+$. If $(M, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I}^*$ is full relative to $(M_i^\gamma)_{i \in I, \gamma < \sigma}$, then $M := \bigcup_{i \in I} M_i$ is a (μ, θ) -limit model.*

Proof. Without loss of generality we may assume that \bar{M} is continuous. Let M' be a (μ, θ) -limit model over M_{i_0} witnessed by $\langle M'_\alpha \mid \alpha < \theta \rangle$. By μ -DAP over limit models, we may assume that $M' \cap M = M_{i_0}$. Since $\theta = \mu \cdot \theta$, we may also arrange things so that the universe of M'_α is $\mu \cdot \alpha$ and $\alpha \in M'_{\alpha+1}$.

We will construct an isomorphism between M and M' by induction on $\alpha < \theta$. Define an increasing and continuous sequence of $\prec_{\mathcal{K}}$ -mappings $\langle h_\alpha \mid \alpha < \theta \rangle$ such that

- (1) $h_\alpha : M_{i_\alpha+j} \rightarrow M'_{\alpha+1}$ for some $j < \mu \cdot \omega$
- (2) $h_0 = \text{id}_{M_{i_0}}$ and
- (3) $\alpha \in \text{rg}(h_{\alpha+1})$.

For $\alpha = 0$ take $h_0 = \text{id}_{M_{i_0}}$. For α a limit ordinal let $h_\alpha = \bigcup_{\beta < \alpha} h_\beta$. Since \bar{M} is continuous, the induction hypothesis gives us that h_α is a $\prec_{\mathcal{K}}$ -mapping from M_{i_α} into M'_α allowing us to satisfy condition (1) of the construction.

Suppose that h_α has been defined. Let $j < \mu \cdot \omega$ be such that $h_\alpha : M_{i_\alpha+j} \rightarrow M'_{\alpha+1}$. There are two cases: either $\alpha \in \text{rg}(h_\alpha)$ or $\alpha \notin \text{rg}(h_\alpha)$. First suppose that $\alpha \in \text{rg}(h_\alpha)$. Since $M'_{\alpha+2}$ is universal over $M'_{\alpha+1}$, it is also universal over $h_\alpha(M_{i_\alpha+j})$. This allows us to extend h_α to $h_{\alpha+1} : M_{i_{\alpha+1}} \rightarrow M'_{\alpha+2}$.

Now consider the case when $\alpha \notin \text{rg}(h_\alpha)$. Since $\langle M_{i_\alpha+j}^\gamma \mid \gamma < \sigma \rangle$ witnesses that $M_{i_\alpha+j}$ is a (μ, σ) -limit model, by Assumption 2.8, there exists $\gamma < \sigma$ such that $\text{ga-tp}(\alpha/M_{i_\alpha+j})$ does not μ -split over $M_{i_\alpha+j}^\gamma$. By our choice of \bar{M}' disjoint from \bar{M} outside of M_{i_0} , we know that $\alpha \notin M_{i_\alpha+j}$. Thus

$\text{ga-tp}(\alpha/M_{i_{\alpha+j}})$ is non-algebraic. By relative fullness of $(\bar{M}, \bar{a}, \bar{N})$, there exists j' with $j \leq j' < i_{\alpha+1}$ such that $(\text{ga-tp}(\alpha/M_{i_{\alpha+j'}}, M_{i_{\alpha+j}}^{\gamma}), M_{i_{\alpha+j}}^{\gamma})$ is parallel to $(\text{ga-tp}(a_{i_{\alpha+1+j'}/M_{i_{\alpha+1+j'}}}, N_{i_{\alpha+1+j'}})$. In particular we have that

$$(*) \quad \text{ga-tp}(a_{i_{\alpha+1+j'}/M_{i_{\alpha+j}}}) = \text{ga-tp}(\alpha/M_{i_{\alpha+j}}).$$

We can extend h_{α} to an automorphism h' of \mathfrak{C} . An application of h' to $(*)$ gives us

$$(**) \quad \text{ga-tp}(h'(a_{i_{\alpha+1+j'}/h_{\alpha}(M_{i_{\alpha+j}})})/h_{\alpha}(M_{i_{\alpha+j}})) = \text{ga-tp}(\alpha/h_{\alpha}(M_{i_{\alpha+j}})).$$

Since $M'_{\alpha+2}$ is universal over $h_{\alpha}(M_{i_{\alpha}})$, we may extend h_{α} to a \mathcal{K} -mapping $h_{\alpha+1} : M_{i_{\alpha+1+j'}} \rightarrow M'_{\alpha+2}$ such that $h_{\alpha+1}(a_{i_{\alpha+1+j'}}) = \alpha$.

