GENERALIZED AMALGAMATION IN SIMPLE THEORIES AND CHARACTERIZATION OF DEPENDENCE IN NON-ELEMENTARY CLASSES

By

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Abstract

We examine the properties of dependence relations in certain non-elementary classes and first-order simple theories. There are two major parts.

The goal of the first part is to identify the properties of dependence relations in certain non-elementary classes that, firstly, characterize the model-theoretic properties of those classes; and secondly, allow to uniquely describe an abstract dependence itself in a very concrete way. I investigate totally transcendental atomic models and finite diagrams, stable finite diagrams, and a subclass of simple homogeneous models from this point of view.

The second part deals with simple first-order theories. The main topic of this part is investigation of generalized amalgamation properties for simple theories. Namely, we are trying to answer the question of when does a simple theory have the property of n-dimensional amalgamation, where 2-dimensional amalgamation is the Independence theorem for simple theories. We develop the notion of n-simplicity and strong n-simplicity for $1 \leq n \leq \omega$, where both "1-simple" and "strongly 1-simple" is the same as "simple." We present examples of simple unstable theories in each subclass and prove a characteristic property of n-simplicity in terms of n-dividing, a strengthening of the dependence relation called dividing in simple theories. We prove 3-dimensional amalgamation property for 2-simple theories, and, under an additional assumption, a strong $(n+1)$-dimensional amalgamation property for strongly n-simple theories.

Stable theories are strongly ω-simple, and the idea behind developing extra simplicity conditions is to show that, for instance, ω-simple theories are almost as nice as stable theories. The third part of the thesis contains an application of ω-simplicity to construct a Morley sequence without the construction of a long independent sequence.

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Contents

Introduction

The goal of this introduction is to describe how this work fits into the classification program for first-order theories and not first-order, or non-elementary, classes.

The classification program for first-order theories began with M. Morley's proof of Łoš' conjecture, the statement asserting that if a countable theory has one model up to isomorphism of *some* uncountable size, then *any* two models of the same uncountable size must be isomorphic. The transition from this success to a systematic *classification theory* was accomplished by Saharon Shelah in [29].

Many mathematical objects cannot be completely described by their first-order properties, so it is natural to look at the classes of models defined in some, not necessarily first-order, way. The classification task becomes much more difficult because the familiar tools (most notably, compactness) fail beyond the first-order context. The reader is referred to the survey [10] discussing the progress of classification in non-elementary classes.

The method of the classification program is to identify meaningful dividing lines in the class of all complete first-order theories and non-elementary classes. We refer the reader to Section 5 in [32] for an in-depth discussion of what is meant by "meaningful." A somewhat simplified view is that a dividing line should split the class of objects in such a way that a structure theory is possible on the "good" side of the dividing line, and there is a clear reason why such structure theory is impossible on the "bad" side.

Examples of such dividing lines in the first-order case are *stable/unstable* theories, where a theory is stable if it does not interpret an infinite linear ordering; or *simple/non-simple* theories, where a theory is simple if it does not interpret a certain tree structure. The importance of these dividing lines, each of which can be characterized in a great variety of ways, becomes obvious when one is able to develop positive results for, in these examples, stable or simple theories as well as negative results for the theories that are not stable or not simple. Many of the positive results became a foundation for development of important fields within model theory, such as geometric stability theory.

Our approach is based on the observation that in all the cases the analysis of "good" theories (or classes of models) is possible because one can define a dependence relation, a necessary tool in studying these objects. To illustrate what is meant by a dependence relation in the model-theoretic context, let us give some examples.

Examples. (1) Let $\mathbb{C}$ be an algebraically closed field. Let $A, B, C \subset \mathbb{C}$ be such that $C \subset A, B$. We say that A is independent from B over C and write $A \underset{C}{\downarrow} B$ if $\text{acl}(A)$ is linearly disjoint from $\text{acl}(B)$ over $\text{acl}(C)$.

(2) Let V be a vector space, and $A, B, C \subset V$ are such that $C \subset A, B$. Then $A \underset{C}{\downarrow} B$ if $\text{Span}(A) \cap \text{Span}(B) \subset \text{Span}(C)$.

(3) Let X be an infinite set, and $A, B, C \subset X$ are such that $C \subset A, B$. Then $A \underset{C}{\downarrow} B$ if $A \cap B \subset C$.

The common idea is that the relation $A \underset{C}{\downarrow} B$ roughly means "B does not have more information about A than C does."

In the first-order cases, the properties of dependence relations characterize the model-theoretic properties of the classes. For example, in [3], it is shown that a first-order theory is stable if and only if in its models one can define a dependence relation with certain "stable" properties, and in addition that dependence relation must coincide with the *forking* dependence relation developed by Shelah for all stable theories in general. In particular, this result shows that the dependence relations in Examples 1–3 are all instances of forking in those particular contexts.

B. Kim and A. Pillay showed in [22] that, for simple theories, forking satisfies almost all the properties it has for stable theories. Moreover, they showed that a first order theory must be simple if it has an (abstract) dependence relation with certain properties of forking. To prove the last fact, it was shown that any abstract dependence relation with certain properties must actually coincide with forking.

The first part of this thesis establishes that model-theoretic properties of non-elementary classes too can be completely characterized by the properties of the dependence relations, and that the dependence relations there are unique. That is, we obtain analogs of the two first-order results mentioned above.

In the second part we attempt to draw some dividing lines in the class of first order simple unstable theories. As the guiding principle we use a family of properties that the forking dependence relation may (or may not) possess in simple unstable theories. The motivation for considering these properties comes from a non first-order context of *excellent classes* developed by Shelah in [31]. Excellent classes received much attention recently due to work of Boris Zilber [34]. One of the key tools in the context of excellent classes is that of n-dimensional amalgamation, for any $n < \omega$. One of the characterizing properties of forking in simple theories is called the Independence theorem, established by B. Kim and A. Pillay in 1995. It gives a two-dimensional type amalgamation property for all simple theories, and it natural to ask whether generalized amalgamation properties would hold. It turns out that the answer is "no", and so the family of amalgamation properties gives rise to natural dividing lines within the class of all simple unstable theories. The direction of our work is to find alternative characterizations of these properties, for example the appropriate syntactic conditions that would give n-dimensional amalgamation.

Research shows that there are different strengths of the n-dimensional amalgamation conditions that hold in simple theories. This gives rise to several related families of simplicity conditions. Two of these families are studied in Chapters II and III.

The thesis is divided into three chapters.

Chapter I of this thesis is devoted to characterizing dependence relations in some non-elementary classes. Namely, we identify the properties of dependence relations that allow us to conclude from existence of a dependence relation on a non-elementary class that the class has certain model-theoretic properties. Another part establishes the uniqueness of a "nice" dependence relation for the class. We isolate the properties that allow us to describe any abstract dependence relation with those properties in a concrete way. The chapter contains an introduction and is divided into five sections, the first section contains the definitions of abstract dependence relations and each subsequent section is devoted to a particular non first-order context.

In Chapter II we begin our analysis of simple unstable first-order theories from the point of view of n-dimensional amalgamation properties. We start by defining a family of syntactic properties

The definitions of n-simplicity are refined in Chapter III. There we prove the 3-dimensional amalgamation property for 2-simple theories.

Chapter 1

Characterization of Abstract Dependence

Introduction

In the last 25 years, significant effort was made to develop classification theory for non-elementary classes. While for the general case (the abstract elementary classes) existence of a satisfactory dependence relation is a major open question, good dependence relations were defined and used in several non-first order frameworks. In this chapter we study dependence relations in the following non-elementary classes:

(1) *totally transcendental classes of atomic models and homogeneous finite diagrams.* The known dependence relation in atomic models was developed by S. Shelah in [28, 31]; it is called a stable amalgamation. For homogeneous finite diagrams, it was introduced by O. Lessmann in [24] via an appropriate 2-rank.

(2) *stable homogeneous finite diagrams.* The dependence relation is strong splitting, introduced and studied by S. Shelah in [27], with extensions in, for example, [12, 18, 17].

(3) *simple homogeneous models.* The dependence relation is dividing, due to S. Buechler and O. Lessmann in [7].

The goal is to characterize dependence relations for these classes in the following two ways.

First, we identify the properties of dependence relations that allow us to conclude from existence of such a dependence relation on a non-elementary class that the class has certain model-theoretic properties (e.g., totally transcendental, stable, etc.). For the first order case, the work in this direction was started in 1974 by J. Baldwin and A. Blass. In [4] they deal with axiomatization of rank function, and with the question of what do the properties of the rank imply about the theory. In 1978 S. Shelah introduced axiomatizations of various isolation notions in his book [29]. The axiomatization of $\mathbf{F}^f$ is an implicit axiomatization of forking for stable theories. Axiomatic treatment of forking in stable theories appeared in [14]. Abstract dependence relations were systematically studied in the book [3] by J. Baldwin that appeared in 1988. In 1996, B. Kim and A. Pillay showed in [22] that, for simple theories, forking satisfies almost all the properties it has for stable theories. Moreover, they showed that a first order theory must be simple if it has an (abstract) dependence relation with certain properties of forking. To prove the last fact, it was shown that any abstract dependence relation with certain properties must actually coincide with forking.

This brings us to the second aspect of our study: Determine whether or not the specific dependence relation used in analysis of a non-elementary class is the unique "nice" dependence relation for the class. We isolate the properties that allow us to uniquely describe any abstract dependence relation with those properties in a concrete way. For stable first order theories, such a characterization of forking was derived from [23] by J. Baldwin in [3]. For simple first order theories, the characterization of forking was obtained by B. Kim and A. Pillay in [22]. Their analysis was useful in particular as a tool to establish that a certain theory is simple, see for example [9]. On the non-first order front, a characterization of dependence was obtained by T. Hyttinen and O. Lessmann in [17] for homogeneous finite diagrams that are both simple and stable.

The abstract approach to dependence relations goes back to the works of Van der Waerden. In model theory, the abstract treatment of dependence was introduced in [2] by J. Baldwin, with many extensions in [3]. This part of the thesis was inspired by [13], some results from which were presented by R. Grossberg in a model theory course at Carnegie Mellon.

This chapter is organized as follows. In **Section 1** we describe the general context in which we define the notion of an abstract dependence relation and identify the properties of abstract dependence that allow us to characterize totally transcendental, stable, and simple classes. As we show later, the abstract dependence has to coincide with the specific dependence relations introduced for the classes, i.e., is unique in certain sense.

Section 2 deals with totally transcendental classes of atomic models. We present motivation, basic definitions, and dependence relation for this case. The dependence relation is not defined for all sets, it is restricted to *good Tarski-Vaught pairs* of sets. We discuss the reasons for such restrictions. We then prove that a class of atomic models with an abstract dependence relation must be totally transcendental. Moreover, we prove that Shelah's *stable amalgamation* relation must be the only "reasonable" dependence relation in atomic models.

In **Section 3** we discuss a similar case of totally transcendental homogeneous finite diagrams. We find the situation there is analogous to the atomic case. The major differences between the contexts are that the homogeneous finite diagrams have a monster model that is a member of the class (while atomic models do not), but the types in homogeneous finite diagrams are not necessarily isolated, as they are in atomic case.

In **Section 4** we prove that a homogeneous finite diagram is stable if and only if it has a "stable" dependence relation. Moreover, we show that, over models, any stable dependence relation must coincide with (non-)strong splitting. As a byproduct of our study, we conclude that the strong splitting relation is optimal in the sense that it has the smallest local character possible for a stable dependence relation.

Section 5 is devoted to analysis of dependence relations in a simple homogeneous model with type amalgamation over all small sets. We prove an analogous result to the characterization of forking and simplicity obtained for the first order case by B. Kim and A. Pillay; the main difficulty is getting around the failure of the compactness theorem, as compactness is heavily used in the first order case.

1.1 Abstract dependence relations

We first describe the general context for the notion of an abstract dependence relation. The context generalizes the cases of atomic models and homogeneous finite diagrams that we study here. Background and motivation remarks for the classes of atomic models and finite diagrams are postponed to the sections in which those classes are studied.

1.1.1 Preliminary definitions

Fix a complete first order theory T, let $\mathfrak{C}$ be a monster model of T.

Definition 1.1.1. For a set $A \subset \mathfrak{C}$, the set of types $D(A) := \{\text{tp}(\bar{a}/\emptyset) \mid \bar{a} \in A\}$ is called *the diagram of* A. The diagram of T is $D(T) := D(\mathfrak{C})$, where $\mathfrak{C}$ is the monster model of the first order theory T.

For a fixed $D \subset D(T)$, we call A a D*-set* if $D(A) \subset D$. If $M \models T$ and $D(M) \subset D$, we call M a D*-model*.

The object of our study is essentially the class of D-submodels of $\mathfrak{C}$ for a fixed diagram D, with some extra assumptions either on the diagram D (e.g., D is atomic) or on the class of D-models. We restrict ourselves to those subsets of $\mathfrak{C}$ because even though the underlying theory T may be too complex from the classification theory point of view, the collection of D-models could well have nice model-theoretic properties.

Definition 1.1.2. We denote by $S^n_D(A)$ the collection of all complete types in n variables such that for all $\bar{c} \models p$ the set $A \cup \bar{c}$ is a D-set. Accordingly, $S_D(A) := \bigcup_{n<\omega} S^n_D(A)$.

A D-model M is called (D, λ)*-homogeneous* if M realizes all the types $\{p \in S_D(A) \mid A \subset M, |A| < \lambda\}$.

The compactness theorem no longer holds in our context, so the first-order intuition, and many of the methods, do not work. In particular, it is not always possible to realize D-types over sets in some D-model containing the set without any additional assumptions on the class of all the D-structures. Two particular cases of D-structures studied are (1) atomic models, when D is the collection of atomic types; and (2) homogeneous finite diagrams, under the extra assumption of existence of a monster D-model, i.e., (D, χ)-homogeneous model for some very large χ. In recent literature, homogeneous finite diagrams are called *homogeneous models*, and the subject homogeneous model theory. Everywhere below, when we are talking about a finite diagram, we always mean a finite diagram with the extra homogeneity assumption.

1.1.2 Abstract dependence relation

It is natural to define the dependence relation only on the "relevant" sets and models in our context. Let A, B, and C be D-sets such that $A \cup B \cup C$ is a D-set as well. The expression $A \underset{C}{\overset{(A)}{\perp}} B$ reads "A is independent from B over C," we use the superscript (A) to distinguish an abstract dependence relation from a concrete dependence relation in each context.

Definition 1.1.3. We call a relation $\overset{(A)}{\perp}$ on triples of D-sets $A, B, C \subset \mathfrak{C}$ *totally transcendental* if it satisfies the following conditions:

(1) **Invariance:** If $f \in \operatorname{Aut}(\mathfrak{C})$, then

$$A \underset{C}{\overset{(A)}{\perp}} B \quad \text{if and only if} \quad f(A) \underset{f(C)}{\overset{(A)}{\perp}} f(B).$$

(2) **Monotonicity:** Suppose $A \underset{C}{\overset{(A)}{\perp}} B$. For any B', C' such that $C \subseteq C' \subseteq B' \subseteq B$ we have $A \underset{C'}{\overset{(A)}{\perp}} B'$.

(3) **Finite Character:** If $A \underset{C}{\overset{(A)}{\not\perp}} B$, then there are finite tuples $\bar{a} \in A$, $\bar{b} \in B$ such that $\bar{a} \underset{C}{\overset{(A)}{\not\perp}} \bar{b}$.

(4) **Stationarity over finite subsets of models:** Suppose that M is a $(D, \aleph_0)$-homogeneous model, and $\bar{a}M$ is a D-set. There is a finite tuple $\bar{c} \in M$ such that $\bar{a} \underset{\bar{c}}{\overset{(A)}{\downarrow}} M$ and for any D-set B containing $\bar{c}$ the type $\mathrm{tp}(\bar{a}/\bar{c})$ can be uniquely extended to a $\overset{(A)}{\downarrow}$-independent D-type over B.

We use the name "totally transcendental" for such a dependence relation because we prove in the subsequent sections that existence of such a relation implies that a class of D-structures is totally transcendental as defined in Sections 1.2.3 and 1.3.

The properties (1)–(4) imply other properties of dependence (such as Extension, Symmetry, and Transitivity). We say more about this after Definition 1.1.6.

In the totally transcendental case, the stationarity property holds over $(D, \aleph_0)$-homogeneous models; when we go to the stable case, stationarity can be guaranteed for a smaller class of D-structures.

Definition 1.1.4. Let A be a D-set, suppose $\bar{a}$, $\bar{b}$ are finite tuples such that $A\bar{a}\bar{b}$ is a D-set as well. We say that $\bar{a}$ and $\bar{b}$ *have the same Lascar strong type over* A and write $\mathrm{lstp}(\bar{a}/A) = \mathrm{lstp}(\bar{b}/A)$ if $\bar{a}E\bar{b}$ for every A-invariant equivalence relation E with fewer than $|\,\mathfrak{C}\,|$ equivalence classes.

A D-model is *Lascar* (D, λ)*-homogeneous* if it realizes all the Lascar strong types over its subsets of size less than λ.

Remark 1.1.5. In [18], the term "a-saturated" is used for Lascar (D, κ)-homogeneous, for certain κ. We want the cardinal explicitly mentioned in the property.

Definition 1.1.6. Let A, B, C be as above. A relation $\overset{(A)}{\downarrow}$ is *stable* if it has the invariance, monotonicity, and finite character properties and in addition it satisfies:

Local Character: There is a cardinal κ such that for $A = \bar{a}$, there is $C \subset B$, $|C| < \kappa$, with $\bar{a} \underset{C}{\overset{(A)}{\downarrow}} B$.

Stationarity over models: Let κ be as above and suppose that M is a Lascar (D, κ)-homogeneous model, and $\bar{a}M$ is a D-set. Then for every D-set $B \supset M$ there is a unique $p \in S_D(B)$ such that for all $\bar{a}' \models p$ we have $\bar{a}' \underset{M}{\overset{(A)}{\downarrow}} B$.

If $\kappa = \aleph_0$, we call the above relation *superstable*.

The symmetry and transitivity properties for stable and totally transcendental dependence relations do hold. We show this in a rather indirect way. We prove in Subsections 1.2.5, 1.3.2 and 1.4.2 that abstract dependence has to coincide with the relations of strong splitting (for the stable case) and splitting (for totally transcendental). For those specific relations symmetry and transitivity hold, hence we can conclude that abstract dependence relations must have them as well. For the first order case, such an approach was used by J. Baldwin (see [3], Chapter 7). Beyond the stable context, symmetry can no longer be derived from other properties.

In the case of simple D-structures, we work in the context described by S. Buechler and O. Lessmann in [7]. They require the extension property over all the D-sets. In totally transcendental and stable case extension is a part of stationarity, and is guaranteed to hold over certain models only (or only for certain pairs of sets, see Fact 1.2.13(10)).

Definition 1.1.7. Let A, B, C be as above. For an infinite cardinal κ, s relation $\overset{(A)}{\smile}$ is κ-*simple* if it has the invariance, monotonicity, finite and κ-local character properties and in addition it satisfies:

Extension: If $\bar{a} \underset{B}{\overset{(A)}{\smile}} A$, $B \subset A$, then for all C there is $\bar{a}' \models \mathrm{tp}(\bar{a}/A)$ such that $\bar{a}' \underset{B}{\overset{(A)}{\smile}} C$.