Let $h := \bigcup_{\alpha < \theta} h_{\alpha}$. Clearly $h : M \rightarrow M'$. To see that h is an isomorphism, notice that condition (3) of the construction forces h to be surjective. \dashv

5. UNIQUENESS OF LIMIT MODELS

We now begin the construction of our array of models and M^* . Let θ be an ordinal as in the previous section. The goal is to build an array of models with $\omega + 1$ rows so that the bottom row of the array is a relatively full tower indexed by a set of cofinality θ . To do this, we will be adding elements to the index set of towers row by row so that at stage n of our construction the tower that we build is indexed by I_n described here:

Notation 5.1. The index sets I_n will be defined inductively so that $\langle I_n \mid n < \omega + 1 \rangle$ is an increasing and continuous chain of well-ordered sets. We fix I_0 to be an index set of order type $\theta + 1$ and will denote it by $\langle i_{\alpha} \mid \alpha \leq \theta \rangle$. We will refer to the members of I_0 by name in many stages of the construction. These indices serve as anchors for the members of the remaining index sets in the array. Next we demand that for each $n < \omega$, $\{j \in I_n \mid i_{\alpha} < j < i_{\alpha+1}\}$ has order type $\mu \cdot n$ such that each I_n has supremum i_{θ} . An example of such $\langle I_n \mid n \leq \omega \rangle$ is $I_n = \theta \times (\mu \cdot n) \cup \{i_{\theta}\}$ ordered lexicographically, where i_{θ} is an element \geq each $i \in \bigcup_{n < \omega} I_n$. Also, let $I = \bigcup_{n < \omega} I_n$.

To prove the main theorem of the paper, we need to prove that for a fixed $M \in \mathcal{K}$ of cardinality μ any (μ, θ) -limit and (μ, ω) -limit model over M are isomorphic over M . Let us begin by fixing $M \in \mathcal{K}_{\mu}$ and θ such that $\mu \cdot \theta = \theta$. Without loss of generality, M is a limit model. We define by induction on $n \leq \omega$ a $<$ -increasing and continuous sequence of towers $(\bar{M}, \bar{a}, \bar{N})$ such that

$$(1) \quad (\bar{M}, \bar{a}, \bar{N})^0 \text{ is a tower with } M_0^0 = M.$$

- (2) $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I_n}^*$
- (3) For every $(p, N) \in \mathfrak{St}(M_i^n)$ with $i_\alpha \leq i < i_{\alpha+1}$ there is $j \in I_{n+1}$ with $i < j < i_{\alpha+1}$ so that $(\text{ga-tp}(a_j/M_j^{n+1}), N_j^{n+1})$ and (p, N) are parallel.
- (4) $M_{i_{\alpha+1}}^{n+1}$ is a (μ, μ) -limit model over $\bigcup_{j < i_{\alpha+1}} M_j^{n+1}$.

Given M , we can find a tower $(\bar{M}, \bar{a}, \bar{N})^0 \in \mathcal{K}_{\mu, I_0}^*$ with $M_0^0 = M$ because of the existence of universal extensions and because of Assumption 2.8.4b. The last pages (Page 22 onward) of this section provide a picture of this construction of an array of models, explanations for carrying out the final stage of the construction and a proof that this is sufficient to prove the main theorem. We spend most of the remainder of this section verifying that it is possible to carry out the induction step of the construction. This is a particular case of Theorem II.7.1 of [Va1]. But since our context is somewhat easier, we do not encounter so many obstacles as in [Va1] and we provide a different, more direct proof here:

Theorem 5 (Dense $<$ -extension property). *Given $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I_n}^*$ there exists $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I_{n+1}}^*$ such that $(\bar{M}, \bar{a}, \bar{N}) < (\bar{M}', \bar{a}, \bar{N})$ and for each $(p, N) \in \mathfrak{St}(M_i)$ with $i_\alpha \leq i < i_{\alpha+1}$, there exists $j \in I_{n+1}$ with $i < j < i_{\alpha+1}$ such that $(\text{ga-tp}(a_j/M'_j), N_j)$ and (p, N) are parallel. Here, the M_i 's are defined for $i \in I_n$ and the M'_j are defined for $j \in I_{n+1}$.*

Before we prove Theorem 5, we prove a slightly weaker extension property, one in which we can find an extension of the tower $(\bar{M}, \bar{a}, \bar{N})$ of the same index set:

Lemma 5.2 ($<$ -extension property). *Given $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I}^*$, there exists a $<$ -extension $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I}^*$ of $(\bar{M}, \bar{a}, \bar{N})$ such that for each limit i , M'_i is a (μ, μ) -limit model over $\bigcup_{j < i} M'_j$.*

Proof. Given $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I}^*$ we will define a $<$ -extension $(\bar{M}', \bar{a}, \bar{N})$ by induction on $i \in I$. Notice that a straightforward induction proof is not sufficient here for if we have defined $\langle M_j \mid j \leq i \rangle$ as a tower extending $(\bar{M}, \bar{a}, \bar{N})$ restricted to $\langle j \mid j \leq i \rangle$ and are at the stage of defining M'_{i+1} , we may be faced with an impossible task: during our construction we may have inadvertently placed inside M'_i witnesses for the splitting of the type of a_{i+1} over N_{i+1} ; this would prevent us from extending M'_i to M'_{i+1} so that $\text{ga-tp}(a_{i+1}/M'_{i+1})$ does not μ -split over N_{i+1} . Therefore, we will instead define approximations, M_i^+ , for M'_i by induction on $i \in I$ and at each

stage i of the induction we will make adjustments of the previously defined approximation M_j^+ for $j < i$. This leads us into defining M_i^+ and a directed system of $\prec_{\mathcal{K}}$ -embeddings $\langle f_{j,i} \mid j < i \in I \rangle$ such that for $i \in I$, $M_i \prec_{\mathcal{K}} M_i^+$ for $j \leq i$, $f_{j,i} : M_j^+ \rightarrow M_i^+$ and $f_{j,i} \upharpoonright M_j = \text{id}_{M_j}$. We further require that M_{i+1}^+ is a limit model over $f_{i,i+1}(M_i^+)$ and $\text{ga-tp}(a_i/f_{i,i+1}(M_i^+))$ does not μ -split over N_i . When i is a limit, we choose M_i^+ to be a (μ, μ) -limit model over $\bigcup_{j < i} f_{j,i}(M_j^+)$.

This construction is done by induction on $i \in I$ using the existence of non- μ -splitting extensions. Suppose that $\langle M_k^+ \mid k \leq i \rangle$ and $\langle f_{k,l} \mid k \leq l \leq i \rangle$ have been defined. We explain how to define M_{i+1}^+ and $f_{i,i+1}$. The rest of the definitions required for the $i + 1^{\text{st}}$ stage are dictated by the requirement that we are forming a directed system. Let M_{i+1}^* be a limit model over both M_i^+ and M_{i+1} . Since $\text{ga-tp}(a_{i+1}/M_{i+1})$ does not μ -split over N_{i+1} , by Fact 2.4 there exists $f \in \text{Aut}_{M_{i+1}}(\mathfrak{C})$ so that $\text{ga-tp}(a_{i+1}/f(M_{i+1}^*))$ does not μ -split over N_{i+1} . Take $M_{i+1}^+ := f(M_{i+1}^*)$ and $f_{i,i+1} := f \upharpoonright M_i^+$.

At limit stages we take direct limits so that $f_{j,i} \upharpoonright M_j = \text{id}_{M_j}$. This is possible by Subclaims II.7.10 and II.7.11 of [Va1] or see Claim 2.17 of [GrVa2]. Take an extension of the direct limit that is both universal over M_i and is a (μ, μ) -limit over $\bigcup_{j < i} f_{j,i}(M_j)$ and call this M_i^+ . Notice that we do not obtain a continuous tower; continuity will be recovered later using reduced towers.

Let $f_{j,\text{sup}\{I\}}$ and $M'_{\text{sup}\{I\}}$ be the direct limit of this system such that $f_{j,\text{sup}\{I\}} \upharpoonright M_j = \text{id}_{M_j}$. We can now define $M'_j := f_{j,\text{sup}\{I\}}(M_j^+)$ for each $j \in I$. By construction, we have that $\text{ga-tp}(a_i/f_{i,i+1}(M_i^+))$ does not μ -split over N_i . Mapping into $M_{\text{sup}\{I\}}$ by $f_{i+1,\text{sup}\{I\}}$, and noting that both a_i and N_i are fixed by $f_{i+1,\text{sup}\{I\}}$, we conclude that $\text{ga-tp}(a_i/M'_i)$ does not μ -split over N_i as required. ⊣

We can now use the extension property for towers of the same index set from Lemma 5.2 to prove the dense extension property which allows us to grow the index set as we add elements to the models in the extension.

Proof of Theorem 5. Given $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I_n}^*$, let $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I_n}^*$ be an extension of $(\bar{M}, \bar{a}, \bar{N})$ as in Lemma 5.2 so that each $M'_{i_{\alpha+1}}$ is a (μ, μ) -limit model over $\bigcup_{j < i_{\alpha+1}} M'_j$.

For each i_α , let $\langle M'_l \mid l \in I_{n+1}, i_\alpha + \mu \cdot n < l < i_{\alpha+1} \rangle$ witness that $M'_{i_{\alpha+1}}$ is a (μ, μ) -limit model over $\bigcup_{j < i_{\alpha+1}} M'_j$. Without loss of generality we may assume that each of these M'_l is a limit model over its predecessor.

Fix $\{(p, N)_{i_\alpha}^l \mid i_\alpha + \mu \cdot n < l < i_{\alpha+1}\}$ an enumeration of $\bigcup\{\mathfrak{St}(M_i) \mid i \in I_n, i_\alpha \leq i < i_{\alpha+1}\}$. By our choice of I_{n+1} and stability in μ , such an enumeration is possible. Since $M'_{\text{succ}_{I_{n+1}}(l)}$ is universal over M'_l , there exists a realization in $M'_{\text{succ}_{I_{n+1}}(l)}$ of the non- μ -splitting extension of $p_{i_\alpha}^l$ to M'_l . Let a_l be this realization and take $N_l := N_{i_\alpha}^l$.

Notice that $(\langle M'_j \mid j \in I_{n+1} \rangle, \langle a_j \mid j \in I_{n+1} \rangle, \langle N_j \mid j \in I_{n+1} \rangle)$ provide the desired extension of $(\bar{M}, \bar{a}, \bar{N})$ in $\mathcal{K}_{\mu, I_{n+1}}^*$. \dashv

We are almost ready to carry out the complete construction. However, notice that Theorem 5 does not provide us with a continuous extension. Therefore the bottom (i.e. the $\omega + 1^{\text{st}}$) row of our array may not be continuous which would prevent us from applying Theorem 4 to conclude that M^* is a (μ, θ) -limit model. So we will further require that the towers that occur in the rows of our array are all continuous. This can be guaranteed by restricting ourselves to reduced towers as in [ShVi] and [Val].