Symmetry:

$$A \underset{C}{\overset{(A)}{\smile}} B \quad \text{if and only if} \quad B \underset{C}{\overset{(A)}{\smile}} A.$$

Transitivity: If $B \subset C \subset D$, then

$$A \underset{B}{\overset{(A)}{\smile}} C \text{ and } A \underset{C}{\overset{(A)}{\smile}} D \quad \text{if and only if} \quad A \underset{B}{\smile} D.$$

Type amalgamation: Suppose $\bar{a}_1$, $\bar{a}_2$ are tuples of length less than κ; $\bar{b}_1$, $\bar{b}_2$ are tuples of arbitrary size. If $\mathrm{lstp}(\bar{a}_1/C) = \mathrm{lstp}(\bar{a}_2/C)$, $\bar{b}_1 \underset{C}{\overset{(A)}{\smile}} \bar{b}_2$, and $\bar{a}_i \underset{C}{\overset{(A)}{\smile}} \bar{b}_i$, $i = 1, 2$, then there is $\bar{a} \models \mathrm{lstp}(\bar{a}_1/C\bar{b}_1) \cup \mathrm{lstp}(\bar{a}_2/C\bar{b}_2)$ such that $\bar{a} \underset{C}{\overset{(A)}{\smile}} \bar{b}_1\bar{b}_2$.

If $\kappa = \aleph_0$, we call the above relation *supersimple*. If $\overset{(A)}{\perp}$ is κ-simple for some κ, we call the relation *simple*.

It is easy to see that if $\overset{(A)}{\perp}$ is a totally transcendental dependence relation, then it is stable (in fact, superstable). However, it does not follow that it is supersimple or even simple. The problem is that extension (and other properties) hold over *arbitrary* sets in the simple case.

1.2 Atomic models

In this section we give some background information about totally transcendental classes of atomic models, in particular, we present the dependence relation for the classes introduced by Shelah in [31]. We then prove that existence of an (abstract) totally transcendental dependence relation on a class of atomic models implies that the class is totally transcendental and that the abstract relation coincides with that defined by Shelah.

1.2.1 Motivation

A major motivating question for developing classification theory in non-first order situation is Shelah's categoricity conjecture for $L_{\omega_1,\omega}$: if a sentence $\psi \in L_{\omega_1,\omega}$ is categorical in some cardinality above $\beth_{\omega_1}$, then it is categorical in every cardinality above $\beth_{\omega_1}$.

One of the tools to deal with this context was introduced by Shelah in [28]. There he suggests to expand the language with predicates that isolate complete types in the sense of $L_{\omega_1,\omega}$, and deal with atomic sets, types, and models of the appropriate first order theory. Of course, the class of atomic models is closely tied to the original class of models as described in Theorem 1.2.1.

In [28] Shelah showed the following result (Lemmas 2.5 and 3.1); we state it in the form closer to that in [31]. For a class of models $\mathcal{K}$, let $I(\lambda, \mathcal{K})$ denote the number of non-isomorphic models in $\mathcal{K}$ of cardinality λ.

Theorem 1.2.1 (Shelah). *Let ψ be a complete $L_{\omega_1,\omega}$ sentence, and suppose that in some uncountable $M^* \models \psi$ only countably many $L_{\omega_1,\omega}$ types are realized. Then there is a first order theory T in an expanded language $L(T)$ such that, letting $\mathcal{K}$ be the class of atomic models of T,*

(1) every formula in $L(T)$ is equivalent to an atomic formula modulo T;

(2) $\mathcal{K}$ has an uncountable model, and if $\mathrm{Mod}(\psi)$ has arbitrarily large models, then so does $\mathcal{K}$;

(3) every model in $\mathcal{K}$ can be made into a model of ψ, so $I(\lambda, \mathcal{K}) \leq I(\lambda, \mathrm{Mod}(\psi))$ for all λ. More precisely, $I(\lambda, \mathcal{K}) = I(\lambda, \{M \models \psi \mid M \equiv_{\infty,\omega} M^*\}$.

The assumption on countably many $L_{\omega_1,\omega}$ types in an uncountable model is not too restricting. Shelah proved that this holds if ψ has countably many non-isomorphic models in $\aleph_1$, or if it has less than $2^{\aleph_1}$ non-isomorphic models (the latter requires a mild set-theoretic assumption). It is easy to see that if ψ has arbitrarily large models, the assumption also holds no matter how many models there are in $\aleph_1$.

Recall that for a first order theory T and a set of types Γ in the language of T, an $EC(T, \Gamma)$ class is a class of models of T that omit the types in Γ. For an $EC(T, \Gamma)$ class, Theorem 1.2.1 works when the class has arbitrarily large models; or if it can be axiomatized by an $L_{\omega_1,\omega}$ sentence with properties implying existence of an uncountable model with countably many $L_{\omega_1,\omega}$ types.

Assumptions 1.2.2. For the rest of this section, we fix a first order theory T in a relational language such that every formula is equivalent to an atomic formula modulo T. Let $\mathfrak{C}$ be the monster model of T, and let $\mathcal{K}$ be the class of atomic elementary submodels of $\mathfrak{C}$. By default, $\models \varphi$ means satisfaction in $\mathfrak{C}$, and all the sets and elements are in $\mathfrak{C}$. By a "model" we mean a model in $\mathcal{K}$.

We further assume that $\mathcal{K}$ has an $\aleph_0$-amalgamation property, that is for all countable models $M \prec M_0, M_1$ there is a countable model N and elementary embeddings $f_i : M_i \to N$, $i = 0, 1$ that coincide on M. The amalgamation property holds for instance, if $I(\aleph_1, \mathcal{K}) < 2^{\aleph_1}$ and $2^{\aleph_0} < 2^{\aleph_1}$ by [31] for $\aleph_0$-categorical $\mathcal{K}$.

1.2.2 Preliminary results and definitions

We introduce now some important definitions and basic results for the context. All of the definitions and almost all the results are due to Shelah [31]. We will use them extensively in this section.

Definition 1.2.3. (1) A set B is *constructible over* A if $B = A \cup \{b_i \mid i < \alpha\}$, where for all $i < \alpha$ the type $\mathrm{tp}(b_i/A \cup \{b_j \mid j < i\}$ is isolated. A constructible model M over A is called *primary over* A.

(2) A model M is *universal over* A if $A \subset M$, $\|M\| = |A|$, and every $N \supset A$ of the same cardinality can be elementarily embedded into M over A.

(3) An atomic set $A \subset \mathfrak{C}$ is *good* if for each $\bar{a} \in A$ if $\models \exists \bar{x} \varphi(\bar{x}, \bar{a})$, then $\varphi(\bar{x}, \bar{a})$ belongs to a type $p \in S_D(A)$ (i.e., to an isolated type over A).

In [31], the set of types $S_D(A)$ is denoted D_A.

Remark 1.2.4. If T is an $\aleph_0$-stable countable first order theory, then every set $A \subset \mathfrak{C}$ is good. In our context, it is possible to have a situation when the class $\mathcal{K}$ is $\aleph_0$-stable (i.e., $|S_D(M)| = \aleph_0$ for all countable atomic M), but the underlying theory T is not. The example in [15] shows in particular that not every set is good in general for an $\aleph_0$-stable class $\mathcal{K}$. A simpler example of a non-good set can be found in [1].

Definition 1.2.5. The pair (A, B), $A \subset B$, satisfies the *Tarski-Vaught condition* if for every $\bar{b} \in B$ and $\bar{a} \in A$ if $\models \varphi[\bar{b}, \bar{a}]$, then there is $\bar{b}' \in A$ such that $\models \varphi[\bar{b}', \bar{a}]$.

We call such pair (A, B) a *Tarski-Vaught pair* and write $A \subset_{TV} B$.

We list some properties of Tarski-Vaught pairs that we will use later.

Proposition 1.2.6. *If $M \in \mathcal{K}$ and $M \cup B$ is atomic, then $(M, M \cup B)$ satisfies the Tarski-Vaught condition.*

Proof. Suppose $\bar{a} \in |M|$, $\bar{b} \in B$ and $\models \varphi[\bar{a}, \bar{b}]$. Since $M \cup B$ is atomic, there is $\psi(\bar{x}, \bar{y})$ isolating the type $\mathrm{tp}(\bar{a}\bar{b}/\emptyset)$. Since $\models \exists \bar{y} \psi(\bar{a}, \bar{y})$, there is $\bar{b}' \in |M|$ such that $M \models \psi[\bar{a}, \bar{b}']$. Clearly, $M \models \psi[\bar{a}, \bar{b}']$. $\dashv$

Claim 1.2.7. *Suppose* $B \subset_{TV} C$*,* $\bar{d}C$ *is atomic, and assume* $\mathrm{tp}(\bar{d}/B)$ *is isolated by* $\varphi(\bar{x}, \bar{b})$ *for some* $\bar{b} \in B$*. Then* $\varphi(\bar{x}, \bar{b})$ *isolates the type* $\mathrm{tp}(\bar{d}/C)$*.*

Proof. Let $\bar{d} \models \psi(\bar{x}, \bar{c})$ for some $\bar{c} \in C$. Since $B \subset_{TV} C$, there is $\bar{c}' \in B$ such that $\mathrm{tp}(\bar{c}'/\bar{b}) = \mathrm{tp}(\bar{c}/\bar{b})$. Since $\models \forall\bar{x}\varphi(\bar{x}, \bar{b}) \to \psi(\bar{x}, \bar{c}')$, conjugating $\bar{c}'$ to $\bar{c}$ over $\bar{b}$ we get $\models \forall\bar{x}\varphi(\bar{x}, \bar{b}) \to \psi(\bar{x}, \bar{c})$. ⊣

Remark 1.2.8. The dependence relation for the class $\mathcal{K}$ that we describe in the next subsection does not have the extension property in general. We can show that extension holds for pairs that satisfy the Tarski-Vaught condition, and get a nice description of the relation for this case. Shelah defined the dependence relation for pairs that satisfy the Tarski-Vaught condition in [31]. Getting the extension property is probably the main reason for introducing the concept in this context.

1.2.3 Rank and dependence relation in atomic models

Let $\mathcal{K}$ be a class of atomic models of a first order theory T such that Assumptions 1.2.2 hold. We give the definition of a rank function (it is due to Shelah [28], though we present it in a slightly different form) and describe the resulting dependence relation for the class.

Definition 1.2.9. Let M be a model, let p be a finite type over (finite) $B \subset |M|$.

(1) $R_M[p] \geq 0$ if p is realized in M.

(2) for α limit ordinal, $R_M[p] \geq \alpha$ if $R_M[p] \geq \beta$ for all $\beta < \alpha$.

(3) $R_M[p] \geq \alpha + 1$ if

  (a) there are $\varphi(\bar{x}, \bar{y})$ and $\bar{a} \in M$ such that

$$R_M[p \cup \varphi(\bar{x}, \bar{a})] \geq \alpha \quad \text{and} \quad R_M[p \cup \neg\varphi(\bar{x}, \bar{a})] \geq \alpha;$$

  (b) for every $\bar{b} \in |M|$ there is a complete formula $\psi(\bar{x}, \bar{y})$ such that

$$R_M[p \cup \psi(\bar{x}, \bar{b})] \geq \alpha.$$

As usual, we say
$R_M[p] = -1$ if $R_M[p] \not\geq 0$;
$R_M[p] = \alpha$ if $R_M[p] \geq \alpha$ and $R_M[p] \not\geq \alpha + 1$;
$R_M[p] = \infty$ if $R_M[p] \geq \alpha$ for all $\alpha \in \mathrm{On}$;
for an arbitrary type q over a subset of M, we let

$$R_M[q] := \mathrm{Min}\{R_M[p] \mid p \subseteq q, p \text{ finite}\}.$$

If the model $M \in \mathcal{K}$ is clear from the context, we omit the subscript M in the notation for the rank.

The following properties of the rank appeared in [28].

Fact 1.2.10 (Properties of the rank). *(1)* ***Invariance:*** *if $f \in \mathrm{Aut}(\mathfrak{C})$, then $R_M[p] = R_M[f(p)]$;*

(2) ***Monotonicity:*** *If $p \vdash q$, then $R_M[p] \leq R_M[q]$;*

(3) If $R_M[p] \geq \omega_1$, then $R_M[p] = \infty$;

(4) ***Stationarity:*** *Suppose $R_M[\bar{x} = \bar{x}] < \infty$. Let $N \in \mathcal{K}$ be a submodel of M and let p be a complete type over N. Then there exist $\bar{b} \in N$ and a formula φ such that $R_M[p] = R_M[\varphi(\bar{x}, \bar{b})]$. Moreover, if $A \subset M$, $\bar{b} \in A$, then there is a unique type p_A containing $\varphi(\bar{x}, \bar{b})$ with the same rank.*

Certainly, the rank can be unbounded in general. To define a dependence relation, we need to assume that the rank is bounded. Boundedness of the rank can be obtained if, for instance, the class comes from an $L_{\omega_1,\omega}$ sentence that has less than $2^{\aleph_1}$ models of cardinality $\aleph_1$ under the assumption $2^{\aleph_0} < 2^{\aleph_1}$ (it was done in [28] under CH, and in [31] under weak CH). The class $\mathcal{K}$ is called *totally transcendental* if R is bounded. For the rest of this subsection, we assume that the class $\mathcal{K}$ is totally transcendental.

Facts 1.2.11. *(1) In this context, if M is a primary model over A, then it is unique over A. If M is primary over A, then it is prime over A (i.e., can be elementarily embedded in any $N \supset A$ over A).*

(2) Let A be a countable set. A is good if and only if there is a countable primary model over A.

(3) If A is countable, then A is not good if and only if $|S_D(A)| = 2^{\aleph_0}$.

(4) Let A be countable good set. There is a countable model N that is $(D_A, \aleph_0)$-homogeneous. The model is unique and universal over A. Moreover, a countable A is good if and only if there is a countable universal model over it.

The usual way to define a dependence relation from a rank is by saying A is independent of B over C if for all $\bar{a} \in A$ the ranks of $\mathrm{tp}(\bar{a}/B)$ and $\mathrm{tp}(\bar{a}/C)$ coincide. The limitation is of course that the rank is computed inside a model in $\mathcal{K}$. In addition, one can get the extension property for the rank only for types over models.

The following dependence relation for $\mathcal{K}$ was suggested by Shelah (see [31]). It is defined only for good sets, essentially because these are the sets of interest here and since one cannot expect a good dependence relation for all atomic sets. We discuss this further in the next section.

Definition 1.2.12. Suppose $A \cup B \cup C$ is atomic and C is good. Then A is *independent of B over C* (we write $A \underset{C}{\downarrow} B$) if for each $\bar{a} \in A$, $\mathrm{tp}(\bar{a}/B)$ does not split over some finite subset of C.

We summarize below the properties of this dependence relation. Many of the properties were defined in Section 1, we restate some of them for this situation to avoid any ambiguities.

Fact 1.2.13 (Properties of $\downarrow$). *The relation $\downarrow$ has the following properties:*

(1) Invariance;

(2) Monotonicity;

(3) Local Character: For all $\bar{a}$ and B such that $\bar{a} \cup B$ is atomic and B is good, there is a finite $\bar{b} \in B$ such that $\bar{a} \underset{\bar{b}}{\downarrow} B$.

(4) Existence: For all A, C such that $A \cup C$ is atomic and C is good we have $A \underset{C}{\downarrow} C$.

(5) *Extension: If* $\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}_{A} B$ *and* C *is a good set such that* $B \subset C$ *and* $A \subset_{TV} C$*, then there is* $\bar{a}' \models \mathrm{tp}(\bar{a}/B)$ *such that* $\bar{a}' \mathop{\smile\!\!\!\!\!\!\,|}_{A} C$*. In particular, there is always an independent extension of a type over a model.*

(6) *Finite Character;*

(7) *Symmetry over models;*

(8) *Transitivity over Tarski-Vaught pairs: Suppose* $B \subset_{TV} C \subset_{TV} D$*, and* B, C *are good. If* $\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}_{B} C$ *and* $\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}_{C} D$*, then* $\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}_{B} D$*.*

(9) *Stationarity over finite subsets of models: Suppose* $M \in \mathcal{K}$*,* $\bar{a} \cup M \cup B$ *is atomic. There is a finite* $\bar{c} \in M$ *such that* $\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}_{\bar{c}} M$*. Moreover, if* B *is atomic,* $B \supset M$*, then there is* $\bar{a}' \models \mathrm{tp}(\bar{a}/M)$ *such that* $\bar{a}' \mathop{\smile\!\!\!\!\!\!\,|}_{\bar{c}} B$ *and such an extension is unique over* $\bar{c}$*. We say that* $\mathrm{tp}(\bar{a}/\bar{c})$ *is stationary, and the extension to* B *is a stationarization of* $\mathrm{tp}(\bar{a}/\bar{c})$ *in this situation.*

(10) *Weak stationarity over good sets: Suppose* A *is good and* $\bar{a}A$ *is atomic. There is finite* $\bar{c} \in A$ *such that* $\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}_{\bar{c}} A$ *and for any atomic set* B*,* $A \subset_{TV} B$*, the type* $\mathrm{tp}(\bar{a}/\bar{c})$ *can be uniquely extended to an independent atomic type over* B*. We say that* $\mathrm{tp}(\bar{a}/\bar{c})$ *is weakly stationary over* $\bar{c}$*, and the unique extension is the weak stationarization in this case.*

(1) and (2) are obvious from the definition. (3) is proved in Lemma 2.2(1), last sentence, in [31] and (4) follows at once from (3). (5) is proved in Lemma 2.10(1) in [31]. (6) is immediate by the definition. (7) appears in Theorem 1.4.1(c) in [31], for a complete proof see [1]. (8) is in Lemma 2.10(1), second paragraph, in [31]. (9) follows from Theorem 1.4.1(b) in [31]; (10) from local character and Lemma 2.10(1) in Shelah.

1.2.4 Some negative results

The purpose of this subsection is to complement the next section, where we prove a uniqueness result assuming existence of a dependence relation with certain properties. Here we present various results showing that requiring less from an abstract dependence relation is unreasonable.

We start by pointing out that we do need to restrict the dependence relation to good sets. Suppose A is countable, not good. Then $D_A = \{tp(\bar{a}/A) \mid A\bar{a} \text{ is atomic}\}$ is uncountable by Facts 1.2.11 (in fact, it has size continuum). So we get an instability phenomenon in this situation. The reason is that the theory T need not be $\aleph_0$-stable (or stable at all). Our intention, however, is to investigate the totally transcendental atomic part of T, so we need to choose carefully which sets to deal with.

Next we show that one can define two dependence relations that coincide on Tarski-Vaught pairs, but differ on other (countable) sets.

Claim 1.2.14. *Let B, C be good countable with $B \subset_{TV} C$. Suppose further that $\bar{a}C$ is atomic. Let M_B, M_C be primary models over B and C respectively. Suppose $\bar{a} \underset{M_B}{\perp} M_C$. Then $\bar{a} \underset{B}{\perp} C$.*

Proof. By Theorem 1.6(1) of [31], the primary model M_B is also prime over B, so we may assume that $M_B \prec M_C$. Let $\bar{d} \in M_B$ be such that $\mathrm{tp}(\bar{a}/\bar{d})$ is stationary. Since $\bar{d} \in M_B$ and M_B is primary over B, the type of $\bar{d}$ over B is isolated. Let $b \in B$ and φ be such that $\varphi(\bar{x}, \bar{b}) \vdash \mathrm{tp}(\bar{d}/B)$. Since $B \subset_{TV} C$ and certainly $\bar{d}C$ is atomic, by Claim 1.2.7 we get that $\varphi(\bar{x}, \bar{b})$ isolates $\mathrm{tp}(\bar{d}/C)$.

Now by a standard argument we conclude that since $\mathrm{tp}(\bar{a}/M_C)$ is stationary over $\bar{d}$, and $\varphi(\bar{x}, \bar{b})$ isolates $\mathrm{tp}(\bar{d}/C)$, we have $\mathrm{tp}(\bar{a}/C)$ does not split over $\bar{b}$. $\dashv$

So we can define another dependence relation on good atomic subsets.