Definition 5.3. A tower $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I}^*$ is said to be *reduced* provided that for every $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I}^*$ with $(\bar{M}, \bar{a}, \bar{N}) \leq (\bar{M}', \bar{a}, \bar{N})$ we have that for every $i \in I$,

$$(*)_i \quad M'_i \cap \bigcup_{j \in I} M_j = M_i.$$

If we take a $<$ -increasing chain of reduced towers, the union will be reduced. The following fact appears as Theorem 3.1.14 of [ShVi]. We provide the proof for completeness.

Fact 5.4. Let $\langle (\bar{M}, \bar{a}, \bar{N})^\gamma \in \mathcal{K}_{\mu, I_\gamma}^* \mid \gamma < \beta \rangle$ be a $<$ -increasing and continuous sequence of reduced towers such that the sequence is continuous in the sense that for a limit $\gamma < \beta$, the tower $(\bar{M}, \bar{a}, \bar{N})^\gamma$ is the union of the towers $(\bar{M}, \bar{a}, \bar{N})^\zeta$ for $\zeta < \gamma$. Then the union of the sequence of towers $\langle (\bar{M}, \bar{a}, \bar{N})^\gamma \in \mathcal{K}_{\mu, I_\gamma}^* \mid \gamma < \beta \rangle$ is itself a reduced tower.

Proof. Suppose that $(\bar{M}, \bar{a}, \bar{N})^\beta$ is not reduced. Let $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I_\beta}^*$ witness this. Then there exists an $i \in I_\beta$ and an element b such that $b \in (M'_i \cap \bigcup_{j \in I_\beta} M_j^\beta) \setminus M_i^\beta$. There exists $\gamma < \beta$ such that $b \in \bigcup_{j \in I_\gamma} M_j^\gamma \setminus M_i^\gamma$. Notice that $(\bar{M}', \bar{a}, \bar{N}) \upharpoonright I_\gamma$ witnesses that $(\bar{M}, \bar{a}, \bar{N})^\gamma$ is not reduced. \dashv

The following appears in [ShVi] (Theorem 3.1.13).

Fact 5.5 (Density of reduced towers). *There exists a reduced \prec -extension of every tower in $\mathcal{K}_{\mu, I}^*$.*

Proof. Assume for the sake of contradiction that no \prec -extension of $(\bar{M}, \bar{a}, \bar{N})$ is reduced. This allows us to construct a \leq -increasing and continuous sequence of towers $\langle (\bar{M}, \bar{a}, \bar{N})^\zeta \in \mathcal{K}_{\mu, I}^* \mid \zeta < \mu^+ \rangle$ such that $(\bar{M}, \bar{a}, \bar{N})^{\zeta+1}$ witnesses that $(\bar{M}, \bar{a}, \bar{N})^\zeta$ is not reduced. The construction is done inductively in the obvious way.

For each $b \in \bigcup_{\zeta < \mu^+, i \in I} M_i^\zeta$ define

$$i(b) := \min \{i \in I \mid b \in \bigcup_{\zeta < \mu^+} \bigcup_{j \leq i} M_j^\zeta\} \text{ and}$$

$$\zeta(b) := \min \{\zeta < \mu^+ \mid b \in M_{i(b)}^\zeta\}.$$

$\zeta(\cdot)$ can be viewed as a function from μ^+ to μ^+ . Since $|I| = \mu$ and each M_i^ζ has cardinality μ , there exists a club $E = \{\delta < \mu^+ \mid \forall b \in \bigcup_{i \in I} M_i^\delta, \zeta(b) < \delta\}$. Actually, all we need is for E to be non-empty.

Fix $\delta \in E$. By construction $(\bar{M}, \bar{a}, \bar{N})^{\delta+1}$ witnesses the fact that $(\bar{M}, \bar{a}, \bar{N})^\delta$ is not reduced. So we may fix $i \in I$ and $b \in M_i^{\delta+1} \cap \bigcup_{j \in I} M_j^\delta$ such that $b \notin M_i^\delta$. Since $b \in M_i^{\delta+1}$, we have that $i(b) \leq i$. Since $\delta \in E$, we know that there exists $\zeta < \delta$ such that $b \in M_{i(b)}^\zeta$. Because $\zeta < \delta$ and $i(b) \leq i$, this implies that $b \in M_i^\delta$ as well. This contradicts our choice of i and b witnessing the failure of $(\bar{M}, \bar{a}, \bar{N})^\delta$ to be reduced. \dashv

By revising the proof of Lemma 5.2, we can conclude:

Lemma 5.6. *Suppose that $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I}^*$ is reduced. If I_0 is an initial segment of I , then $(\bar{M}, \bar{a}, \bar{N}) \upharpoonright I_0$ is reduced.*

Proof. Suppose that $(\bar{M}, \bar{a}, \bar{N}) \upharpoonright I_0$ is not reduced. Let $(\bar{M}', \bar{a} \upharpoonright I_0, \bar{N} \upharpoonright I_0)$ and $\delta < j \in I_0$ with $b \in (M'_\delta \cap M_j) \setminus M_\delta$ witness this. We can apply the inductive step of Lemma 5.2 (replacing an initial segment of the construction there with \bar{M}'), to find $(\bar{M}'', \bar{a}, \bar{N})$ an extension of $(\bar{M}, \bar{a}, \bar{N})$ such that there is a $\prec_{\mathcal{K}}$ -mapping f from the models of \bar{M}' into the models of \bar{M}'' with $f \upharpoonright M_j = \text{id}_{M_j}$. Notice that $(\bar{M}'', \bar{a}, \bar{N})$ and b, δ, j will witness that $(\bar{M}, \bar{a}, \bar{N})$ is not reduced. \dashv

The following theorem makes use of the unidimensionality assumption. This generalizes a special case of the uniqueness of limit models result in

the series of papers [Va1] and [Va2] by replacing the assumption of categoricity in μ^+ with the weaker unidimensionality assumption. Further work of VanDieren in [Va3] weakens this assumption further for tame classes.