Definition 1.2.15. For B, C good, $\bar{a}C$ atomic $\bar{a}$ is independent from C over B if there are primary models $M_B \prec M_C$ over B and C respectively and $\bar{a} \underset{M_B}{\perp} M_C$ ($\perp$ is in the sense of non-splitting); if there is no primary model over one of the sets (they are not guaranteed to exist for uncountable good sets), then we leave the relation as was defined before.

The new dependence relation coincides with the standard one for Tarski-Vaught pairs by Claim 1.2.14. However, for a countable good B and all $\bar{a}$ such that $\bar{a}B$ is atomic we have $\bar{a}$ is independent from M_B over B according to Definition 1.2.15 (the independence would mean $\bar{a} \underset{M_B}{\perp} M_B$, which holds for $\perp$). If B is not a model and M_B contains at least two realizations $\bar{a}$, $\bar{b}$ of an isolated type over B, then certainly $\operatorname{tp}(\bar{a}/M_B)$ splits over B and hence, it splits over every finite subset of B. We illustrate this on the following simple example.

Example 1.2.16. Let φ be a Scott sentence for an algebraically closed field of characteristic zero of infinite transcendence degree. Clearly, φ has models in every infinite cardinality and is totally categorical. Let $\mathcal{K}$ be the class of atomic models in the expanded language constructed as described in Theorem 1.1. The situation in class $\mathcal{K}$ is actually very close to that in first order algebraically closed fields. Letting $\bar{a} := i$, $B := \mathbb{Q}$, and M_B an algebraic closure of $\mathbb{Q}$ of infinite transcendence degree, we see that $\bar{a}$ is independent from M_B over B according to Definition 1.2.15 of dependence. It is also clear that $\operatorname{tp}(\bar{a}/M_B)$ splits over B.

The example shows that, without restriction to the Tarski-Vaught pairs, one cannot hope to uniquely characterize the dependence relations in atomic models.

We also can look at this example from another angle. Every type over a good set can be split into a stationary and an isolated part (see Fact 1.2.17). As we show in the next section, the dependence relation for stationary types can be uniquely characterized (so the notion of stationarity is invariant in certain sense). Our example also shows that dependence for isolated types can be decided positively or negatively without affecting the stationary part.

Fact 1.2.17 (Shelah [31]). *Suppose A is good. A type p is atomic over A if and only if there are $\bar{a}$ and $\bar{d}$ such that $p = \operatorname{tp}(\bar{a}/A)$, the type $\operatorname{tp}(\bar{a}/\bar{d})$ is stationary, $\operatorname{tp}(\bar{a}/A\bar{d})$ is the non-splitting extension of $\operatorname{tp}(\bar{a}/\bar{d})$, and $\operatorname{tp}(\bar{d}/A)$ is isolated.*

1.2.5 Abstract dependence characterization

In this subsection, we prove that if $\mathcal{K}$ has an abstract totally transcendental dependence relation in the sense of the Definition 1.1.3, then the rank R has to be bounded for $\mathcal{K}$, i.e., $\mathcal{K}$ is totally transcendental. We also show that the relation $\overset{(A)}{\smile}$ coincides with the dependence relation introduced above over models. We show that a stronger stationarity assumption for $\overset{(A)}{\smile}$ implies that it coincides with $\smile$ over all good sets.

Theorem 1.2.18. *Suppose $\mathcal{K}$ has an abstract totally transcendental dependence relation. Then the rank function R is bounded on $\mathcal{K}$.*

Proof. If the rank $R_M[p]$ is unbounded, then by Lemma 4.2 in [28] there is a countable model M such that $|S_D(M)| \geq \aleph_1$. By stationarity, each of those types is $\overset{(A)}{\smile}$-independent over a finite subset of M. By the pigeonhole principle, there are at least $\aleph_1$ types in $S_D(M)$ that are independent over the same subset of M. Since there are only countably many D-types over a finite set, by pigeonhole principle again we conclude that there are $\aleph_1$ independent extensions of the same type over the stationarity base. Contradiction to the stationarity over finite subsets of models. ⊣

The following definition will facilitate the proofs characterizing the abstract dependence relation in terms of $\smile$ over models and over good sets in general.

Definition 1.2.19. Let C be a good set and $\bar{a}$ be such that $C\bar{a}$ is atomic. We say that $\mathrm{tp}(\bar{a}/C)$ is $\overset{(A)}{\smile}$*-weakly stationary* if there is $\bar{c} \in C$ such that $\bar{a} \underset{\bar{c}}{\overset{(A)}{\smile}} C$ and for all atomic B containing $\bar{c}$, $C \subset_{TV} B \cup C$, the type $\mathrm{tp}(\bar{a}/\bar{c})$ can be uniquely extended to a $\overset{(A)}{\smile}$-independent atomic type over B.

Remarks 1.2.20. (1) If $\mathcal{K}$ has an abstract totally transcendental dependence relation, and C is the universe of a model, then every type over C is $\overset{(A)}{\smile}$-weakly stationary for all atomic $B \supset C$. That is simply because of the stationary over models property.

(2) If $\mathcal{K}$ is totally transcendental, then types over good sets are $\smile$-weakly stationary.

We first prove that the relations are the same for $\overset{(A)}{\smile}$-*weakly stationary* types. This will imply that stationarity in the sense of $\overset{(A)}{\smile}$ and in the sense of non-splitting $\smile$ is the same thing. So the dependence relation is unique for types over models.

Theorem 1.2.21. *Let $\overset{(A)}{\smile}$ be an abstract totally transcendental dependence relation on $\mathcal{K}$. Suppose $\bar{a}$ is a finite atomic tuple, C is a good set, and B is atomic with $C \subset_{TV} B$. Suppose that $\mathrm{tp}(\bar{a}/C)$ is $\overset{(A)}{\smile}$-weakly stationary. Then*

$$\bar{a} \underset{C}{\overset{(A)}{\smile}} B \quad \textit{if and only if} \quad \bar{a} \underset{C}{\smile} B.$$

Proof. ($\Rightarrow$) Suppose $\bar{a} \underset{C}{\overset{(A)}{\smile}} B$. Let $\bar{c}$ be as in the definition of weak stationarity. Then we have $\bar{a} \underset{\bar{c}}{\overset{(A)}{\smile}} B$ by uniqueness of independent extension. We prove that $\mathrm{tp}(\bar{a}/B)$ does not split over $\bar{c}$. Suppose not; let $\bar{b}_1, \bar{b}_2 \in B$ be such that $\mathrm{tp}(\bar{b}_1/\bar{c}) = \mathrm{tp}(\bar{b}_2/\bar{c})$, but the types $\mathrm{tp}(\bar{a}\bar{b}_1/\bar{c})$ and $\mathrm{tp}(\bar{a}\bar{b}_2/\bar{c})$ are different. By monotonicity, we have

$$\bar{a} \underset{\bar{c}}{\overset{(A)}{\smile}} \bar{c}\bar{b}_1 \quad \text{and} \quad \bar{a} \underset{\bar{c}}{\overset{(A)}{\smile}} \bar{c}\bar{b}_2.$$

Let $f \in \mathrm{Aut}_{\bar{c}}(\mathfrak{C})$ be such that $f(\bar{b}_1) = \bar{b}_2$. Let $\bar{a}_1 := f(\bar{a})$. Then by invariance $\bar{a}_1 \underset{\bar{c}}{\overset{(A)}{\smile}} \bar{c}\bar{b}_2$. Thus we get two distinct $\overset{(A)}{\smile}$-independent extensions of $\mathrm{tp}(\bar{a}/\bar{c})$ to $\bar{c}\bar{b}_2$. Since $C \subset_{TV} C \cup \bar{b}_2$, we get a contradiction to weak stationarity.

($\Leftarrow$) Suppose $\mathrm{tp}(\bar{a}/B)$ does not split over a finite subset $\bar{c}_1$ of C. Suppose for contradiction that $\bar{a} \underset{C}{\overset{(A)}{\not\smile}} B$. Let $\bar{a}' \models p$, $p = \mathrm{tp}(\bar{a}/C)$ be such that $\bar{a}' \underset{C}{\overset{(A)}{\smile}} B$ (possible to find by extension requirement). By the first part of the proof, $\mathrm{tp}(\bar{a}'/B)$ does not split over a finite subset $\bar{c}_2 \in C$. By invariance, $\mathrm{tp}(\bar{a}/B) \neq \mathrm{tp}(\bar{a}'/B)$, so there are $\varphi(\bar{x}, \bar{y})$ and $\bar{d} \in B$ such that $\varphi(\bar{x}, \bar{d}) \in \mathrm{tp}(\bar{a}/B)$ and $\neg\varphi(\bar{x}, \bar{d}) \in \mathrm{tp}(\bar{a}'/B)$. Since $C \subset_{TV} B$, we can find $\bar{e} \in C$ such that $\mathrm{tp}(\bar{d}/\bar{c}_1 \cup \bar{c}_2) = \mathrm{tp}(\bar{e}/\bar{c}_1 \cup \bar{c}_2)$. Since $\varphi(\bar{x}, \bar{d}) \in \mathrm{tp}(\bar{a}/B)$, necessarily $\neg\varphi(\bar{x}, \bar{e}) \notin p$ (otherwise, $\mathrm{tp}(\bar{a}/B)$ splits over $\bar{c}_1$). Similarly, $\varphi(\bar{x}, \bar{e}) \notin p$ by $\neg\varphi(\bar{x}, \bar{d}) \in \mathrm{tp}(\bar{a}'/B)$. So we get a contradiction to completeness of the type p. $\dashv$

Now from the remarks above and Theorem 1.2.21 we conclude:

Corollary 1.2.22. *If $\mathcal{K}$ has a totally transcendental dependence relation $\overset{(A)}{\downfree}$, $M \in \mathcal{K}$ is a model, $B \supset M$, and $\bar{a}B$ is atomic, then*

$$\bar{a} \underset{M}{\overset{(A)}{\downfree}} B \quad \textit{if and only if} \quad \bar{a} \underset{M}{\downfree} B.$$

Note that we did not require the symmetry property for the abstract dependence relation. The following is a (rather indirect) proof that symmetry property over models follows.

Corollary 1.2.23 (Symmetry property). *Suppose $\mathcal{K}$ has an abstract dependence relation $\overset{(A)}{\downfree}$ satisfying (1)–(4). For $M \in \mathcal{K}$, $\bar{a}$, $\bar{b}$ such that $M\bar{a}\bar{b}$ is atomic we have*

$$\bar{a} \underset{M}{\overset{(A)}{\downfree}} M\bar{b} \quad \Longleftrightarrow \quad \bar{b} \underset{M}{\overset{(A)}{\downfree}} M\bar{a}.$$

Proof. By Theorem 1.2.18 the rank R is bounded for the class $\mathcal{K}$. By Lemmas 4.2 and 6.4 in [28], the dependence relation $\downfree$ has the symmetry property. By Corollary 1.2.22, $\overset{(A)}{\downfree}$ has the symmetry over models property as well. ⊣

Suppose now that $\overset{(A)}{\downfree}$ satisfies the following additional property:

Uniqueness of extension:

Assume that C is good, $\bar{a}C$ is atomic. There is finite $\bar{c} \in C$ such that $\bar{a} \underset{\bar{c}}{\overset{(A)}{\downfree}} C$ and for any atomic set B, $C \subset_{TV} B \cup C$, the type $\mathrm{tp}(\bar{a}/\bar{c})$ can be uniquely extended to a $\overset{(A)}{\downfree}$-independent atomic type over B.

Remarks 1.2.24. (1) Certainly, the Uniqueness of extension property implies Stationarity over models. We have seen above that having Stationarity over models property already allows to draw many conclusion about the class $\mathcal{K}$. Perhaps, in the non-first order situation, types over models are the right objects to look at.

(2) Assuming Uniqueness property, we characterize the dependence not only for stationary types, but also for Tarski-Vaught good pairs. That is because with Uniqueness, the types are $\overset{(A)}{\downfree}$-weakly stationary.

Corollary 1.2.25. *If $\mathcal{K}$ has a totally transcendental dependence relation satisfying the Uniqueness property, C is good and $C \subset_{TV} B$. If $\bar{a}B$ is atomic, then*

$$\bar{a} \overset{(A)}{\underset{C}{\downarrow}} B \quad \textit{if and only if} \quad \bar{a} \underset{C}{\downarrow} B.$$

1.3 Totally transcendental finite diagrams

The goal of this section is to establish results parallel to those of Section 2 in the context of finite diagrams. The subject here is a class of models of a first order theory T such that each model omits a set of types Γ. Such classes are denoted $EC(T, \Gamma)$. A homogeneous finite diagram is a class $EC(T, \Gamma)$ with one extra assumption: everything happens inside a big homogeneous model that also belongs to the class. In other words we assume existence of a monster model that omits all the types in Γ. Another way to look at the class $EC(T, \Gamma)$ is to treat it as a class of D-models, where $D = D(T) \setminus \Gamma$. So the extra assumptions translate into existence of a large homogeneous D-model. As the result of this assumption, we can, for instance, realize unions of D-types over sets of cardinality less than the cardinality of the monster model. For the remainder of the section, we agree to use the symbol $\mathfrak{C}$ for the monster D-model.

Homogeneous finite diagrams in the stable context were introduced by S. Shelah in [27], with recent extensions due to R. Grossberg and O. Lessmann [12] and T. Hyttinen and S. Shelah [18, 19]. The totally transcendental case was studied by O. Lessmann in [24].

Although there is a similarity in methods in the contexts of atomic models and homogeneous finite diagrams, it is not true that either one case is a subcase of the other, so we cannot directly apply the results of the previous section.

1.3.1 Rank and dependence relation

A dependence relation for $\aleph_0$-stable finite diagrams was introduced by Olivier Lessmann in [24]. The dependence relation is defined via the rank function. The challenge is to make sure that unbounded rank gives uncountably many types over a countable set, and that the types are realized in the monster model. This is achieved by adding an extra condition to the definition of the 2-rank (due to Lessmann) that we give now. Note the similarity between this rank and the rank in Subsection 1.2.3

Definition 1.3.1. Let p be a type over a finite $B \subset |\mathfrak{C}|$.

(1) $R[p] \geq 0$ if p is realized in $\mathfrak{C}$.

(2) for α limit ordinal, $R[p] \geq \alpha$ if $R[p] \geq \beta$ for all $\beta < \alpha$.

(3) $R[p] \geq \alpha + 1$ if

(a) there are $\varphi(\bar{x}, \bar{y})$ and $\bar{a} \in \mathfrak{C}$ such that

$$R[p \cup \varphi(\bar{x}, \bar{a})] \geq \alpha \quad \text{and} \quad R[p \cup \neg\varphi(\bar{x}, \bar{a})] \geq \alpha;$$

(b) for every $\bar{b} \in |\mathfrak{C}|$ there is a complete type $q(\bar{x}, \bar{y}) \in D$ such that

$$R[p \cup q(\bar{x}, \bar{b})] \geq \alpha.$$

As usual,
$R[p] = -1$ if $R[p] \not\geq 0$;
$R[p] = \alpha$ if $R[p] \geq \alpha$ and $R[p] \not\geq \alpha + 1$;
$R[p] = \infty$ if $R[p] \geq \alpha$ for all $\alpha \in \mathrm{On}$;
if q is a type over a subset of $\mathfrak{C}$ which is not necessarily finite, we let

$$R[q] := \mathrm{Min}\{R[p] \mid p \subseteq q, \mathrm{dom}(p) \text{ finite}\}.$$

The rank function has similar properties to the one defined in the previous section. For the rest of this subsection we assume that the diagram is totally transcendental, that is, the rank is bounded. Thus we can define the dependence relation by equality of the ranks:

Definition 1.3.2. Suppose A, B, C are subsets of $\mathfrak{C}$ such that $B \subset A$. Then $A \underset{B}{\perp} C$ if and only if for all $\bar{a} \in A$ $R[\mathrm{tp}(\bar{a}/B)] = R[\mathrm{tp}(\bar{a}/C)]$.

One of the key notions is stationarity. We give a definition formally different from the one suggested by Olivier Lessmann although the concepts we define are the same.

Definition 1.3.3. A D-type p is *stationary over* $B \subseteq \mathrm{dom}(p)$ if it has a unique $\smile$-independent extension to any superset of B.

For finite diagrams, stationary types are the most natural candidates for studying dependence relations. As well as in the atomic case, every type "splits" into stationary and isolated parts, for the right notion of isolation. We give now some definitions and facts, all of them due to Olivier Lessmann [24].

Definition 1.3.4. A type $p \in S_D(A)$ is D^s_λ*-isolated* if there is $B \subset A$, $|B| < \lambda$, such that for any $q \in S_D(A)$ extending $p \upharpoonright B$ we have $p = q$.

Facts 1.3.5. *(1) Let* $p \in S_D(A)$ *realized by* $\bar{a}$*. There is* $\bar{d}$ *such that the type* $\mathrm{tp}(\bar{a}/\bar{d})$ *is stationary,* $\mathrm{tp}(\bar{a}/A\bar{d})$ *is the unique independent extension of* $\mathrm{tp}(\bar{a}/\bar{d})$*, and* $\mathrm{tp}(\bar{d}/A)$ *is* $D^s_{\aleph_0}$*-isolated.*

(2) If M *is a* $(D, \aleph_0)$*-homogeneous model and* $p \in S_D(M)$*, then* p *is stationary over a finite subset of* M*.*

(3) If $\mathrm{tp}(\bar{a}/B)$ *is stationary, then* $\bar{a} \underset{B}{\smile} C$ *if and only if* $\mathrm{tp}(\bar{a}/C)$ *does not split over a finite subset of* B*.*

1.3.2 Abstract dependence characterization

Fact 1.3.6. *The dependence relation defined in [24] satisfies the properties of a totally transcendental abstract dependence relation. In addition, this dependence relation has symmetry and transitivity properties.*

Same theorems we proved in the previous section are true here as well. The proofs are almost the same, so we just state the results.

Theorem 1.3.7. *Suppose* $\mathfrak{C}$ *has a totally transcendental dependence relation. Then the rank function* R *is bounded on* $\mathfrak{C}$*.*

Theorem 1.3.8. *If $\mathfrak{C}$ has a totally transcendental dependence relation, M is a $(D, \aleph_0)$-homogeneous model, and $B \supset M$, then*

$$\bar{a} \overset{(A)}{\underset{M}{\perp}} B \quad \textit{if and only if} \quad \bar{a} \underset{M}{\perp} B.$$

Note that the stationary bases may be different in different dependence relations. As before, we also get symmetry property for types over models.

Corollary 1.3.9 (Symmetry property). *Suppose $\overset{(A)}{\perp}$ is a totally transcendental dependence relation on $\mathfrak{C}$. If M is a $(D, \aleph_0)$-homogeneous model, then*

$$\bar{a} \overset{(A)}{\underset{M}{\perp}} M\bar{b} \quad \Longleftrightarrow \quad \bar{b} \overset{(A)}{\underset{M}{\perp}} M\bar{a}.$$

We can extend our results to more sets, similar to what was done in the first section. We need the notion of a Tarski-Vaught pair. In finite diagrams, it translates to a relative saturation requirement; but we keep the Tarski-Vaught name.

Definition 1.3.10. We say that a pair of sets (A, B), $A \subset B$, satisfies the *D-Tarski-Vaught condition* if for every $\bar{b} \in B$, $\bar{a} \in A$, and $q(\bar{x}, \bar{y}) \in D$ if $\bar{b} \models q(\bar{x}, \bar{a})$, then there is $\bar{b}' \in A$ such that $\bar{b}' \models q(\bar{x}, \bar{a})$. We write $A \subset_{TV} B$.