Theorem 6 (Reduced towers are continuous). *If $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I}^*$ is reduced, then it is continuous, namely for each limit i in I , $M_i = \bigcup_{j < i} M_j$.*

Proof of Theorem 6. Suppose the theorem fails for μ . Let δ be the minimal limit ordinal such that there exists an index set I and $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, I}^*$ a reduced tower which is discontinuous at the δ^{th} element of I . We can apply Lemma 5.6 to assume without loss of generality that $I = \delta + 1$. Fix $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, \delta+1}^*$ reduced and discontinuous at δ with $b \in M_\delta \setminus \bigcup_{i < \delta} M_i$. By Fact 2.6, there exists a minimal type p over M_0 . So by our unidimensionality Assumption 3, we know that the Galois type of p must be realized in $M_\delta \setminus \bigcup_{i < \delta} M_i$. Therefore, we may assume that $b \models p$.

Claim 5.7. *There exists a $<$ -extension of $(\bar{M}, \bar{a}, \bar{N}) \upharpoonright \delta$, containing b . We will refer to such a tower in $\mathcal{K}_{\mu, \delta}^*$ as $(\bar{M}', \bar{a} \upharpoonright \delta, \bar{N} \upharpoonright \delta)$. Furthermore, b may be assumed to be an element of M'_0 .*

Proof of Claim 5.7. We use the minimality of δ and the $<$ -extension property to find a tower of length δ , $(\bar{M}^*, \bar{a} \upharpoonright \delta, \bar{N} \upharpoonright \delta)$, that is a proper extension of $(\bar{M}, \bar{a}, \bar{N}) \upharpoonright \delta$. By the definition of $<$ -extension, M_0^* is universal over M_0 ; so we can find $b^* \in M_0^* \setminus M_0$ realizing p .

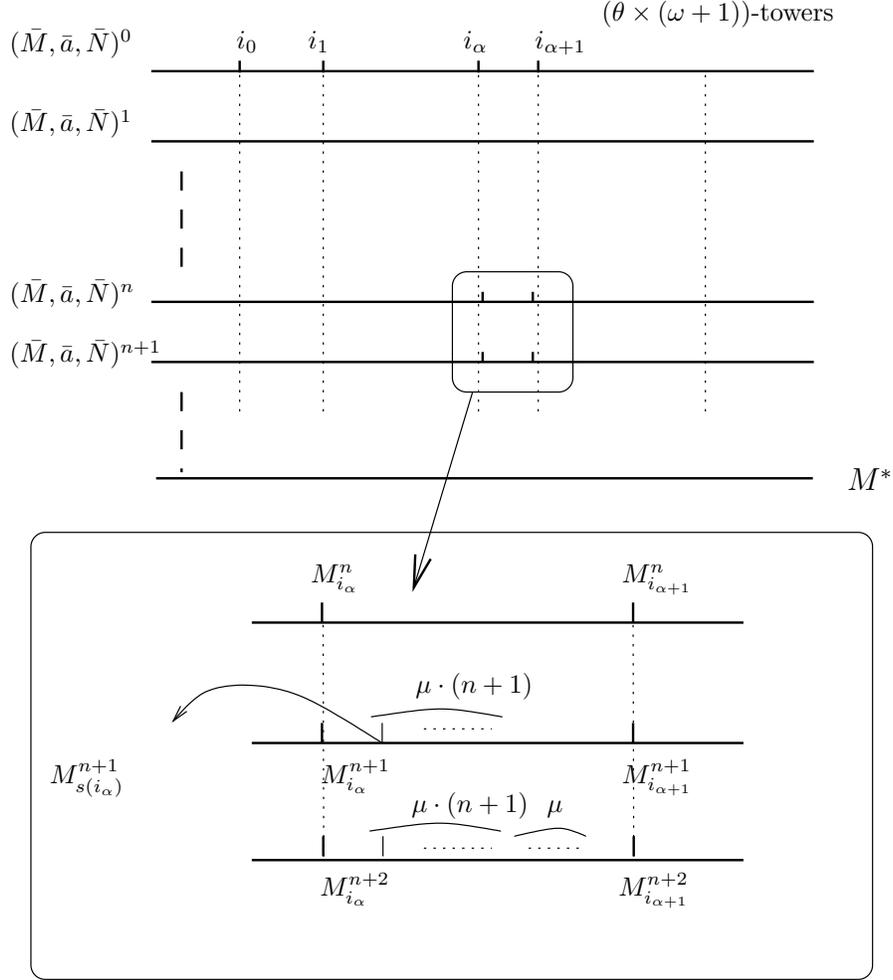
Notice that by Lemma 5.6, $(\bar{M}, \bar{a}, \bar{N}) \upharpoonright \delta$ is reduced. Thus we can conclude that $b^* \in M_0^* \setminus \bigcup_{i < \delta} M_i$ and $\text{ga-tp}(b^* / \bigcup_{i < \delta} M_i)$ is non-algebraic. Since p is minimal, it must be the case that $\text{ga-tp}(b^* / \bigcup_{i < \delta} M_i) = \text{ga-tp}(b / \bigcup_{i < \delta} M_i)$. Let $f \in \text{Aut}_{\bigcup_{i < \delta} M_i} \mathfrak{C}$ take b^* to b .

Consider the image of $(\bar{M}^*, \bar{a}, \bar{N})$ under f ; denote this tower by $(\bar{M}', \bar{a}, \bar{N})$. Because f fixes $(\bar{M}, \bar{a}, \bar{N}) \upharpoonright \delta$, $(\bar{M}', \bar{a}, \bar{N})$ is an extension of $(\bar{M}, \bar{a}, \bar{N}) \upharpoonright \delta$ as required. \dashv

Using $(\bar{M}', \bar{a}, \bar{N})$ from Claim 5.7, define M'_δ to be a limit model of cardinality μ containing $\bigcup_{i < \delta} M'_i$ so that it is universal over M_δ . Notice that the tower $(\bar{M}' \wedge \langle M'_\delta \rangle, \bar{a}, \bar{N})$ extends $(\bar{M}, \bar{a}, \bar{N})$ with $b \in (M'_0 \setminus \bigcup_{i < \delta} M_i) \cap M_\delta$. This contradicts our assumption that $(\bar{M}, \bar{a}, \bar{N})$ is reduced and completes the proof of Theorem 6. \dashv

Corollary 5.8. *In Theorem 5, we can choose $(\bar{M}, \bar{a}, \bar{N})$ to be reduced, and hence continuous.*

Now we return to the construction in the proof of the Main Theorem.



Corollary 5.8 tells us that the construction of our array of models as an increasing sequence of towers is possible in successor cases. In the limit case, let $I_\omega = \bigcup_{m < \omega} I_m$, and simply define $(\bar{M}, \bar{a}, \bar{N})^\omega \in \mathcal{K}_{\mu, I_\omega}^*$ to be the union of the towers $(\bar{M}, \bar{a}, \bar{N})^n$.

To see that the construction satisfies our requirements, first notice that the last column of the array, $\langle M_{i_\theta}^n \mid n < \omega \rangle$, witnesses that M^* is a (μ, ω) -limit model. In light of Theorem 4 we need only verify that the last row of the array is a relatively full tower of cofinality θ .

Claim 5.9. $(\bar{M}, \bar{a}, \bar{N})^\omega$ is full relative to $(M_i^n)_{n < \omega, i \in I_\omega}$.

Proof. Given i with $i_\alpha \leq i < i_{\alpha+1}$, let (p, M_i^n) be some strong type in $\mathfrak{St}(M_i^\omega)$. Notice that by monotonicity of non-splitting $(p \upharpoonright M_i^{n+1}, M_i^n) \in \mathfrak{St}(M_i^{n+1})$. By construction there is a $j \in I_{n+1}$ with $i < j < i_{\alpha+1}$ such that $(\text{ga-tp}(a_j/M_j^{n+2}), N_j^{n+2})$ is parallel to $p \upharpoonright M_i^{n+1}$. We will show that $(\text{ga-tp}(a_j/M_j^\omega), N_j^\omega)$ is parallel to (p, N) .

First notice that $\text{ga-tp}(a_j/M_j^\omega)$ does not μ -split over $N_j^\omega = N_j^{n+2}$ because $(\bar{M}, \bar{a}, \bar{N})^\omega$ is a tower. Since $(\text{ga-tp}(a_j/M_j^{n+2}), N_j^{n+2})$ is parallel to $(p \upharpoonright M_i^{n+1}, M_i^n)$ there is $q \in \text{ga-S}(M_j^\omega)$ such that q extends both $p \upharpoonright M_i^{n+1}$ and $\text{ga-tp}(a_j/M_j^{n+2})$. By two separate applications of the uniqueness of non- μ -splitting extensions we know that $q \upharpoonright M_i^\omega = p$ and $q = \text{ga-tp}(a_j/M_j^\omega)$. To see that (q, N_j^ω) is parallel to (p, M_i^n) , let M' be an extension of M_j^ω of cardinality μ . Since $(p \upharpoonright M_i^{n+1}, M_i^n)$ and $(q \upharpoonright M_j^{n+2}, N_j^{n+2})$ are parallel, there is $q' \in \text{ga-S}(M')$ extending both $p \upharpoonright M_i^{n+1}$ and $q \upharpoonright M_j^{n+2}$ and not μ -splitting over both M_i^n and N_j^{n+2} . By the uniqueness of non- μ -splitting extensions, we have that q' is also an extension of q and p . Thus q' witnesses that (q, N_j^ω) and (p, M_i^n) are parallel. \dashv

This completes the proof of Theorem 3.

6. CONCLUDING REMARKS

In this section we discuss other results related to Question 1.9. First to understand the boundaries of Question 1.9, consider the elementary case. Limit models are not necessarily unique even for first order complete stable theories.