Now if we replace Stationarity over finite subsets of models in the definition of a totally transcendental dependence relation by a stronger condition

Uniqueness of extension:
Assume in addition that $A =: \bar{a}$ is finite. There is finite $\bar{c} \in C$ such that $\mathrm{tp}(\bar{a}/C)$ is weakly stationary over $\bar{c}$. That is, for any B, $C \subset_{TV} B$, there is $\bar{a}' \models \mathrm{tp}(\bar{a}/C)$ such that $\bar{a}' \overset{(A)}{\underset{\bar{c}}{\perp}} B$, and such an extension is unique over $\bar{c}$.
then we can get a stronger result:

Theorem 1.3.11. *Suppose $\overset{(A)}{\perp}$ is a totally transcendental dependence relation on $\mathfrak{C}$. Suppose $\bar{a}$ is a finite tuple, C is a set, and B is such that $C \subset_{TV} B$. Then*

$$\bar{a} \overset{(A)}{\underset{C}{\perp}} B \quad \textit{if and only if} \quad \bar{a} \underset{C}{\perp} B.$$

1.4 Stable homogeneous finite diagrams

In this section, we prove that a finite diagram is stable if and only if it has a stable dependence relation (see Definition 1.1.6). Moreover, we show that, over models, any stable dependence relation must coincide with (non) strong splitting.

1.4.1 Preliminary results

Fix a diagram D, and let $\mathfrak{C}$ be a monster D-model. The following definitions are due to Shelah.

Definition 1.4.1. (1) The diagram D is *stable in* λ if for every $A \subset \mathfrak{C}$ of cardinality at most λ we have $|S_D(A)| \leq \lambda$.

(2) The diagram D is *stable* if it is stable in some λ; D is *superstable* if there is λ such that D is stable in μ for all $\mu \geq \lambda$.

(3) A type $\mathrm{tp}(\bar{c}/A)$ *splits strongly* over $B \subset A$ if there is an indiscernible sequence $\{a_i \mid i < \omega\}$ over B such that $\bar{a}_0 \in A$ and for some $\varphi(\bar{x}, \bar{y})$ we have $\bar{c} \models \varphi(\bar{x}, \bar{a}_0) \wedge \neg\varphi(\bar{x}, \bar{a}_1)$.

(4) We write $A \underset{C}{\perp} B$ if for every finite $\bar{a} \in A$, $\mathrm{tp}(\bar{a}/B)$ does not split strongly over C.

Much is known about the structure/non-structure theory of stable finite diagrams (see [27, 12, 19]) as well as dependence relation of strong splitting ([18]). It is worth pointing out that the dependence relation in [18] is slightly different from the one we define. In [18] extension property is a part of the definition. Therefore, the Existence property ($\bar{a} \underset{B}{\perp} B$) and the more so Local Character hold only over extension bases. For the strong splitting relation, both properties hold over arbitrary D-sets in a stable finite diagram.

Fact 1.4.2. *If D is a stable finite diagram, then the relation $\perp$ is a stable dependence relation. In addition, if D is superstable, then the local character of strong splitting is $\aleph_0$.*

The invariance, monotonicity, and finite character properties are clear; local character is established in a convenient for us form in [12], Theorem 4.11 (see also [27, 18]). The local character of non strong splitting κ_s is less than or equal to the least stability cardinal. Stationarity is established in [18], Lemma 3.4 (remember that a-saturated is Lascar (D, κ)-homogeneous in our terminology).

Under the assumption of stability with local character $\kappa = \kappa_s$, there are many Lascar (D, κ)-homogeneous models. Namely, the following holds.

Facts 1.4.3. *(1) (Stability Spectrum theorem) If D is stable, λ_D is the least stability cardinal, and $\mu \geq \lambda_D$, then D is stable in μ if and only if $\mu^{<\kappa_s} = \mu$.*

(2) If D is stable in λ and $\lambda^{<\kappa} = \lambda$ for a regular κ, then every set of cardinality at most λ is contained in a Lascar (D, κ)-homogeneous model of cardinality λ.

(1) is presented in [27, 12]; (2) is essentially Lemma 1.9(ii) in [18].

In the next subsection, we will need to use stationarity over Lascar (D, κ)-homogeneous models, for κ the local character of $\overset{(A)}{\smile}$, without the stability assumption (our goal is to deduce stability from existence of a stable dependence relation).

Accordingly, we need to know that such models exist. First we state a few facts.

Facts 1.4.4 ([7]). *(1) If I is an indiscernible sequence over A, then* $\text{lstp}(\bar{a}/A) = \text{lstp}(\bar{b}/A)$ *for all $\bar{a}, \bar{b} \in I$.*

(2) There are fewer than $\beth_{(2^{|A|})^+}$ distinct Lascar strong types (in finitely many variables) over a set A.

(3) Equality of Lascar strong types over a set A is the finest bounded A-invariant equivalence relation over A.

Lemma 1.4.5. *Let κ be a fixed cardinal. For every set A, there is a Lascar (D, κ)-homogeneous model M containing A. If in addition $|A| \geq \beth_{(2^\kappa)^+}$ and $|A|^{<\kappa} = |A|$, then M could be chosen of the same cardinality as A.*

Proof. Let $\lambda := \max\{\beth_{(2^\kappa)^+}, |A|\}$. Construct $\{M_i \mid i < \kappa^+\}$ such that

(1) $M_0 := A$, $|M_i| \leq \lambda$;

(2) M_{i+1} is a model that contains M_i and realizes all Lascar strong types in finitely many variables over all the subsets of M_i of size less than κ;

(3) if i is a limit ordinal, $M_i := \bigcup_{j<i} M_j$.

To carry out the construction at the successor step, observe that there are at most λ many subsets of M_i of size less than κ since the cofinality of λ is at least κ. Over each such subset, there are at most λ many Lascar strong types. So a set A_{i+1} of representatives of each Lascar strong types over all subsets of M_i of size less than κ has cardinality at most λ. We let M_{i+1} be a model containing A_{i+1}, $|M_{i+1}| \leq \lambda$.

Let $M := \bigcup_{i<\kappa^+} M_i$. Clearly, M is as needed: if $B \subset M$, $|B| < \kappa$, then $B \subset M_i$ for some $i < \kappa$, and so all Lascar strong types over B are realized in M. ⊣

Note that if κ is a regular cardinal, then it is enough to construct M_i for $i < \kappa$.

1.4.2 Abstract dependence characterization

Proposition 1.4.6. *If the finite diagram D has a stable dependence relation $\overset{(A)}{\mathrel{\perp}}$, then D is stable.*

Proof. Let κ be the local character of $\overset{(A)}{\mathrel{\perp}}$ and let $\lambda := \beth_{(2^\kappa)^+}$. We prove that D is stable in $\mu := 2^\lambda$. Suppose for contradiction that there is A is such that $|A| = \mu$, and $|S_D(A)| \geq \mu^+$. By Lemma 1.4.5 we may assume that A is the universe of a Lascar (D, κ)-homogeneous model.

Let $\{a_i \mid i < \mu^+\}$ be realizations of distinct types over A. By local character, there are $\{B_i \mid i < \mu^+\}$ such that $|B_i| < \kappa$ and $\bar{a}_i \underset{B_i}{\overset{(A)}{\mathrel{\perp}}} A$ for all $i < \mu^+$. Since there are μ many subsets of A of size less than κ, by the pigeonhole principle we may assume that for some $B \subset A$, $|B| < \kappa$, $\bar{a}_i \underset{B}{\overset{(A)}{\mathrel{\perp}}} A$ for all $i < \mu^+$. Let M be a Lascar (D, κ)-homogeneous model containing B. By Lemma 1.4.5 M could be chosen so that $|M| \leq \lambda$, and $M \subset A$. Monotonicity now gives $\bar{a}_i \underset{M}{\overset{(A)}{\mathrel{\perp}}} A$ for $i < \mu^+$. Since there are at most μ different types over M, by the pigeonhole principle we may assume that all $\{\bar{a}_i \mid i < \mu^+\}$ realize the same type $p \in S_D(M)$. This contradicts the stationarity property. ⊣

We now prove that the stable dependence relation $\overset{(A)}{\perp}$ is exactly that of non strong splitting over Lascar (D, κ)-homogeneous models.

Lemma 1.4.7. *Let $\overset{(A)}{\perp}$ be a stable dependence relation. Suppose A, B are D-sets and a model M is Lascar (D, κ)-homogeneous. If $A \overset{(A)}{\underset{M}{\perp}} B$, then $A \underset{M}{\perp} B$.*

Proof. By finite character, we may assume $A = M\bar{a}$, $B = M\bar{b}$ for finite $\bar{a}$, $\bar{b}$. Assume $\bar{a} \overset{(A)}{\underset{M}{\perp}} M\bar{b}$, but $\text{tp}(\bar{a}/M\bar{b})$ strongly splits over M. Let $\{\bar{b}_i \mid i < \omega\}$ witness strong splitting, with $\bar{b}_0 = \bar{b}$. By extension, there is $\bar{a}' \models \text{tp}(\bar{a}/M\bar{b})$ such that $\bar{a}' \overset{(A)}{\underset{M}{\perp}} M\{\bar{b}_i \mid i < \omega\}$. Let $f \in \text{Aut}_M(\mathfrak{C})$ be such that $f(\bar{b}_0) = \bar{b}_1$. By monotonicity, we have

$$\bar{a}' \overset{(A)}{\underset{M}{\perp}} M\bar{b}_0 \quad \text{and} \quad \bar{a}' \overset{(A)}{\underset{M}{\perp}} M\bar{b}_1.$$

Let $\bar{a}_1 := f(\bar{a}')$. Then by invariance $\bar{a}_1 \overset{(A)}{\underset{M}{\perp}} M\bar{b}_1$. Thus we get two distinct $\overset{(A)}{\perp}$-independent extensions of $\text{tp}(\bar{a}/M)$ to $M\bar{b}_1$. Contradiction to stationarity over models. ⊣

To prove the converse, we need to establish the connection between the local character κ of $\overset{(A)}{\perp}$ and the local character κ_s of non-strong splitting. Without loss of generality, we may assume that κ is a regular cardinal (clearly, κ-local character implies κ^+-local character).

Lemma 1.4.8. *Suppose D has a stable dependence relation $\overset{(A)}{\perp}$. Let κ be the (regular) local character cardinal of $\overset{(A)}{\perp}$. Then $\kappa_s \leq \kappa$.*

Proof. Suppose for contradiction that $\kappa < \kappa_s$. By Proposition 1.4.6, the finite diagram D is stable. Let λ_D be the least stability cardinal. We know that $\kappa_s < \lambda_D$, so $\kappa < \lambda_D$. Let $\mu > \lambda_D$ be a cardinal such that $\mu^{<\kappa} = \mu$ and $\mu^{<\kappa_s} > \mu$ (for example, the κth successor of λ_D will work, here we use regularity of κ). By Stability Spectrum theorem, D is unstable in μ, so let A be a set of cardinality μ such that $|S_D(A)| \geq \mu^+$.

Claim 1.4.9. *We may assume that A has the following property. For every $B \subset A$, $|B| < \kappa$, there is a Lascar (D, κ)-homogeneous model $M \subset A$, $|M| = \lambda_D$ containing B.*

Proof. Construct a sequence $\{A_i \mid i < \kappa\}$ such that

(1) $A_0 := A$, $|A_i| = \mu$;

(2) for every $B \subset A_i$ of size less than κ, A_{i+1} contains a Lascar (D, κ)-homogeneous model $M \supset B$.

(3) if i is a limit ordinal, $A_i := \bigcup_{j<i} A_j$.

For the successor step, let $\{B_\alpha \mid \alpha < \mu\}$ be an enumeration of all the subsets of A_i of size less than κ (there are μ many of them since $\mu^{<\kappa} = \mu$. By stability in λ_D we have $\lambda_D^{<\kappa} \leq \lambda_D^{<\kappa_s} = \lambda_D$, so the conditions of Fact 1.4.3(2) are satisfied. Therefore, for every $\alpha < \mu$, there is a Lascar (D, κ)-homogeneous model M_α containing B_α such that $|M_\alpha| = \lambda_D$. Let $A_{i+1} := \bigcup_{\alpha<\mu} M_\alpha$. Clearly, A_{i+1} is as needed.

The set A_κ has the property required in the claim: if $B \subset A_\kappa$, $|B| < \kappa$, then $B \in A_i$ for some $i < \kappa$ (here we use regularity of κ again). So A_{i+1} (and therefore A_κ) contain the needed model. It is clear that $|A_\kappa| = \mu$, and since $A \subset A_\kappa$, there are at least μ^+ many types over A_κ. So we may take A_κ in place of A. ⊣

Let $\{a_i \mid i < \mu^+\}$ be an enumeration of distinct types over A. By local character, there are $\{B_i \subset A \mid i < \mu^+\}$, $|B_i| < \kappa$, such that $\bar{a}_i \underset{B_i}{\overset{(A)}{\perp}} A$. By pigeonhole principle, we may assume that for all $i < \mu^+$ $\bar{a}_i \underset{B}{\overset{(A)}{\perp}} A$ for some $B \subset A$. Let $M \subset A$ be a Lascar (D, κ)-homogeneous model of cardinality λ_D containing B. By stability in λ_D, there are at most λ_D types over M. By pigeonhole principle we may assume that $\{\bar{a}_i \mid i < \mu^+\}$ realize the same type over M. By monotonicity, $\bar{a}_i \underset{M}{\overset{(A)}{\perp}} A$ for each $i < \mu^+$, so we get a contradiction to stationarity over models. ⊣

Theorem 1.4.10. *Suppose D has a stable dependence relation $\overset{(A)}{\downarrow}$. Suppose A, B are D-sets and a model M is Lascar (D,κ)-homogeneous. Then $A \underset{M}{\downarrow} B$ if and only if $A \overset{(A)}{\underset{M}{\downarrow}} B$.*

Proof. One direction is established in Lemma 1.4.7.

We may assume $A = M\bar{a}$ for finite $\bar{a}$. Suppose $\bar{a} \underset{M}{\downarrow} B$ but $\bar{a} \overset{(A)}{\underset{M}{\not\downarrow}} B$. Since $\bar{a} \overset{(A)}{\underset{M}{\downarrow}} M$, by extension there is $\bar{a}' \models \mathrm{tp}(\bar{a}/M)$ such that $\bar{a}' \overset{(A)}{\underset{M}{\downarrow}} B$. Since $\bar{a} \overset{(A)}{\underset{M}{\not\downarrow}} B$ by invariance, $\mathrm{tp}(\bar{a}'/B) \neq \mathrm{tp}(\bar{a}/B)$. Now Lemma 1.4.7 gives $\bar{a}' \underset{M}{\downarrow} B$. Since $\kappa_s \leq \kappa$, M is also Lascar (D,κ_s)-homogeneous. So we get a contradiction to stationarity over models of non strong splitting. ⊣

Since strong splitting has symmetry and transitivity properties over Lascar (D,κ_s)-homogeneous models, we also get the properties for $\overset{(A)}{\downarrow}$.

Corollary 1.4.11. *The relation $\overset{(A)}{\downarrow}$ is symmetric and transitive over Lascar (D,κ)-homogeneous models.*

If we let $\kappa := \aleph_0$, we get the following

Corollary 1.4.12. *A finite diagram D is superstable if and only if it has a superstable dependence relation $\overset{(A)}{\downarrow}$. Moreover, the relation must coincide with that of non-strong splitting over Lascar $(D,\aleph_0)$-homogeneous models. In particular, $\overset{(A)}{\downarrow}$ must be symmetric and transitive over those models.*

1.5 Simple homogeneous models

1.5.1 Preliminaries

We work in the context described in the paper [7] by Steven Buechler and Olivier Lessmann. The class of structures we are dealing with here is formally larger than in the homogeneous finite diagrams framework. Namely, we study logical structures

$(M, \mathcal{R})$, where M is a structure in a first order language and $\mathcal{R}$ is a collection of finitary relations on M closed under some reasonable operations.

We further assume that $(M, \mathcal{R})$ is *strongly λ-homogeneous* for a large λ, see Definition 1.3 in [7]. By *large* we mean the following: if we are interested in types over set of size at most π in less than π many variables, then λ should be at least $\beth_{(2^\pi)^+}$. The strong homogeneity assumption means that, for practical purposes, we can treat a logical structure as a homogeneous finite diagram.

For the rest of the section, we fix π and agree to consider the types in less than π many variables over sets of cardinality at most π. Furthermore, we identify the sets of size less than π with some enumeration of those sets.

The following is an important property of strongly homogeneous structures (it appears in [7] in Lemmas 1.3 and 1.4; see also assumption Π in Section 2 there).

Fact 1.5.1. *Let $(M, \mathcal{R})$ be a strongly λ-homogeneous structure. There is a cardinal $\pi' \leq \lambda$ such that for every type $p(\bar{x})$ over a A set of cardinality less than π in less than π many variables, if $\{\bar{a}_i \mid i \in X\}$ is a sequence of realizations of p indexed by a linear order X, $|X| \geq \pi'$, then for every linear order Y, $|Y| \leq \lambda$ there is an indiscernible over A sequence $\{\bar{b}_i \mid i \in Y\}$ with $\text{tp}(b_{i_0}, \dots, \bar{b}_{i_n}/A)$ realized by some increasing sequence $\{\bar{a}_{j_0}, \dots, \bar{a}_{j_n}\}$ for all $n < \omega$.*

The following concept is a substitute for algebraic types in our context.

Definition 1.5.2. A type p is *small*, if the set of realizations of p has cardinality less than π' (from Fact 1.5.1). A type p is called *large* otherwise.

The dependence relation in this situation is given by dividing.

Definition 1.5.3. (1) A type $p(\bar{x}, \bar{b})$ *divides over A*, if there is an infinite indiscernible sequence $\{b_i \mid i \in X\}$ such that $\bar{b} = \bar{b}_i$ for some $i \in X$ and the type $\bigcup_{i \in X} p(\bar{x}, \bar{b})$ is inconsistent.

(2) We say that A is *free from B over C* ($A \underset{C}{\perp} B$) if for all finite tuples $\bar{a} \in A$ and $\bar{b} \in B \cup C$ we have $\text{tp}(\bar{a}, \bar{b})$ does not divide over C.

It is clear that if $\text{tp}(\bar{b}/A)$ is small, then $p(\bar{x}, \bar{b})$ does not divide over A.

In [7], it is established that in a κ-simple homogeneous model dividing has the all properties of a simple dependence relation, except type amalgamation, which does hold in the following "local" form.

Fact 1.5.4 (Buechler, Lessmann). *Suppose* $|C| < \kappa$ *and* $\bar{a}_i$, $\bar{b}_i$, $i = 1, 2$, *are tuples of length less than* κ. *If* $\text{lstp}(\bar{a}_1/C) = \text{lstp}(\bar{a}_2/C)$, $\bar{b}_1 \overset{(A)}{\underset{C}{\perp}} \bar{b}_2$, *and* $\bar{a}_i \overset{(A)}{\underset{C}{\perp}} \bar{b}_i$, $i = 1, 2$, *then there is* $\bar{a} \models \text{lstp}(\bar{a}_1/C\bar{b}_1) \cup \text{lstp}(\bar{a}_2/C\bar{b}_2)$ *such that* $\bar{a} \overset{(A)}{\underset{C}{\perp}} \bar{b}_1\bar{b}_2$.

Remark 1.5.5. In the definition of a simple dependence relation we require the type amalgamation to hold over any small $\bar{b}_i$, $i = 1, 2$, and C not necessarily of size less than κ. It is not clear whether this type amalgamation property would follow from the local version in general.

However, the local type amalgamation implies type amalgamation over all small sets for compact homogeneous models; and any counterexample would have to be quite exotic: it will be an example of a κ-simple homogeneous model which is not κ'-simple for some $\kappa' > \kappa$.

Definition 1.5.6. We say that a strongly λ-homogeneous structure $(M, \mathcal{R})$ is *κ-simple with type amalgamation over all small sets* if $(M, \mathcal{R})$ is κ-simple and the type amalgamation property holds for $\bar{b}_1$, $\bar{b}_2$, and C of arbitrary small size.

Fact 1.5.7 (Buechler, Lessmann). *If the homogeneous structure is κ-simple with type amalgamation over all small sets, then dividing is a simple dependence relation with local character κ.*

Remark 1.5.8. In [7], the extension property for large types is required by the definition of a simple structure. For small types, extension for non-dividing holds trivially by Lemma 2.6 in [7] and transitivity. For the purpose of characterizing the dependence relations, it is essential to formulate the extension for both large and small types.

Even for first order simple theories, one can define a dependence relation different from dividing that satisfies all the properties of a simple relation, but not the extension property for small (algebraic, in first order) types.

1.5.2 Abstract dependence characterization.

The following result generalizes Kim-Pillay's theorem for the simple case. Though the idea of the proof is similar, there are some added difficulties due to the failure of compactness and a different definition of a Lascar strong type.

Theorem 1.5.9. *A strongly λ-homogeneous logical structure $(M, \mathcal{R})$ is simple with type amalgamation over all small sets if and only if it has a simple dependence relation. In addition, the abstract dependence relation coincides with the one defined by dividing.*

Proof. One direction is given by Fact 1.5.7.

Suppose now that we have a simple dependence relation $\overset{(A)}{\perp}$. First we prove that $\overset{(A)}{\perp}$ relation coincides with the one defined by dividing. By Finite Character of $\overset{(A)}{\perp}$, it is enough to show that for all finite $\bar{a}, \bar{b} \in M$, $A \subset M$

$$\bar{a} \overset{(A)}{\underset{A}{\perp}} A\bar{b} \text{ if and only if } \operatorname{tp}(\bar{a}/A\bar{b}) \text{ does not divide over } A.$$

We split the proof into several lemmas.

Lemma 1.5.10 (Existence of $\overset{(A)}{\perp}$-Morley sequences). *Let $\bar{a} \in M$, $B \subset M$, and $A \subset B$ be such that $\bar{a} \overset{(A)}{\underset{A}{\perp}} B$. Let X be an infinite order. If $\operatorname{tp}(\bar{a}/B)$ is large, then M contains a $\overset{(A)}{\perp}$-Morley sequence $I = \{\bar{a}_i \mid i \in X\}$ for $\operatorname{tp}(\bar{a}/B)$ over A.*

Proof. Use the same argument as in Lemma 2.4 in [7]. $\dashv$

Lemma 1.5.11. *If $\bar{a} \overset{(A)}{\underset{A}{\not\perp}} A\bar{b}$, then $\operatorname{tp}(\bar{a}/A\bar{b})$ divides over A.*

Proof. Denote $p(\bar{x}, \bar{b}) := \mathrm{tp}(\bar{a}/A\bar{b})$. First we prove that $\bar{a} \underset{A}{\overset{(A)}{\not\downarrow}} A\bar{b}$ implies that $\mathrm{tp}(\bar{b}/A)$ is large. Suppose $\mathrm{tp}(\bar{b}/A)$ is small. Let D be the set of all realizations of $\mathrm{tp}(\bar{b}/A)$, let $\bar{b}' \models \mathrm{tp}(\bar{b}/A)$ be such that $\bar{b}' \underset{A}{\overset{(A)}{\downarrow}} A \cup D$. Since $\bar{b}' \in D$, we get $\bar{b}' \underset{A}{\overset{(A)}{\downarrow}} A\bar{b}'$ by monotonicity, so invariance implies $\bar{b} \underset{A}{\overset{(A)}{\downarrow}} A\bar{b}$. By extension property, there is $\bar{b}'' \models \mathrm{tp}(\bar{b}/A\bar{b})$ such that $\bar{b}'' \underset{A}{\overset{(A)}{\downarrow}} A\bar{b}$. Clearly, $\bar{b}'' = \bar{b}$, so we have $\bar{b} \underset{A}{\overset{(A)}{\downarrow}} A\bar{a}\bar{b}$, and by symmetry $\bar{a} \underset{A}{\overset{(A)}{\downarrow}} A\bar{b}$, contradiction.

Since $\mathrm{tp}(\bar{b}/A)$ is large, we can find $I := \{\bar{b}_i \mid i < \kappa\}$ a $\overset{(A)}{\downarrow}$-Morley sequence for $\mathrm{tp}(\bar{b}/A)$. We claim that $\bigcup_{i<\kappa} p(\bar{x}, \bar{b}_i)$ is inconsistent.

Suppose for contradiction that it is consistent and let $\bar{a}' \models \bigcup_{i<\kappa} p(\bar{x}, \bar{b}_i)$. By invariance, $\bar{a} \underset{A}{\overset{(A)}{\not\downarrow}} A\bar{b}$ implies $\bar{a}' \underset{A}{\overset{(A)}{\downarrow}} A\bar{b}_i$ for all $i < \kappa$. On the other hand, by local character $\bar{a}' \underset{A \cup J}{\overset{(A)}{\downarrow}} A \cup I$ for some $J \subset I$, $|J| < \kappa$. Let $i < \kappa$ be such that $J < i$, then by symmetry and transitivity of $\overset{(A)}{\downarrow}$ we have $\bar{a}' \underset{A}{\overset{(A)}{\downarrow}} A\bar{b}_i$, contradiction. ⊣

Lemma 1.5.12. *If* $\bar{a} \underset{A}{\overset{(A)}{\downarrow}} A\bar{b}$*, then* $\mathrm{tp}(\bar{a}/A\bar{b})$ *does not divide over* A*.*

Proof. If $\mathrm{tp}(\bar{b}/A)$ is small, then $\mathrm{tp}(\bar{a}/A\bar{b})$ does not divide over A, so we are done.

Suppose now that $\mathrm{tp}(\bar{b}/A)$ is large. Let $I = \{\bar{b}_i \mid i \in X\}$ be an indiscernible sequence in $\mathrm{tp}(\bar{b}/A)$, with $\bar{b}_0 = \bar{b}$. We need to prove that $\bigcup_{i \in X} p(\bar{x}, \bar{b}_i)$ is consistent.

Take a long extension of the sequence $\{\bar{b}_i \mid i \in X\}$: let $\bar{X}$ be a linear order that extends X, we take it to be κ^+ copies of X, where κ is the local character of $\overset{(A)}{\downarrow}$, with an extra last element i^*. Accordingly, $\bar{I}$ is an indiscernible sequence that has κ^+ copies of I, with an extra element $\bar{b}_{i^*}$.

By the local character of $\overset{(A)}{\downarrow}$, there is a subsequence $I' \subset \bar{I}$, $|I'| < \kappa$, such that $\bar{b}_{i^*} \underset{AI'}{\overset{(A)}{\downarrow}} \bar{I}$. By regularity of κ^+, there is $\delta < \kappa^+$ such that $I' \subset \{\bar{I}_\alpha \mid \alpha < \delta\}$, where $\bar{I}_\alpha$ is the αth copy of I in $\bar{I}$. By monotonicity,

$$\bar{b}_{i^*} \underset{A\{\bar{I}_\alpha \mid \alpha < \delta\}}{\overset{(A)}{\downarrow}} \bar{I}_\delta. \qquad (*)$$

Since $\bar{I}_\delta$ is a copy of I, we may assume that in fact $\bar{I}_\delta = \{\bar{b}_i \mid i \in X\}$, i.e., $\bar{I}_\delta = I$.

For a subset $Y \subset X$, and an index $i \in X$ we say $Y < i$ when i is greater than any element in Y; the symbol $\bar{b}_Y$ stands for the sequence $\{\bar{b}_j \mid j \in Y\}$. With these notations, for any $Y \subset X$ and any $i > Y$, by indiscernibility of $\bar{I}$ over A, we have $\mathrm{tp}(\bar{b}_{i^*}/AI'\bar{b}_Y) = \mathrm{tp}(\bar{b}_i/AI'\bar{b}_Y)$. By invariance and $(*)$, from this we can conclude

$$\bar{b}_i \underset{AI'}{\overset{(A)}{\perp}} \bar{b}_Y$$

for any $Y \subset X$ and $i > Y$. Therefore $\{\bar{b}_i \mid i \in X\}$ is a $\overset{(A)}{\perp}$-Morley sequence over AI'. We will need two implications of this fact.

First, all $\bar{b}_i$ realize the same Lascar strong type over AI'. Second, by a standard argument, symmetry and transitivity imply that for any rearrangement of I is a $\overset{(A)}{\perp}$-independent sequence over AI' (of course, not necessarily indiscernible). Let $\lambda := |X|$. Rearranging the elements of I in some order, we may assume that I is a $\overset{(A)}{\perp}$-independent sequence over AI' (not necessarily indiscernible), and that all the elements of I have the same Lascar strong type over AI'.

For $i < \lambda$, let f_i be a strong automorphism over AI' such that $f_i(\bar{b}_0) = \bar{b}_i$. Let $\bar{a}_i := f_i(\bar{a})$. By invariance, we then have $\bar{a}_i \underset{AI'}{\overset{(A)}{\perp}} \bar{b}_i$.

Let $\bar{a}_0'$ be in a $\overset{(A)}{\perp}$-Morley sequence in $\mathrm{tp}(\bar{a}_0/AI'\bar{b}_0)$ over AI'. We can choose $\bar{a}_0'$ so that $\bar{a}_0' \underset{AI'}{\overset{(A)}{\perp}} I$. Then $\bar{a}_0 \underset{AI'}{\overset{(A)}{\perp}} \bar{b}_0\bar{a}_0'$, $\mathrm{lstp}(\bar{a}_0'/AI') = \mathrm{lstp}(\bar{a}_0/AI')$, and $\bar{a}_0'\bar{b}_0 \underset{AI'}{\overset{(A)}{\perp}} \{\bar{b}_j \mid 1 \leq j < \lambda\}$.

Let $q(\bar{x}, \bar{b}_0, \bar{a}_0') := \mathrm{tp}(\bar{a}_0/AI'\bar{b}_0\bar{a}_0')$. By induction on $1 \leq \alpha \leq \lambda$ we construct complete types $q_\alpha(\bar{x})$ such that

(1) $q_1(\bar{x}) := q(\bar{x}, \bar{b}_0, \bar{a}_0')$; $\mathrm{dom}(q_\alpha) = AI'\bar{a}_0'\{\bar{b}_i \mid i < \alpha\}$;

(2) $q_\alpha \subset q_\beta$ for $\alpha < \beta < \lambda$, and $q_\alpha(\bar{x}) \supset \bigcup_{i<\alpha} p(\bar{x}, \bar{b}_i)$;

(3) if $\bar{c}_\alpha \models q_\alpha$, then $\mathrm{lstp}(\bar{c}_\alpha/AI') = \mathrm{lstp}(\bar{a}_0/AI')$ and

(4) $\bar{c}_\alpha \underset{AI'}{\overset{(A)}{\perp}} \bar{a}_0'\{\bar{b}_i \mid i < \alpha\}$;

The case $\alpha = 1$ is clear. Note that the Lascar strong type of $\bar{a}_0$ over AI' became type definable when we added $\bar{a}'_0$. Hence, any independent realization of $q_1(\bar{x})$ will have the same Lascar strong type over AI' as $\bar{a}_0$.

For the successor case, suppose we have a complete $q_\alpha(\bar{x})$ as in (1)–(4) above. By the choice of $\bar{a}'_0$, we have $\mathrm{dom}(q_\alpha) \overset{(A)}{\underset{AI'}{\perp}} \bar{b}_\alpha$. Also $\mathrm{lstp}(\bar{a}_\alpha/AI') = \mathrm{lstp}(\bar{a}_0/AI') = \mathrm{lstp}(\bar{c}_\alpha/AI')$. So we can apply type amalgamation (Property 7) and get $\bar{c}_{\alpha+1}$ that realizes $q_\alpha(\bar{x}) \cup p(\bar{x}, \bar{b}_\alpha)$ such that $\bar{c}_{\alpha+1} \overset{(A)}{\underset{AI'}{\perp}} \bar{a}'_0\{\bar{b}_i \mid i < \alpha+1\}$. Let $q_{\alpha+1} := \mathrm{tp}(\bar{c}_{\alpha+1}/AI'\bar{a}'_0\{\bar{b}_i \mid i < \alpha + 1\})$. Clearly, $q_{\alpha+1}$ is as needed.

For the limit case, we apply Lemma 1.2 in [7] to the sequence of types $\{q_i(\bar{x}) \mid i < \alpha\}$ to conclude that the union $\bigcup_{i<\alpha} q_i(\bar{x})$ is consistent. By the choice of the type $q_1(\bar{x})$, the realization $\bar{c}_\alpha$ has the same Lascar strong type over AI' as does $\bar{a}'_0$, which coincides with the Lascar strong type of $\bar{a}_0$ over AI'. So $\mathrm{lstp}(\bar{c}_\alpha/AI') = \mathrm{lstp}(\bar{a}_0/AI')$. Condition (4) in the construction holds for the union by finite character of $\overset{(A)}{\perp}$.

Finally, the construction gives that the union $\bigcup_{i\in X} p(\bar{x}, \bar{b}_i)$ is consistent. $\dashv$

Thus, we have proved that $\overset{(A)}{\perp}$ relation coincides with the relation given by dividing. Therefore, dividing has the properties of a simple dependence relation, and hence the logical structure $(M, \mathcal{R})$ is simple. $\dashv$

Chapter 2

Strong n-simplicity

Introduction

Simple theories were introduced by Shelah [30] in 1980 as a part of his program to draw further dividing lines within the class of first order theories.

In the early nineties, model-theorists identified a natural unstable first-order theory of algebraically closed fields with a generic automorphism (ACFA) with many stability-like properties. This prompted researchers to look for a wider class of "nice" first-order structures than that of stable theories. An important discovery came in 1996 when B. Kim [20] established that the dependence relation forking has the symmetry property in simple theories. B. Kim and A. Pillay proved in [22] that forking satisfies all the good properties it has in the stable case, except for stationarity, and they found a substitute for stationarity, called the Independence Theorem. The theory of ACFA fits in the class of simple but unstable first order theories. For background on simple theories, we refer the reader to the expositions such as [11, 33].

Following the work of Kim and Pillay, the field started to develop very rapidly. However, some fundamental questions remain open. In paper [32], Shelah conjectured that the class of all simple theories can be split into $\omega + 1$ subclasses by some family of syntactic properties that would reflect the difference in behavior with respect to the following question: Does a model of size λ have a κ-saturated extension of size λ?

The idea is that theories at "simpler" levels would have more pairs (λ, κ) for which the answer is positive.

Our research was originally motivated by an attempt to understand when the n-dimensional amalgamation property (defined in Section 4) holds in a simple theory, and was strongly influenced by Shelah's conjecture. While it is not clear if the $\omega + 1$ subclasses with different saturated pairs correspond to a family of n-simple theories (research in this direction is ongoing), there are some strong connections.

As it turns out, there are several reasonable meanings of the n-dimensional amalgamation property, and accordingly, of what to mean by the corresponding syntactic properties. In this Chapter, we investigate one of the families of properties, that we call *strong n-simplicity.* It offers a convenient test case for developing the tools necessary to prove a rather strong form of n-dimensional amalgamation, and for understanding what n-dimensional amalgamation implies about a simple theory. The next Chapter deals with another family of properties, *n-simplicity.*

In the stable theories, stationarity guarantees that generalized amalgamation properties hold for all dimensions. That the strongly ω-simple (n-simple for all n) theories form an interesting class is supported by the fact that generalized amalgamation properties hold for the theory of ACFA ([8]). In general, existence of the generalized amalgamation properties for all dimensions means that the theory does not interpret *tetrahedron-free hypergraphs* (see Sections 1 and 2 for the definitions) of any dimension. We expect that ω-simple theories will have other good properties that put them a lot closer to the stable theories than all the simple theories in general.

In **Section 1**, we introduce a family of properties of a first order theory T that divide the class of simple theories into $\omega + 1$ subclasses in the following way: We define the family of ranks D^*_n, $1 \leq n < \omega$ that generalize the "simplicity" D-rank. A theory is strongly n-simple is the ranks D^*_k for $k \leq n$ are bounded. If the rank D^*_n is unbounded, it is witnessed by an appropriate *strong n-dimensional tree property.* We develop the basic properties of ranks D^*_n, and prove that all stable theories are strongly ω-simple.

Examples of simple theories in each n-simplicity level are given in the beginning of Section 1 to provide an intuition for the definitions. They are carefully worked out in **Section 2**. The theory of random graphs is an example of an unstable ω-simple theory.

Section 3 deals with the key property of a strongly n-simple theory. We introduce the notion of strong n-dividing and prove a characterization of n-simplicity in terms of it. The implication "non-dividing implies the failure of strong n-dividing" and the resulting conclusions for Lascar strong types proved to be useful in [25].

It is quite clear that strong 2-simplicity has strong implications on the behavior of Lascar strong types. However, it is not entirely clear what additional structure is obtained with strong n-simplicity for $n > 2$.

Another property of strongly 2-simple theories, namely the 3-dimensional amalgamation property for Lascar strong types was also used in [25]. **Section 4** of our paper contains the definition of the (strong) n-dimensional amalgamation property and some its implications.

Section 5 contains helpful results about $(n+1)$-dimensional amalgamation that are used to prove 3-amalgamation for 2-simple theories and, with an additional assumption, the $(n+1)$-dimensional amalgamation for n-simple theories. The proofs of these amalgamation properties are given in **Section 6**. There we also discuss the extra assumptions and the open questions.

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2.1 Strong n-simplicity

In this section we introduce one of the families of properties that divide the class of all simple theories into $\omega + 1$ subclasses. Another family is studied in Chapter 3. Particular examples of theories in each subclass are fully worked out in the next section. We provide the examples here in an abbreviated form for the benefit of the reader.

The theories T_k, $k \geq 3$, below are all simple theories with distinct behaviors.

The definitions of strong n-simplicity (as well as n-simplicity) capture some of the differences. The distinction between the different versions of n-simplicity is more subtle, and becomes apparent only when a part of the general theory is developed. That is why we defer the definition of (the non-strong version of) n-simplicity.

Fix $k \geq 3$, let $L_k := \{P, R, S\}$, where P is an unary predicate, S and R are k-ary predicates. Let T_k be the model completion of the following set of sentences in L_k:

(1) "$R \subset P^k$;"

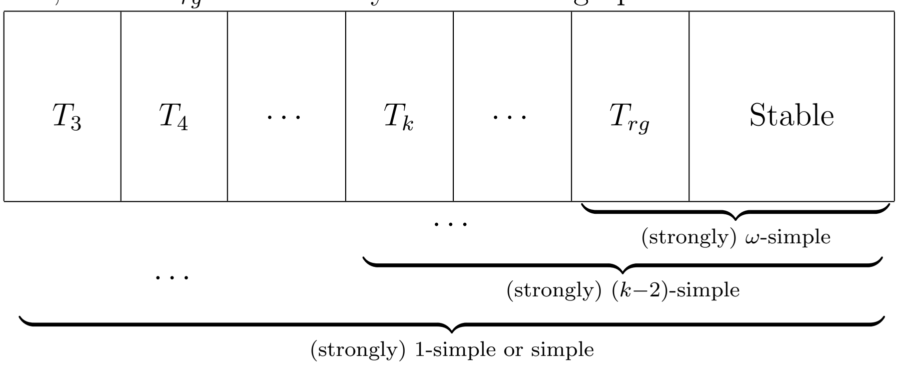
(2) "R is symmetric (with respect to all permutations), irreflexive;"

(3) "$S \subset P^{k-1} \times \neg P$" (we use the notation $\bar{x}\, S\, y$, $\bar{x}$ is understood to be a tuple in P^{k-1});

(4) "S is symmetric irreflexive in the first $k-1$ variables;"

(5) "if $R(x_1, \ldots, x_k)$, then no $y \in \neg P$ is connected via S to *all* the $(k-1)$-element subtuples of $x_1, \ldots, x_k$."