Theorem 7. *Suppose T is a complete, stable theory. Let $\mu \geq 2^{|T|}$ such that $\mu^{|T|} = \mu$. If T is not superstable, then no (μ, ω) -limit model is isomorphic to any (μ, κ) -limit model for any κ with $\text{cf}(\kappa) \geq \kappa(T)$.*

Proof. Let T be a stable, but not superstable, complete theory, and fix κ and μ as in the statement of the theorem. As T is not superstable, by [Sh e, Lemma VII, 3.5 (2)], for $\lambda := (2^\mu)^+$, there are $\langle \bar{a}_\eta | \eta \in \omega^{\geq \lambda} \rangle$ and $\langle \varphi_n(\bar{x}, \bar{y}_n) | n < \omega \rangle$ such that for every $n < \omega$ and all $\eta \in \omega^\lambda$,

$$(\mathfrak{C} \models \varphi_n[\bar{a}_\eta, \bar{a}_\nu]) \iff \nu = \eta \upharpoonright n.$$

By induction on $n < \omega$ define $\langle M_n | n < \omega \rangle$ all of cardinality μ and $\langle \eta_n, \nu_n | n < \omega \rangle$ such that

- (1) M_{n+1} is universal over M_n and saturated of cardinality μ ,
- (2) $\eta_{n+1} > \eta_n$, $\nu_{n+1} > \eta_n$, and $\eta_{n+1} \neq \nu_{n+1}$,

- (3) $\bar{a}_{\eta_{n+1}}, \bar{a}_{\nu_{n+1}} \in M_{n+1}$ and
(4) $\text{tp}(\bar{a}_{\eta_{n+1}}/M_n) = \text{tp}(\bar{a}_{\nu_{n+1}}/M_n)$.

THIS CONSTRUCTION IS ENOUGH: Let $N' \models T$ be a (μ, κ) -limit over M_0 . By Theorem 2, N' must be saturated. Let $N = \bigcup_{n < \omega} M_n$. Clearly N is a (μ, ω) -limit over M_0 . To conclude that N and N' are non-isomorphic, it is enough to show that N is not saturated. Consider $p := \{\varphi_{n+1}(\bar{x}; \bar{a}_{\eta_{n+1}}) \wedge \neg \varphi_{n+1}(\bar{x}; \bar{a}_{\nu_{n+1}}) \mid n < \omega\}$. The set of formulas p is a type since it is realized in \mathfrak{C} by \bar{a}_η where $\eta := \bigcup_{n < \omega} \eta_n$. Notice that N cannot satisfy p . If $\bar{a} \in N$ would satisfy p , then M_n realizes p for some $n < \omega$. Thus by condition (4), we would have

$$\mathfrak{C} \models \varphi_{n+1}[\bar{a}, \bar{a}_{\eta_{n+1}}] \iff \mathfrak{C} \models \varphi_{n+1}[\bar{a}, \bar{a}_{\nu_{n+1}}]$$

which would contradict the assumption that \bar{a} satisfies p .

THIS IS POSSIBLE: By stability and $\mu^{|T|} = \mu$, using the proof of [Sh e, Th. III 3.12], every model of cardinality μ has a saturated proper elementary extension. Let M_0 be a saturated model of cardinality μ and take $\eta_0 = \nu_0 := \langle \rangle$. Given η_n, ν_n, M_n , using Theorem 1 let M^* be universal over M_n of cardinality μ . Let $M^{**} \succ M^*$ of cardinality μ containing \bar{a}_{η_n} and \bar{a}_{ν_n} . By [Sh e, Th. III 3.12], we can take $M_{n+1} \succ M^{**}$ saturated of cardinality μ . Clearly it is universal over M_n . For $n < \omega$, consider $F_n(\alpha) := \text{tp}(\bar{a}_{\eta_n \hat{\ } \alpha}/M_n)$. As λ is regular and $\lambda > |S(M_n)|$, there is $S \subset \lambda$ of cardinality λ such that $\alpha \neq \beta \in S \Rightarrow F_n(\alpha) = F_n(\beta)$. Pick $\alpha \neq \beta \in S$ and define $\eta_{n+1} := \eta_n \hat{\ } \alpha$ and $\nu_{n+1} := \eta_n \hat{\ } \beta$. \dashv

In the non-elementary setting, many authors have considered approximations to Theorem 3. Several authors have proved and used the uniqueness of limit models in AECs under the assumption of categoricity: [Sh 394] [Ba], [KoSh], [Sh 576], [Sh Vi], [Va1], and [Va2]. Also, Shelah's [Sh i] examines (as an aside) the uniqueness of limit models in good frames. Below we briefly describe the results and techniques of these papers and distinguish them from our context.

In Theorem 6.5 of [Sh 394], Shelah claims uniqueness of limit models of cardinality μ for classes with the amalgamation property under little more than categoricity in some $\lambda > \mu > \text{LS}(\mathcal{K})$ together with existence of arbitrarily large models. Shelah's claim in Theorem 6.5 of [Sh 394] (isomorphism over the base) seems too strong for the proof that he suggests. Instead, he proves that (μ, κ) -limit models are Galois saturated, which implies uniqueness only over models of size $< \mu$. The argument in [Sh 394] depends in

a crucial way on an analysis of Ehrenfeucht-Mostowski models. In our paper, we cannot employ Ehrenfeucht-Mostowski machinery because we do not assume here categoricity or the existence of models above the Hanf number.

Under similar categoricity assumptions as those in [Sh 394], more recently, Baldwin in [Ba] (Chapter 11) has used methods based on [Sh 394] to prove that if M_1 and M_2 are (μ, σ_1) - and (μ, σ_2) -limit models over N , respectively, then $M_1 \cong M_2$. Baldwin, however, does not prove that M_1 and M_2 are isomorphic *over* N . Our result is therefore much stronger than that in [Ba].