The main purpose of the first two sections is to show that we have the following picture, where T_{rg} is the theory of a random graph.



One of the equivalent definitions for simple theories is via the boundedness of the rank $D[p, \varphi, k]$ for all formulas φ and natural numbers k. Below, we present one of the "n-dimensional" generalization of the rank D, namely the family of the ranks $D^*_n[p, \varphi, k]$. Then we use them to define the n-simplicity property similar to the way the D-rank defines simplicity.

Notation 2.1.1. Fix $1 \leq n < \omega$. If I is a linearly ordered set, $|I| \geq n$, we use the symbol $[I]^n$ to denote the set $\{(i_0, \ldots i_{n-1}) \mid i_0 < \cdots < i_{n-1} \in I\}$. We denote the elements of $[I]^n$ by bold-face $\bar{\boldsymbol{\imath}}$, $\bar{\boldsymbol{\jmath}}$, etc. Observe that $[I]^n$ lists all the n-element subsets of I without repetitions.

If $\{\bar{a}_i \mid i \in I\}$ is a sequence of tuples of the same length l, and $\bar{\boldsymbol{\imath}} \in [I]^n$, then we agree to use the symbol $\bar{a}_{\bar{\boldsymbol{\imath}}}$ for the tuple $\bar{a}_{\bar{\boldsymbol{\imath}}[0]} \ldots \bar{a}_{\bar{\boldsymbol{\imath}}[n-1]}$ of the length $l \cdot n$.

Let $\varphi(\bar{x}; \bar{y}_0, \ldots, \bar{y}_{n-1})$ be a formula where $\ell(\bar{y}_i) = \ell(\bar{y}_j) = l$ for $i, j < n$, $\{\bar{a}_0 \ldots, \bar{a}_{n-1}\}$ be a sequence of tuples of the length l. We agree to abbreviate $\varphi(\bar{x}; \bar{a}_0, \ldots, \bar{a}_{n-1})$ as $\varphi(\bar{x}; \bar{a}_{\bar{\boldsymbol{n}}})$.

Definition 2.1.2. Let $\{\bar{a}_i \mid i \in I\}$ be a sequence of tuples of length l. We say that the set $\{\varphi(\bar{x}, \bar{a}_{\bar{\boldsymbol{\imath}}}) \mid \bar{\boldsymbol{\imath}} \in [I]^n\}$ is $[k]^n$-contradictory if for any $J \subset I$ of size k, we have

$$\models \neg \exists \bar{x} \bigwedge_{\bar{\boldsymbol{\jmath}} \in [J]^n} \varphi(\bar{x}, \bar{a}_{\bar{\boldsymbol{\jmath}}}).$$

Example 2.1.3. Fix $n \geq 2$. In the monster model of T_{n+1} let $I = \{a_i \mid i < \omega\}$ be an indiscernible sequence such that $R(a_0, \ldots a_n)$. Then the set $\{\bar{a}_{\bar{\boldsymbol{\imath}}}\, S\, x \mid \bar{\boldsymbol{\imath}} \in [\omega]^n\}$ is $[n+1]^n$-contradictory.

The rank we define below generalizes the D-rank for simple theories to "higher dimensions."

For $n \geq 2$, let the symbol $\mathrm{Ind}(\bar{x}; \bar{y}_0, \ldots, \bar{y}_{n-1})$ denote the type expressing that $\bar{y}_0, \ldots, \bar{y}_{n-1}$ are indiscernible over $\bar{x}$.

Definition 2.1.4. Fix $1 \leq n < \omega$. Take a formula $\varphi(\bar{x}; \bar{y}_0, \ldots, \bar{y}_{n-1})$, natural number $k > n$, and a partial type $p(\bar{x})$. Define $D^*_n[p, \varphi, k] \geq \alpha$ by induction on α.

(1) $D^*_n[p, \varphi, k] \geq 0$ if p is consistent.

(2) for α limit, $D^*_n[p, \varphi, k] \geq \alpha$ if $D^*_n[p, \varphi, k] \geq \beta$ for all $\beta < \alpha$;

(3) $D^*_n[p, \varphi, k] \geq \alpha + 1$ if for every finite $r \subseteq p(\bar{x})$ there is a sequence $\{\bar{a}_i \mid i < \omega\}$ such that for all $\bar{\boldsymbol{\imath}} \in [\omega]^n$

$$D^*_n[r \cup \{\varphi(\bar{x}, \bar{a}_{\bar{\boldsymbol{\imath}}})\} \cup \mathrm{Ind}(\bar{x}; \bar{a}_{\bar{\boldsymbol{\imath}}[0]}, \ldots, \bar{a}_{\bar{\boldsymbol{\imath}}[n-1]}), \varphi, k] \geq \alpha$$

and the set $\{\varphi(\bar{x}, \bar{a}_{\bar{\imath}}) \mid \bar{\imath} \in [\omega]^n\}$ is $[k]^n$-contradictory.

The expressions $D^*_n[p, \varphi, k] = \alpha$, $D^*_n[p, \varphi, k] = -1$, and $D^*_n[p, \varphi, k] = \infty$ are defined as usual.

The rank D^*_n can be defined in a more general setting, generalizing the rank $D[p, \Delta, \lambda, k]$ that appears in [30].

When $n = 1$, the indiscernibility requirement in (3) of the above definition collapses, so D^*_1 is the familiar simplicity D-rank. To explain the reason for introducing that requirement at all, we need another definition.

Definition 2.1.5. Fix $1 \leq n < \omega$. Given a formula $\varphi(\bar{x}; \bar{y}_0, \ldots, \bar{y}_{n-1})$ and parameters $\bar{a}_0, \ldots, \bar{a}_{n-1}$, we say that $\varphi(\bar{x}, \bar{a}_0, \ldots \bar{a}_{n-1})$ is *n-admissible over a type $p(\bar{x})$* if there is $\bar{b} \models p(\bar{x}) \cup \{\varphi(\bar{x}, \bar{a}_0, \ldots \bar{a}_{n-1})\}$ such that the sequence $\{\bar{a}_0, \ldots, \bar{a}_{n-1}\}$ is indiscernible over $\bar{b}$.

We make the extra demand in (3) of Definition 2.1.4 to make sure that every $\varphi(\bar{x}; \bar{a}_{\bar{\imath}})$ is n-admissible over r at every successor step. Otherwise, the n-rank can be unbounded for a trivial reason. Consider for example the theory of random graph with the edge relation E. Let $\varphi(x, y_0, y_1) := xEy_0 \wedge \neg(xEy_1)$. If we drop the extra requirement in Definition 2.1.4(1), then we would have $D^*_2[x = x, \varphi, 3] = \infty$, for a purely syntactic reason.

A standard application of compactness theorem gives the following.

Remark 2.1.6. In the Definition 2.1.4(3), we may require the sequence $\{\bar{a}_i \mid i < \omega\}$ to be indiscernible over $\mathrm{dom}(p)$. In addition, the index set for the sequence may be any infinite linearly ordered set.

Definition 2.1.7. Let $\alpha \leq \omega$. We say that a complete theory T is *strongly α-simple* if for all $n < \alpha$, for all $\varphi(\bar{x}, \bar{y}_0, \ldots, \bar{y}_n)$ and $k > n + 1$ the rank $D^*_{n+1}[\bar{x} = \bar{x}, \varphi, k]$ is bounded (i.e., is less than ∞).

So every simple theory is automatically strongly 1-simple; in the next section we show that T_k, $k \geq 3$, is strongly $(k-2)$-simple, but not strongly $(k-1)$-simple, and the theory of random graph is strongly ω-simple.

We now establish some useful properties of the ranks D_n^*.

Proposition 2.1.8 (Basic Properties). *Fix $n < \omega$.*

(1) Monotonicity: If $p_1 \vdash p_2$ and $k_1 \leq k_2$, then

$$D_n^*[p_1(\bar{x}), \varphi, k_1] \leq D_n^*[p_2(\bar{x}), \varphi, k_2].$$

(2) Invariance: If $f \in \mathrm{Aut}(\mathfrak{C})$, then

$$D_n^*[p(\bar{x}), \varphi, k] = D_n^*[f(p(\bar{x}), \varphi, k].$$

(3) Finite Character: For every $p(\bar{x})$ and T, there is finite $r \subset p$ such that

$$D_n^*[p, \varphi, k] = D_n^*[r, \varphi, k].$$

Proof. We prove (1). By induction on α, we show that

$$D_n^*[p_1, \varphi, k_1] \geq \alpha \text{ implies } D_n^*[p_2, \varphi, k_2] \geq \alpha.$$

The cases $\alpha = 0$ and α limit ordinal are obvious. Suppose

$$D_n^*[p_1, \varphi, k_1] \geq \alpha + 1.$$

Take an arbitrary finite $r_2 \subset p_2$. Since $p_1 \vdash p_2$, there is a finite $r_1 \vdash r_2$. By definition of the rank there is a sequence $\{\bar{a}_i \mid i < \omega\}$ such that for all $\bar{\boldsymbol{\imath}} \in [\omega]^n$

$$D_n^*[r_1(\bar{x}) \cup \mathrm{Ind}(\bar{x}; \bar{a}_{\bar{\boldsymbol{\imath}}}) \cup \{\varphi(\bar{x}, \bar{a}_{\bar{\boldsymbol{\imath}}})\}, \varphi, k_1] \geq \alpha,$$

and the set $\{\varphi(\bar{x}, \bar{a}_{\bar{\boldsymbol{\imath}}}) \mid \bar{\boldsymbol{\imath}} \in [\omega]^n\}$ is $[k_1]^n$-contradictory. By induction hypothesis

$$D_n^*[r_2(\bar{x}) \cup \mathrm{Ind}(\bar{x}; \bar{a}_{\bar{\boldsymbol{\imath}}}) \cup \{\varphi(\bar{x}, \bar{a}_{\bar{\boldsymbol{\imath}}})\}, \varphi, k_2] \geq \alpha$$

and since $k_2 \geq k_1$ the set $\{\varphi(\bar{x}, \bar{a}_{\bar{\boldsymbol{\imath}}}) \mid \bar{\boldsymbol{\imath}} \in [\omega]^n\}$ is $[k_2]^n$-contradictory. By definition of the rank it means

$$D_n^*[p_2, \varphi, k_2] \geq \alpha + 1.$$

The proofs of (2) and (3) are routine. ⊣

Lemma 2.1.9 (Ultrametric Property). *For every $m, n, k < \omega$, type $p(\bar{x})$, and formulas $\{\psi(\bar{x}, \bar{b}_l) \mid l < m\}$ we have*

$$D_n^*[p(\bar{x}) \cup \bigvee_{l<m} \psi(\bar{x}, \bar{b}_l), \varphi, k] = \max_{l<m} D_n^*[p(\bar{x}) \cup \psi(\bar{x}, \bar{b}_l), \varphi, k].$$

Proof. By the monotonicity property, for all $l < m$ we have

$$D_n^*[p(\bar{x}) \cup \bigvee_{l<m} \psi(\bar{x}, \bar{b}_l), \varphi, k] \geq D_n^*[p(\bar{x}) \cup \psi(\bar{x}, \bar{b}_l), \varphi, k].$$

Therefore, $D_n^*[p(\bar{x}) \cup \bigvee_{l<m} \psi(\bar{x}, \bar{b}_l), \varphi, k] \geq \max_{l<m} D_n^*[p(\bar{x}) \cup \psi(\bar{x}, \bar{b}_l), \varphi, k]$.

To prove the reverse inequality, we establish that for all $p(\bar{x})$ and $\{\psi(\bar{x}, \bar{b}_l) \mid l < m\}$

$$D_n^*[p(\bar{x}) \cup \bigvee_{l<m} \psi(\bar{x}, \bar{b}_l), \varphi, k] \geq \alpha \quad \text{implies} \quad \max_{l<m} D_n^*[p(\bar{x}) \cup \psi(\bar{x}, \bar{b}_l), \varphi, k] \geq \alpha$$

by induction on α. If $\alpha = 0$ or α is a limit ordinal, the implication is obvious.

Suppose that the statement holds for an ordinal α, and let

$$D_n^*[p(\bar{x}) \cup \bigvee_{l<m} \psi(\bar{x}, \bar{b}_l), \varphi, k] \geq \alpha + 1.$$

Suppose for contradiction that for all $l < m$

$$D_n^*[p(\bar{x}) \cup \{\psi(\bar{x}, \bar{b}_l)\}, \varphi, k] \leq \alpha.$$

By finite character, we can find a finite subset $r_l \subset p$ such that $D_n^*[r_l(\bar{x}) \cup \psi(\bar{x}, \bar{b}_l), \varphi, k] \leq \alpha$. Letting $r(\bar{x}) := \bigcup_{l<m} r_l(\bar{x})$, by monotonicity we have

$$D_n^*[r(\bar{x}) \cup \bigvee_{l<m} \psi(\bar{x}, \bar{b}_l), \varphi, k] \geq \alpha + 1 \text{ but} \quad \max_{l<m} D_n^*[r(\bar{x}) \cup \psi(\bar{x}, \bar{b}_l), \varphi, k] \leq \alpha.$$

By the definition of the rank D_n^*, there is a sequence $\{\bar{a}_i \mid i < \omega\}$ such that

$$D_n^*[r(\bar{x}) \cup \bigvee_{l<m} \psi(\bar{x}, \bar{b}_l) \cup \operatorname{Ind}(\bar{x}; \bar{a}_{\bar{\imath}}) \cup \{\varphi(\bar{x}, \bar{a}_{\bar{\imath}})\}, \varphi, k] \geq \alpha$$

for all $\bar{\imath} \in [\omega]^n$ and the set $\{\varphi(\bar{x}, \bar{a}_{\bar{\imath}}) \mid \bar{\imath} \in [\omega]^n\}$ is $[k]^n$-contradictory. By induction hypothesis, for every $\bar{\imath} \in [\omega]^n$, there is $l(\bar{\imath}) < m$ such that

$$D^*_n[r(\bar{x}) \cup \{\psi(\bar{x}, \bar{b}_{l(\bar{\imath})})\} \cup \mathrm{Ind}(\bar{x}; \bar{a}_{\bar{\imath}}) \cup \{\varphi(\bar{x}, \bar{a}_{\bar{\imath}})\}, \varphi, k] \geq \alpha.$$

By Ramsey's theorem, we may assume that there is $l^* < m$ such that for all $\bar{\imath} \in [\omega]^n$

$$D^*_n[r(\bar{x}) \cup \{\psi(\bar{x}, \bar{b}_{l^*})\} \cup \mathrm{Ind}(\bar{x}; \bar{a}_{\bar{\imath}}) \cup \{\varphi(\bar{x}, \bar{a}_{\bar{\imath}})\}, \varphi, k] \geq \alpha.$$

By the definition of the rank, we have

$$D^*_n[r(\bar{x}) \cup \{\psi(\bar{x}, \bar{b}_{l^*})\}, \varphi, k] \geq \alpha + 1,$$

a contradiction. ⊣

By a standard argument, the Ultrametric property gives the following.

Lemma 2.1.10 (Extension property). *Let $k, n < \omega$ and $\varphi \in L(T)$ be fixed. Let p be a type, possibly with parameters. For every set A, there is a complete type q over A such that*

$$D^*_n[p, \varphi, k] = D^*_n[p \cup q, \varphi, k].$$

It is convenient for many reasons to view the rank D^*_n as the foundation rank on a certain tree. The next lemma describes an appropriate tree for the rank D^*_n.

Lemma 2.1.11 (Tree characterization). *Let $p(\bar{x})$ be a type, k a natural number, and $\alpha \leq \omega$. Then the following are equivalent:*

(1) $D^*_n[p, \varphi, k] \geq \alpha$;

(2) for every $\eta \in ([\omega]^n)^{<\alpha}$ there is a sequence $I_\eta = \{\bar{a}^\eta_i \mid i < \omega\}$ such that

(a) for each $\eta \in ([\omega]^n)^\alpha$, there is

$$\bar{b}_\eta \models p(\bar{x}) \cup \{\varphi(\bar{x}; \bar{a}^{\eta\restriction\beta}_{i_0}, \dots, \bar{a}^{\eta\restriction\beta}_{i_{n-1}}) \mid \bar{a}^{\eta\restriction\beta}_{i_0}, \dots, \bar{a}^{\eta\restriction\beta}_{i_{n-1}} \in I_{\eta\restriction\beta}, \langle i_0, \dots, i_{n-1}\rangle = \eta[\beta], \beta < \alpha\}$$

such that $\{\bar{a}^{\eta\restriction\beta}_{i_0}, \dots, \bar{a}^{\eta\restriction\beta}_{i_{n-1}}\}$, $\langle i_0, \dots, i_{n-1}\rangle = \eta[\beta]$, are indiscernible over $\bar{b}_\eta$ for all $\beta < \alpha$;

(b) *for every* $\eta \in ([\omega]^n)^{<\alpha}$ *the set* $\{\varphi(\bar{x}, \bar{a}^{\eta}_{\bar{\imath}}) \mid \bar{\imath} \in [\omega]^n\}$ *is* $[k]^n$*-contradictory.*

Remark 2.1.12. Similar to our previous notation agreements, we write the sequence $\{\bar{a}^{\eta\restriction\beta}_{i_0}, \ldots, \bar{a}^{\eta\restriction\beta}_{i_{n-1}}\}$, $\langle i_0, \ldots, i_{n-1}\rangle = \eta[\beta]$, simply as $\bar{a}^{\eta\restriction\beta}_{\eta[\beta]}$.

Proof. First we use induction for $\alpha < \omega$. By finite character, it is enough to prove the claim for finite p.

The base case $\alpha = 0$ is clear. For the induction step, using the definition of D^*_n and the induction hypothesis, we get $D^*_n[p, \varphi, k] \geq \alpha + 1$ if and only if there is a sequence $I = \{\bar{a}_i \mid i < \omega\}$ such that for all $\bar{\imath} \in [\omega]^n$ for all $\eta \in ([\omega]^n)^{<\alpha}$ there is a sequence $I_{\eta,\bar{\imath}} = \{\bar{a}^{\eta,\bar{\imath}}_i \mid i < \omega\}$ such that

(a) for each $\eta \in ([\omega]^n)^{\alpha}$ and $\bar{\imath} \in [\omega]^n$ there is

$$\bar{b}_{\eta,\bar{\imath}} \models p(\bar{x}) \cup \{\varphi(\bar{x}, \bar{a}_{\bar{\imath}})\} \cup \{\varphi(\bar{x}; \bar{a}^{\eta\restriction\beta,\bar{\imath}}_{\eta[\beta]}) \mid \beta < \alpha\}$$

such that sequences $\bar{a}_{\bar{\imath}}$ and $\bar{a}^{\eta\restriction\beta,\bar{\imath}}_{\eta[\beta]}$ are indiscernible over $\bar{b}_{\eta,\bar{\imath}}$ for all $\beta < \alpha$;

(b) for every $\eta \in ([\omega]^n)^{<\alpha}$ and $\bar{\imath} \in [\omega]^n$ the set $\{\varphi(\bar{x}, \bar{a}^{\eta,\bar{\imath}}_{\bar{\jmath}}) \mid \bar{\jmath} \in [\omega]^n\}$ is $[k]^n$-contradictory.

In addition, we have $\{\varphi(\bar{x}, \bar{a}_{\bar{\imath}}) \mid \bar{\imath} \in [\omega]^n\}$ is $[k]^n$-contradictory.

It remains to observe that this gives us the desired equivalence for $(\alpha+1)$. Indeed, the sequence I corresponds to $I_{\langle\rangle}$ in the tree characterization for the level $\alpha + 1$. The sequences $I_{\eta,\bar{\imath}}$, for $\eta \in ([\omega]^n)^{<\alpha}$ and $\bar{\imath} \in [\omega]^n$, correspond to $I_{\bar{\imath}^\frown\eta}$ in the tree for $\alpha + 1$. Similarly, the elements $\bar{b}_{\eta,\bar{\imath}}$ are $\bar{b}_{\bar{\imath}^\frown\eta}$.

This completes the induction step. Finally, the claim for $\alpha = \omega$ follows by compactness theorem. ⊣

Using the tree characterization for the rank D^*_n we get the following.

Proposition 2.1.13. *For every* $n < \omega$*, type* $p(\bar{x})$*, and* $k < \omega$*,* $D^*_n[p, \varphi, k] = \infty$ *if and only if* $D^*_n[p, \varphi, k] \geq \omega$*.*

Proof. Necessity is clear, we prove sufficiency. We show by induction on $\alpha \geq \omega$ that

$$D^*_n[p, \varphi, k] \geq \alpha \text{ implies } D^*_n[p\cup, \varphi, k] \geq \alpha + 1.$$

Base case: $\alpha = \omega$. Use Lemma 2.1.11, to find $I_\eta = \{\bar{a}^\eta_i \mid i < \omega\}$ for each $\eta \in ([\omega]^n)^{<\omega}$ with the properties guaranteed by $D^*_n[p, \varphi, k] \geq \omega$. Now fix $\bar{\boldsymbol{\imath}} \in [\omega]^n$. Observe that the sequences $I_{\bar{\boldsymbol{\imath}}\frown\eta}$, $\eta \in ([\omega]^n)^{<\omega}$, and elements $\bar{b}_{\bar{\boldsymbol{\imath}}\frown\eta}$, $\eta \in ([\omega]^n)^\omega$, witness

$$D^*_n[p \cup \{\varphi(\bar{x}; \bar{a}^{\langle\rangle}_{\bar{\boldsymbol{\imath}}})\} \cup \operatorname{Ind}(\bar{x}; \bar{a}_{\bar{\boldsymbol{\imath}}}), \varphi, k] \geq \omega.$$

Since the set $\{\varphi(\bar{x}, \bar{a}^{\langle\rangle}_{\bar{\boldsymbol{\imath}}}) \mid \bar{\boldsymbol{\imath}} \in [\omega]^n\}$ is $[k]^n$-contradictory, we conclude $D^*_n[p, \varphi, k] \geq \omega + 1$.

The rest of the induction is immediate by the definition of the rank. $\dashv$

A useful characterization of simple theories involves the notion of a tree property. In fact, Shelah originally defined simple theories as those without the tree property. In [32], he conjectured that there are syntactic properties that split the class of simple theories into $\omega+1$ subclasses, each class having different *saturated pairs spectrum* (see [32] for the definition.

The family of strong n-tree properties defined shortly characterize strong n-simplicity, and there are examples of strongly n-simple, not strongly $n+1$-simple theories for $n \geq 1$ (we present those in Section 2). It is not yet clear if different levels of simplicity imply different saturated pairs spectra.

Definition 2.1.14. (1) A formula $\varphi(\bar{x}, \bar{y}_0, \ldots, \bar{y}_{n-1})$, a set of sequences $\{I_\eta \mid \eta \in ([\omega]^n)^{<\omega}\}$, and $k < \omega$ *witness the strong n-tree property* if for every $\eta \in ([\omega]^n)^\omega$, the type $\{\varphi(\bar{x}; \bar{a}^{\eta\restriction l}_{\eta[l]}) \mid l < \omega\}$ is realized by $\bar{b}_\eta$ such that sequences $\bar{a}^{\eta\restriction l}_{\eta[l]}$ are indiscernible over $\bar{b}_\eta$ for each $l < \omega$ and for every $\eta \in ([\omega]^n)^{<\omega}$ the set $\{\varphi(\bar{x}, \bar{a}^\eta_{\bar{\boldsymbol{\imath}}}) \mid \bar{\boldsymbol{\imath}} \in [\omega]^n\}$ is $[k]^n$-contradictory.

(2) A theory T has the *strong n-tree property* if there exist a formula, a set of parameters, and a number k witnessing the n-tree property.

Proposition 2.1.15. *A theory T is strongly α-simple if and only if it does not have a strong $(n+1)$-tree property for any $n < \alpha$.*

Proof. Follows from the definition of strong n-simplicity and Lemma 2.1.11. $\dashv$

We finish the section by proving that all stable theories are strongly ω-simple.

Theorem 2.1.16. *Every stable theory T is strongly ω-simple.*

Proof. First we need to isolate a useful combinatorial property.

Fix $1 \leq n < \omega$. Let I be a linearly ordered set. The pair $(\bar{\boldsymbol{\imath}}, \bar{\boldsymbol{\jmath}})$ of tuples in $[I]^n$ is *good* if $\mathrm{tp}_I(\bar{\boldsymbol{\imath}}/\bar{\boldsymbol{\imath}} \cap \bar{\boldsymbol{\jmath}}) = \mathrm{tp}_I(\bar{\boldsymbol{\jmath}}/\bar{\boldsymbol{\imath}} \cap \bar{\boldsymbol{\jmath}})$, where tp_I denotes a type in the structure $\langle I, <_I \rangle$.

Given $k \geq n+1$ and $J \subset I$ of size k, define the graph G_J. The vertex set is $V(G_J) := [J]^n$, and $E(\bar{\boldsymbol{\imath}}, \bar{\boldsymbol{\jmath}})$ if and only if $(\bar{\boldsymbol{\imath}}, \bar{\boldsymbol{\jmath}})$ is a good pair (for the structure $\langle J, < \rangle$). The following is easy to see.

Claim 2.1.17. *If I is an infinite linearly ordered set, then the graph G_J is connected for all $1 \leq n < \omega$ and $k \geq n+1$.*

Continuing with the proof of the theorem, we observe that since the rank $R[p, \varphi, \aleph_0]$ is finite for stable theories, it is enough to prove that for all $n < \omega$, for every φ and $n+1 \leq k < \omega$ we have

$$D_n^*[\bar{x} = \bar{x}, \varphi, k] \leq R[\bar{x} = \bar{x}, \varphi, \aleph_0].$$

By induction on $\alpha \leq \omega$ we show that for all types $p(\bar{x})$, formulas φ, and $k < \omega$

$$D_n^*[p(\bar{x}), \varphi, k] \geq \alpha \text{ implies } R[p(\bar{x}), \varphi, \aleph_0] \geq \alpha.$$

When $\alpha = 0$ or $\alpha = \omega$ the implication is clear.

Suppose $D_n^*[p(\bar{x}), \varphi, k] \geq \alpha+1$, and let $r \subset p$ be a finite subtype. By Remark 2.1.6, there is an indiscernible sequence indexed by the rationals $\{\bar{a}_i \mid i \in \mathbb{Q}\}$ such that $D_n^*[r(\bar{x}) \cup \{\varphi(\bar{x}, \bar{a}_{\bar{\boldsymbol{\imath}}})\}, \varphi, k] \geq \alpha$ for all $\bar{\boldsymbol{\imath}} \in [\mathbb{Q}]^n$ and the set $\{\varphi(\bar{x}, \bar{a}_{\bar{\boldsymbol{\imath}}}) \mid \bar{\boldsymbol{\imath}} \in [\mathbb{Q}]^n\}$ is $[k]^n$-contradictory. Let $A := \{\bar{a}_i \mid i \in \mathbb{Q}\}$. The Extension property now gives the φ-types $q_{\bar{\boldsymbol{\imath}}}$ such that

$$D_n^*[r(\bar{x}) \cup \varphi(\bar{x}, \bar{a}_{\bar{\boldsymbol{\imath}}}), \varphi, k] = D_n^*[r(\bar{x}) \cup \{\varphi(\bar{x}, \bar{a}_{\bar{\boldsymbol{\imath}}})\} \cup q_{\bar{\boldsymbol{\imath}}}(\bar{x}), \varphi, k].$$

Note that $\varphi(\bar{x}, \bar{a}_{\bar{\boldsymbol{\imath}}}) \in q_{\bar{\boldsymbol{\imath}}}(\bar{x})$, so we have $D_n^*[r(\bar{x}) \cup q_{\bar{\boldsymbol{\imath}}}(\bar{x}), \varphi, k] \geq \alpha$. Induction hypothesis now gives $R[r(\bar{x}) \cup q_{\bar{\boldsymbol{\imath}}}(\bar{x}), \varphi, \aleph_0] \geq \alpha$ for all $\bar{\boldsymbol{\imath}} \in [\mathbb{Q}]^n$. Since the set $\{\varphi(\bar{x}, \bar{a}_{\bar{\boldsymbol{\imath}}}) \mid \bar{\boldsymbol{\imath}} \in [\mathbb{Q}]^n\}$

is $[k]^n$-contradictory, for every $J \subset \mathbb{Q}$ of size k, there is at least one pair $(\bar{\imath}, \bar{\jmath})$ in $[J]^n$ such that $q_{\bar{\imath}}$ and $q_{\bar{\jmath}}$ are explicitly contradictory. Since the graph G_J is connected, there must also be a *good* pair $(\bar{\imath}, \bar{\jmath})$ with $q_{\bar{\imath}}$ and $q_{\bar{\jmath}}$ explicitly contradictory. Since $\bar{\imath} \neq \bar{\jmath}$, the common type $\mathrm{tp}_{\mathbb{Q}}(\bar{\imath}/\bar{\imath} \cap \bar{\jmath}) = \mathrm{tp}_{\mathbb{Q}}(\bar{\jmath}/\bar{\imath} \cap \bar{\jmath})$ is non-algebraic (in $\mathbb{Q}^n$). Let $\{\bar{\imath}_n \mid n < \omega\}$ be infinitely many realizations of that type in $\mathbb{Q}^n$. By indiscernibility of $\{\bar{a}_i \mid i \in \mathbb{Q}\}$, we get that the types $\{q_{\bar{\imath}_n} \mid n < \omega\}$ are pairwise explicitly contradictory. By the definition of the rank R, it means that $R[r(\bar{x}), \varphi, \aleph_0] \geq \alpha + 1$. ⊣

2.2 Motivating examples

The theories T_k we present below are all simple theories such that for every $k \geq 3$ the theory T_k is strongly $(k-2)$-simple. The theory T_{rg} is an example of a strongly ω-simple unstable theory.

Fix $k \geq 3$, let $L_k := \{P, R, S\}$, where P is an unary predicate, S and R are k-ary predicates. Let T_k be the model completion of the following set of sentences in L_k:

(1) "$R \subset P^k$;"

(2) "R is symmetric irreflexive;"

(3) "$S \subset P^{k-1} \times \neg P$" (we use the notation $\bar{x}\, S\, y$, $\bar{x}$ is understood to be a tuple in P^{k-1});

(4) "S is symmetric irreflexive in the first $k-1$ variables;"

(5) $\forall x_1 \ldots x_k, y \left[R(x_1, \ldots, x_k) \to \bigvee_{\substack{w \subset \{1,\ldots,k\} \\ |w| = k-1}} \neg(\bar{x}_w\, S\, y) \right]$, where for $w = \{i_1, \ldots, i_{k-1}\}$ we put $\bar{x}_w := x_{i_1} \ldots x_{i_{k-1}}$.

Before we write out the axioms of T_k explicitly, we define *basic formulas*. The intuition is that these formulas isolate all the types in finitely many variables over the empty set in the models of T_k.

Definition 2.2.1. Given $m, n < \omega$, let

$$Q(m,n) :=$$

$$\left\{ \bigwedge_{i<j<n} x_i \neq x_j \wedge \bigwedge_{i<j<m} y_i \neq y_j \wedge \bigwedge_{i<n} P(x_i) \wedge \bigwedge_{j<m} \neg P(y_j) \wedge \bigwedge_{w\in[n]^k} R(\bar{x}_w)^{\text{if } w\in I} \wedge \bigwedge_{j<m} \bigwedge_{u\in[n]^{k-1}} (\bar{x}_u \, S \, y_j)^{\text{if } u\in I_j} \;\middle|\; \begin{array}{l} I \subset [n]^k, \text{ and } I_j \subset [n]^{k-1}, \text{ no } k \text{ elements in } I_j \\ \text{form all the } (k-1)\text{-subsets of any } w \in I \end{array} \right\}$$

If m or n are too small for the R or S parts to make sense, we restrict the formulas in $Q(m,n)$ in the obvious way.

A *basic formula* is a formula of the following sort:

$$q(\bar{x},\bar{y}) \wedge \bigwedge_{n\leq i<N} x_i = x_{n-1} \wedge \bigwedge_{m\leq i<M} y_i = y_{m-1},$$

where $q(\bar{x},\bar{y}) \in Q(m,n)$ and $m \leq M$, $n \leq N$.

We now give a formal definition of T_k:

(1) $\forall x_0,\ldots,x_{k-1}\ R(x_0,\ldots,x_{k-1}) \rightarrow \bigwedge_{i<k} P(x_i)$;

(2) $\forall x_0,\ldots,x_{k-1}\ \bigvee_{i<j<k} x_i = x_j \rightarrow \neg R(x_0,\ldots,x_{k-1})$;

(3) $\forall x_0,\ldots,x_{k-1}\ R(x_0,\ldots,x_{k-1}) \rightarrow \bigwedge_{\sigma\in S_k} R(x_{\sigma(0)},\ldots,x_{\sigma(k-1)})$, where S_k is the set of all permutations of k;

(4) $\forall x_0,\ldots,x_{k-2},y\ (x_0\ldots x_{k-2})\, S\, y \rightarrow \left(\bigwedge_{i<k-1} P(x_i) \wedge \neg P(y)\right)$;

(5) $\forall x_0,\ldots,x_{k-2},y\ \bigvee_{i<j<k-1} x_i = x_j \rightarrow \neg(x_0,\ldots,x_{k-2})\, S\, y$;

(6) $\forall x_0,\ldots,x_{k-2},y\ (x_0,\ldots,x_{k-1})\, S\, y \rightarrow \bigwedge_{\sigma\in S_{k-1}} (x_{\sigma(0)},\ldots,x_{\sigma(k-2)})\, S\, y$;

(7) $\forall x_0,\ldots,x_{k-1},y\ R(x_0,\ldots,x_{k-1}) \rightarrow \bigvee_{\substack{w\subset k\\ |w|=k-1}} \neg(\bar{x}_w\, S\, y)$;

(8) for each $q(\bar{x},v,\bar{y}) \in Q(m+1,n) \cup Q(m,n+1)$, $m,n<\omega$: $\forall \bar{x},\bar{y}\, \exists v q(\bar{x},v,\bar{y})$

It is not hard to see that T_k is a consistent theory for every $k \geq 3$. Namely, one can build a chain of finite approximations M_i, $i < \omega$, of an infinite model of T_k. Let $M_0 := \emptyset$; and having constructed M_i, let $|M_{i+1}| := |M_i| \cup A_i$, where A_i is the set of witnesses for $\models \exists v q(\bar{a}, v, \bar{b})$. Here $\bar{a} = P(M_i)$, $\bar{b} = \neg P(M_i)$, and $q(\bar{x}, v, \bar{y}) \in Q(m+1, n) \cup Q(m, n+1)$, where m is the length of $\bar{a}$, n is the length of $\bar{b}$, and we expand R and S on A in any way consistent with Axioms (1)–(7). The union $M := \bigcup M_i$ is clearly a model of T_k.

Claim 2.2.2. *For $k \geq 3$, the theory T_k admits elimination of quantifiers. Moreover, every type in finitely many variables over the empty set is isolated by a basic formula.*

Proof. The following are easy to observe:

(1) For a fixed pair M, N, there are finitely many distinct basic formulas.

(2) Every finite tuple $\bar{d}$ in a model of T_k satisfies a basic formula (possibly after a renumbering the elements of $\bar{d}$).

(3) If $\varphi(x_0, \ldots, x_{n-1})$ is a quantifier-free formula that is satisfied in some model of T_k, then for all basic $\psi(x_0, \ldots, x_{n-1})$ we have $T_k \vdash \psi \to \varphi$ or $T_k \vdash \psi \to \neg\varphi$.

From (1)–(3), it follows that the quantifier-free type of every finite tuple in a model of T_k is isolated by a basic formula. In addition, every quantifier-free formula is equivalent modulo T_k to either $x_0 \neq x_0$ or to a finite disjunction of basic formulas. Indeed, if φ is inconsistent, then φ is equivalent to $x_0 \neq x_0$. Otherwise, for $\bar{d} \models \varphi$, find $\psi_{\bar{d}}$ isolating the quantifier-free type of $\bar{d}$. The set $\{\psi_{\bar{d}} \mid \bar{d} \models \varphi\}$ is finite, let θ be the disjunction of those formulas. Clearly, θ is equivalent to φ modulo T_k.

To show that T_k has the quantifier elimination property, it is enough to show that the formula $\exists v \psi(\bar{x}, \bar{y})$ is T_k-equivalent to a quantifier-free formula for a basic ψ where v is possibly among the free variables $\bar{x}$, $\bar{y}$ of ψ. Moreover, we may assume that $\psi(\bar{x}, \bar{y}) \in Q(m, n)$ for some $m, n < \omega$. If v is not among $\bar{x}$, $\bar{y}$, then the formula is equivalent to $\psi(\bar{x}, \bar{y})$, and we are done. Otherwise, by axioms 7 and 8 $\exists v \psi(\bar{x}, \bar{y})$ is either true for any choice of $\bar{x}$, $\bar{y}$ or is false. In either case, it is equivalent to a quantifier-free formula modulo T_k.

We have shown that T_k admits elimination of quantifiers. To derive the second statement of the claim, it is enough to observe that the basic formulas isolate the quantifier-free types over the empty set. $\dashv$

By Ryll–Nardzewski's theorem it follows that T_k is an $\aleph_0$-categorical theory. In particular, T_k is complete.

We summarize the main results of the first two sections in the following theorem.

Theorem 2.2.3. *The notion strong n-simplicity divides the class of all simple theories into $\omega + 1$ many subclasses in the following way.*

(1) All simple theories are strongly 1-simple.

(2) For $m > n \geq 1$, T strongly m-simple implies that T is strongly n-simple.

(3) The ω subclasses are: for $1 \leq n < \omega$ we have strongly n-simple, not strongly $(n+1)$-simple theories. The theory T_{n+2} serves as an example of an strongly n-simple, not strongly $(n+1)$-simple theory.

(4) The ωth subclass contains the theories that are strongly n-simple for all $1 \leq n < \omega$. All stable theories are strongly ω-simple. An example of an strongly ω-simple unstable theory is the theory T_{rg}.

(1) and (2) and the first part of (4) are immediate from the definitions. We proved that all stable theories are strongly ω-simple in Theorem 2.1.16. We split the proof of (3) into several claims.

Proposition 2.2.4. *Fix $n \geq 2$. The theory T_{n+1} is not strongly n-simple.*

Proof. Fix the theory T_{n+1} for some $n \geq 2$. We claim that the formula $\bar{x}\, S\, y$ has the strong n-tree property for y.