Kolman and Shelah in [KoSh] prove the uniqueness of limit models of cardinality μ in λ -categorical AECs that are axiomatized by a $L_{\kappa, \omega}$ -sentence where $\lambda > \mu$ and κ is a measurable cardinal. Then Kolman and Shelah use this uniqueness result to prove that amalgamation occurs below the categoricity cardinal in $L_{\kappa, \omega}$ -theories with κ measurable. Both the measurability of κ and the categoricity are used integrally in their proof of uniqueness.

Shelah in [Sh 576] (see Claim 7.8) proved a special case of the uniqueness of limit models under the assumption of μ -AP, categoricity in μ and in μ^+ as well as assuming $K_{\mu^{++}} \neq \emptyset$. In that paper Shelah needs to produce *reduced types* and use some of their special properties.

In [ShVi], Shelah and Villaveces attempted to prove a uniqueness theorem without assuming any form of amalgamation; however, they assumed that \mathcal{K} is categorical in some sufficiently large λ , that every model in \mathcal{K} has a proper extension and that $2^\lambda < 2^{\lambda^+}$. VanDieren in [Va1] and [Va2] managed to prove the uniqueness statement under the assumptions of [ShVi] together with the additional assumptions that the class is categorical in μ^+ and $\mathcal{K}^{am} := \{M \in \mathcal{K}_\mu \mid M \text{ is an amalgamation base}\}$ is closed under unions of increasing $\prec_{\mathcal{K}}$ chains.

In [Sh i] the most important new concept is that of a λ -good frame, which is an axiomatization of the notion of superstability, *with hypothesis on just one cardinal* λ . Its full definition is more than a page long. Shelah's assumptions on the AEC include, among other things, the amalgamation property, the existence of a forking like dependence relation and of a family of types playing a role akin to that of regular types in first order superstable theories – Shelah calls them *bs*-types. One of the axioms of a good frame is the existence of a non-maximal super-limit model. This axiom along with μ -stability implies the uniqueness of limit models of cardinality μ . In Lemma II.4.8 of [Sh i] he states that in a good frame, limit models are unique. (While we don't claim that we understand Shelah's proof or believe in its

correctness, he explicitly uses the interplay between bs -types and the forking notion as well as no long forking chains and continuity of forking.)

The formal differences between our approach and Shelah's [Sh i] can be summarized as follows:

- (1) Suppose that \mathcal{K} is an AEC with no maximal models satisfying the disjoint amalgamation property over limit models and is categorical in λ^+ for some $\lambda > \text{LS}(\mathcal{K})$; we then get uniqueness of limit models. By way of comparison, in order to get a uniqueness of limit models, Shelah needs results of [Sh 576] (a 99 pages-long paper) and significant parts of his book [Sh i] along with the stronger assumptions of categoricity in several consecutive cardinals together with several additional set-theoretic axioms. All our results are in ZFC.
- (2) When specialized to the case where \mathcal{K} is the class of models of a complete first order theory T , Shelah's proof in [Sh i, Lemma II.4.8] really uses the full power of assuming that T is *superstable*. The proof of uniqueness in this paper just needs, in addition to the stability and unidimensionality of T , no splitting chains of length ω . As the main interest of our theorem is for the general case of AEC, rather than just for first order theories, the difference between this paper and [Sh i, Lemma II.4.8] is clearer when understood in light of the greater picture.

We are particularly interested in Theorem 3 not only for the sake of generalizing Shelah's result from [Sh 576] but due to the fact that the first and second author originally used an earlier draft of this uniqueness theorem (which did not assume unidimensionality) along with tools from [Sh 394] in a crucial step to prove:

Theorem 8 (Upward categoricity theorem, [GrVa2]⁴). *Suppose that \mathcal{K} has arbitrarily large models, is χ -tame and satisfies the amalgamation and joint embedding properties. Let λ be such that $\lambda > \text{LS}(\mathcal{K})$ and $\lambda \geq \chi$. If \mathcal{K} is categorical in λ^+ then \mathcal{K} is categorical in all $\mu \geq \lambda^+$.*

After the addition of the unidimensionality assumption in 2014 to resolve an error found in 2012 in the proof of Theorem 6, Grossberg and VanDieren have revisited the proof of Theorem 8 to insure that the upward categoricity

⁴Some time after Grossberg and VanDieren announced Theorem 8, Baldwin circulated an alternative proof of Theorem 8 that eventually appeared in [Ba]. Lessmann in [Les05] proved the result for \mathcal{K} with $\text{LS}(\mathcal{K}) = \aleph_0$ beginning with categoricity in \aleph_1 .

transfer still holds [GrVa3]. Grossberg and VanDieren’s initial use of the uniqueness of limit models in this theorem hints at a connection between classical definitions of superstability in first order logic and the uniqueness of limit models. This link is explored in further work of VanDieren [Va3].

It is worth mentioning that the links between classical notions of superstability from first order logic and the uniqueness of limit models have also produced interesting insights in the connections between “continuous model theory” and so-called “metric AECs”. The work of Villaveces and Zambrano [ViZa] has extended notions of independence akin to those used here to the metric AEC context, and at the same time explored various consequences of assuming forms of uniqueness of limit models in that metric (continuous) context.

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