Construct the sequences $\{I_\eta \mid \eta \in ([\omega]^n)^{<\omega}$ by induction on the length of η. Let $I_{\langle\rangle} := \{a_i^{\langle\rangle} \mid i < \omega\}$ be such that $R(a_{i_0}^{\langle\rangle}, \ldots, a_{i_n}^{\langle\rangle})$ for all $i_0 < \cdots < i_n < \omega$. Having constructed I_η for all sequences of length up to k, take $\nu = \eta\hat{\ }\bar{\boldsymbol{\imath}}$ for some $\bar{\boldsymbol{\imath}} \in [\omega]^n$. Let $I_{\eta\hat{\ }\bar{\boldsymbol{\imath}}} := \{a_i^\nu \mid i < \omega\}$ be such that

(1) $R(a^\nu_{i_0}, \ldots, a^\nu_{i_n})$ for all $i_0 < \cdots < i_n < \omega$;

(2) I_ν is disjoint from the set $\{\bar{a}^{\nu\restriction l}_{\nu[l]} \mid l < k+1\}$.

Condition 2 and genericity imply that for every $\eta \in ([\omega]^n)^\omega$, the type $\{\bar{a}^{\eta\restriction l}_{\eta[l]} \, S \, y \mid l < \omega\}$ is realized by some element b_η, since none of the sequences $\{\bar{a}^{\eta\restriction l}_{\eta[l]} \mid l < \omega\}$ form all the n-element subsets of an $(n+1)$-element set.

From the formula isolating the type $\mathrm{tp}(b_\eta/\bar{a}^{\eta\restriction l}_{\eta[l]})$ for any $l < \omega$ we see that each sequence $\bar{a}^{\eta\restriction l}_{\eta[l]}$ is indiscernible over b_η.

Finally, Condition 1 and the axioms of T_{n+1} imply that the set $\{\bar{a}^\eta_{\bar{\boldsymbol{\imath}}} \, S \, y \mid \bar{\boldsymbol{\imath}} \in [\omega]^n\}$ is $[n+1]^n$-contradictory for every $\eta \in ([\omega]^n)^{<\omega}$. $\dashv$

Our next goal is to prove that for any $n \geq 2$ the theory T_{n+1} is strongly $(n-1)$-simple. We find it convenient to separate the (1-) simplicity case and deal with it first.

Claim 2.2.5. *Fix $n \geq 2$. The theory T_{n+1} is simple.*

Proof. We claim that actually $D[\bar{v} = \bar{v}, \varphi(\bar{v}, \bar{p}), 2] \leq 1$ for any φ.

Suppose $D[\bar{v} = \bar{v}, \varphi(\bar{v}, \bar{p}), 2] > 1$ as witnessed by an indiscernible over the empty set sequence $\{\bar{a}_i \mid i < \omega\}$.

By the structure of T_{n+1}, $\varphi(\bar{v}, \bar{p})$ is equivalent to a disjunction of basic formulas, $\varphi(\bar{v}, \bar{p}) \iff \bigvee_l \varphi_l(\bar{v}, \bar{p})$. Since the set $\{\varphi(\bar{v}, \bar{a}_i) \mid i < \omega\}$ is pairwise contradictory, for each l the set $\{\varphi_l(\bar{v}, \bar{a}_i) \mid i < \omega\}$ is pairwise contradictory, for a basic $\varphi_l(\bar{v}, \bar{p})$.

Our next step is to prove that if $\varphi_l(\bar{v}, \bar{p})$ is a basic formula such that $\{\varphi_l(\bar{v}, \bar{a}_i) \mid i < \omega\}$ is pairwise contradictory for some $\{\bar{a}_i \mid i < \omega\}$ indiscernible over the empty set, then $\varphi_l(\bar{v}, \bar{p}) \vdash v_i = p_j$ for some $v_i \in \bar{v}$, $p_j \in \bar{p}$.

Suppose this is not the case. We may assume then that φ_l does not have positive equalities at all. Indeed, if there are equalities of the form $v_i = v_j$ or $p_i = p_j$, then we can ignore the "extra" variables, and we supposed for contradiction that there are no equalities if the form $v_i = p_j$. Note that since φ_l is basic, it isolates a complete type over the empty set.

So we have $\varphi_l(\bar{v}, \bar{a}_0)$ is consistent, but the conjunction $\varphi_l(\bar{v}, \bar{a}_0) \wedge \varphi_l(\bar{v}, \bar{a}_1)$ is not. Let $\psi_l(\bar{p}_0, \bar{p}_1)$ isolate the type $\mathrm{tp}(\bar{a}_0\bar{a}_1/\emptyset)$. Then certainly

$$\psi_l(\bar{p}_0, \bar{p}_1) \wedge \varphi_l(\bar{v}, \bar{p}_0) \wedge \varphi_l(\bar{v}, \bar{p}_1) \qquad (*)$$

is inconsistent. Since the formula φ_l does not contain positive equalities, there could be only two reasons for the inconsistency of $(*)$: (1) there is an atomic formula and its negation inside $(*)$ and (2) inconsistency coming from Axiom 7.

(1) cannot be the case, since $\varphi_l(\bar{v}, \bar{p}_0)$ and $\varphi_l(\bar{x}, \bar{p}_1)$ are both consistent with $\psi(\bar{p}_0, \bar{p}_1)$.

We now deal with (2). Renumbering the variables if necessary we may assume that $(*)$ has the following subformulas:

$$\bar{x}_{\bar{\boldsymbol{\imath}}}\, S\, y \text{ for } \bar{\boldsymbol{\imath}} \in [n+1]^n, \quad R(x_0, \ldots, x_n). \qquad (**)$$

There are 3 terms in the conjunction $(*)$ and $n+2 \geq 4$ formulas in $(**)$, so at least two subformulas in $(**)$ must be subformulas in one of the terms in $(*)$. Observe now that any two terms in $(**)$ involve all of the variables $x_0, \ldots, x_n$ and y. Since each formula in $(*)$ isolates a complete type and (1) is not the case, then the subformula in $(*)$ that has at least two of the terms in $(**)$ must have all the terms in $(**)$. This contradicts the consistency of $\varphi_l(\bar{v}, \bar{a}_i)$.

Thus we proved that a basic formula with no equalities of the form $v_i = p_j$ cannot be 2-contradictory for any choice of parameters.

Let $\{\bar{a}_i \mid i < \omega\}$ be such that $\{\varphi(\bar{v}, \bar{a}_i) \mid i < \omega\}$ is pairwise contradictory, with $\bar{a}_0 = \bar{a}$. By the above and using monotonicity of D we have

$$D[\bigvee_l v_{i_l} = a_{j_l}, \varphi(\bar{v}, \bar{p}), 2] \geq D[\varphi(\bar{v}, \bar{a}), \varphi(\bar{v}, \bar{p}), 2].$$

It remains to observe that modulo $\bigvee_l v_{i_l} = a_{j_l}$, the formula $\varphi(\bar{v}, \bar{p})$ is equivalent to a disjunction of basic formulas with no positive equalities. Hence,

$$0 = D[\bigvee_l v_{i_l} = a_{j_l}, \varphi(\bar{v}, \bar{p}), 2] \geq D[\varphi(\bar{v}, \bar{a}), \varphi(\bar{v}, \bar{p}), 2]$$

and therefore, $D[\bar{v} = \bar{v}, \varphi(\bar{v}, \bar{p}), 2] \leq 1$. $\dashv$

Proposition 2.2.6. *Fix $n \geq 2$. The theory T_{n+1} is strongly $(n-1)$-simple.*

Proof. We prove that $D^*_k[\bar{v} = \bar{v}, \varphi(\bar{v}, \bar{p}_0 \ldots, \bar{p}_{k-1}), n] = 0$ for $1 < k < n$.

Claim 2.2.7. *It is enough to prove that $D^*_k[\bar{v} = \bar{v}, \varphi(\bar{v}, \bar{p}_0 \ldots, \bar{p}_{k-1}), k+1] = 0$ for all formulas φ.*

Proof. Let m be minimal such that there is $\varphi(\bar{v}, \bar{p}_0 \ldots, \bar{p}_{k-1})$ with

$$D^*_k[\bar{v} = \bar{v}, \varphi(\bar{v}, \bar{p}_0 \ldots, \bar{p}_{k-1}), m] > 0.$$

Suppose for contradiction that $m > k+1$.

Take an indiscernible sequence $\{\bar{a}_{\bar{\imath}} \mid i < \omega\}$ and $\bar{b}$ witnessing

$$D^*_k[\bar{v} = \bar{v}, \varphi(\bar{v}, \bar{p}_0 \ldots, \bar{p}_{k-1}), m] \geq 1.$$

That is $\{\varphi(\bar{x}; \bar{a}_{\bar{\imath}}) \mid \bar{\imath} \in [\omega]^k\}$ is $[m]^k$-contradictory and $\bar{b} \models \varphi(\bar{v}, \bar{a}_0, \ldots, \bar{a}_{k-1})$ is such that $\{\bar{a}_0, \ldots, \bar{a}_{k-1}\}$ is indiscernible over $\bar{b}$. Let $p(\bar{x}; \bar{a}_0, \ldots, \bar{a}_{k-1}) := \text{tp}(\bar{b}/\bar{a}_0, \ldots, \bar{a}_{k-1})$.

Case 1. The union $\bigcup_{\bar{\imath} \in [k+1]^k} p(\bar{x}, \bar{a}_{\bar{\imath}})$ is inconsistent. Let $\{\psi(\bar{x}, \bar{a}_{\bar{\imath}}) \mid \bar{\imath} \in [n+1]^n\}$ be a witness for it. Then $D^*_k[\bar{v} = \bar{v}, \psi(\bar{v}, \bar{p}_0 \ldots, \bar{p}_{k-1}), k+1] \geq 1$ as witnessed by $\bar{b}$ and $\{\bar{a}_i \mid i < \omega\}$, and we get a contradiction to $m > k+1$.

Case 2. Otherwise let $\bar{b}' \models \bigcup_{\bar{\imath} \in [k+1]^k} p(\bar{x}, \bar{a}_{\bar{\imath}})$. Note that $\{\bar{a}_0, \ldots, \bar{a}_k\}$ is indiscernible over $\bar{b}'$. Let $\bar{c}_i := \bar{a}_0 \bar{a}_{i+1}$, $i < \omega$. Then $\{\bar{c}_0, \ldots, \bar{c}_{k-1}\}$ are indiscernible over $\bar{b}'$. Let

$$\psi(\bar{x}; \bar{c}_0, \ldots, \bar{c}_{k-1}) := \bigwedge_{\bar{\imath} \in [k+1]^k} \varphi(\bar{x}; \bar{a}_{\bar{\imath}}).$$

Then $D^*_k[\bar{v} = \bar{v}, \psi, m-1] \geq 1$ as witnessed by $\bar{b}'$ and $\{\bar{c}_i \mid i < \omega\}$. Contradiction to minimality of m. ⊣

Continuing with the proof of the proposition, suppose for contradiction that $D^*_k[\bar{v} = \bar{v}, \varphi(\bar{v}, \bar{p}_0, \ldots, \bar{p}_{k-1}), k+1] \geq 1$ and take $\{\bar{a}_i \mid i < \omega\}$ and $\bar{b}$ witnesses for it. Since φ is a disjunction of basic formulas each of which isolates a complete type, $\bar{b}$ satisfies exactly one of those formulas. Since each of the basic formulas must be also $[k+1]^k$-contradictory, we may assume that there is a *basic* $\varphi(\bar{v}, \bar{p}_0, \ldots, \bar{p}_{k-1})$ such that $D^*_k[\bar{v} = \bar{v}, \varphi(\bar{v}, \bar{p}_0, \ldots, \bar{p}_{k-1}), k+1] \geq 1$.

We agree to write $\varphi(\bar{v}, \bar{p}_0, \ldots, \bar{p}_{k-1})$ as $\varphi(\bar{v}, \bar{m}, \bar{p}_0, \ldots, \bar{p}_{k-1})$, where $\bar{m}$ is the common part of parameters $\{\bar{a}_i \mid i < \omega\}$. That is, if $\bar{a}_i = \bar{a}\bar{c}_i$ for $i < \omega$, where the sets of elements in $\bar{c}_i$'s are pairwise disjoint, then the variable $\bar{m}$ is reserved for $\bar{a}$, and the variables $\bar{p}_i$ for $\bar{c}_i$'s.

If one of the conjunctive terms in the basic formula $\varphi(\bar{v}, \bar{m}, \bar{p}_0, \ldots, \bar{p}_{k-1})$ is of the form $v_i = v_j$ or $v_i = m_j$, then we can ignore the "extra" variables, as it does not change the value of the rank. If we write $\bar{p}_i$ as $p_i^0 \ldots p_i^{l-1}$, then in view of the agreement in the paragraph above, φ cannot have equalities of the form $p_i^s = p_j^t$ for $i \neq j$. If $\varphi \vdash p_i^s = p_i^t$ for some $i < k$, then $\varphi \vdash p_j^s = p_j^t$ for all $j < k$ by indiscernibility of $\{\bar{a}_i \mid i < \omega\}$ over the empty set. In this case too we simply ignore the extra variables. It remains to note that φ cannot have equalities of the form $v_i = p_j^s$ for any i, s. Otherwise, by indiscernibility of $\{\bar{a}_i \mid i < \omega\}$ over $\bar{b}$ we would have $v_i = p_j^s$ for all $j < k$, and since $k \geq 2$ this would imply $p_i^s = p_j^s$ for $i \neq j < k$, which is impossible by our agreement.

Thus, we get a basic formula φ with no positive equalities, an indiscernible sequence $\{\bar{a}_i \mid i < \omega\}$ such that $\{\bar{a}_0, \ldots, \bar{a}_{k-1}\}$ is indiscernible over $\bar{b} \models \varphi(\bar{v}, \bar{a}_0, \ldots, \bar{a}_{k-1})$.

So the conjunction

$$\bigwedge_{\bar{\boldsymbol{\imath}} \in [k+1]^k} \varphi(\bar{v}, \bar{a}_{\bar{\boldsymbol{\imath}}})$$

is inconsistent. Let $\psi(\bar{p}_0, \ldots, \bar{p}_k)$ isolate the type $\mathrm{tp}(\bar{a}_0, \ldots, \bar{a}_k/\emptyset)$. Then certainly

$$\psi(\bar{p}_0, \ldots, \bar{p}_k) \wedge \bigwedge_{\bar{\boldsymbol{\imath}} \in [k+1]^k} \varphi(\bar{v}, \bar{p}_{\bar{\boldsymbol{\imath}}}) \qquad (*)$$

is inconsistent. As before, since the formula φ does not contain positive equalities, there could be only two reasons for the inconsistency of $(*)$: (1) there is an atomic formula and its negation inside $(*)$ and (2) inconsistency coming from Axiom 7.

If (1) is the case, since $\varphi(\bar{v}, \bar{p}_{\bar{\boldsymbol{\imath}}}) \wedge \psi(\bar{p}_0, \ldots, \bar{p}_k)$ is consistent for all $\bar{\boldsymbol{\imath}} \in [n]^{n-1}$, the "bad" atomic formula θ and its negation must be inside $\varphi(\bar{v}, \bar{p}_{\bar{\boldsymbol{\imath}}})$ and $\varphi(\bar{v}, \bar{p}_{\bar{\boldsymbol{\jmath}}})$ respectively for some $\bar{\boldsymbol{\imath}} \neq \bar{\boldsymbol{\jmath}}$. Without loss of generality, we may assume that $\bar{\boldsymbol{\imath}} = \langle 0, \ldots, k-1 \rangle$, $\bar{\boldsymbol{\jmath}} = \langle 1, \ldots, k \rangle$. So the variables of the formula θ must be among $\bar{v}$ and $\bar{p}_1, \ldots, \bar{p}_{k-1}$, so $\theta = \theta(\bar{v}, \bar{p}_1, \ldots, \bar{p}_{k-1})$. Since φ is a conjunction of atomic formulas, we have

$\vdash \varphi(\bar{v}, \bar{p}_0, \ldots, \bar{p}_{k-1}) \to \theta(\bar{v}, \bar{p}_1, \ldots, \bar{p}_{k-1})$ and $\vdash \varphi(\bar{v}, \bar{p}_1, \ldots, \bar{p}_k) \to \neg\theta(\bar{v}, \bar{p}_1, \ldots, \bar{p}_{k-1})$. From the last implication we get $\vdash \varphi(\bar{v}, \bar{p}_0, \ldots, \bar{p}_{k-1}) \to \neg\theta(\bar{v}, \bar{p}_0, \ldots, \bar{p}_{k-2})$. Thus we get that $\{\bar{p}_0, \ldots, \bar{p}_{k-2}\}$ and $\{\bar{p}_1, \ldots, \bar{p}_{k-1}\}$ have different types over $\bar{v}$, which contradicts the assumption. Therefore, (1) can never be the case.

We now deal with (2). Renumbering the variables if necessary we may assume that $(*)$ has the following subformulas:

$$\bar{x}_{\bar{\boldsymbol{\imath}}}\, S\, y \text{ for } \bar{\boldsymbol{\imath}} \in [n+1]^n, \quad R(x_0, \ldots, x_n). \qquad (**)$$

Since $k < n$, there are $k+2 \leq n+1$ terms in the conjunction $(*)$ and $n+2$ formulas in $(**)$, so at least two subformulas in $(**)$ must be subformulas in one of the terms in $(*)$. The proof now reduces to checking that in this case $\varphi(\bar{v}, \bar{p}_{\bar{\boldsymbol{\jmath}}}) \wedge \psi(\bar{p}_0, \ldots, \bar{p}_{n-1})$ is inconsistent for some $\bar{\boldsymbol{\jmath}}$, which contradicts the assumptions on the formula φ and the sequence $\{\bar{a}_i \mid i < \omega\}$.

Since any two terms in $(**)$ involve all of the variables $x_0, \ldots, x_n$ and y, each formula in $(*)$ isolates a complete type, and (1) is not the case, then the subformula in $(*)$ that has at least two of the terms in $(**)$ must have all the terms in $(**)$. This contradicts the consistency of $\varphi(\bar{v}, \bar{a}_0, \ldots, \bar{a}_{k-1})$.

To sum up, we know that the rank D_1^* is at most 1 for the theory T_{n+1}, and the ranks D_k^*, $1 < k < n$ are all 0. Therefore, T_{n+1} is strongly $(n-1)$-simple. $\dashv$

Proof of Theorem 2.2.3. As we remarked earlier, (1) and (2) are immediate from the definitions. Statement (3) follows from the two propositions above.

The theory of a random graph is unstable because of the independence property. It is straightforward that the rank D_1^* (of any parameters) is at most 1. The ranks D_n^*, for $1 < n < \omega$, are 0. The verification is similar to, and simpler than, what we did in the proposition above. $\dashv$

Definition 2.2.8. A first order theory T is *strongly n-supersimple* if it is supersimple and strongly n-simple.

Remark 2.2.9. From analysis of T_k, $k \geq 3$, it follows that the theory is supersimple, the only formula witnessing dividing is the equality. So the dependence relation

$A \underset{C}{\perp} B$ is just $A \cap (B \cup C) \subset A \cap C$. Thus, the theory T_{n+2} is strongly n-supersimple, not strongly $n+1$-simple.

Every set in a model of T_k is algebraically closed. By the known facts (e.g. [Bu], [Shami]), in T_k the Lascar strong types coincide with the strong types, so we have independent amalgamation over arbitrary sets.

2.3 A key property of strongly n-simple theories

Definition 2.3.1. For $n < \omega$, we say that a formula $\varphi(\bar{x}, \bar{a}_0, \ldots, \bar{a}_{n-1})$ *strongly n-divides over A* if there is an indiscernible over A sequence $\{\bar{a}_i \mid i < \omega\}$ that starts with $\{\bar{a}_0, \ldots, \bar{a}_{n-1}\}$ and there is $\bar{b} \models \varphi(\bar{x}, \bar{a}_0, \ldots, \bar{a}_{n-1})$ such that $\{\bar{a}_0, \ldots, \bar{a}_{n-1}\}$ are indiscernible over $\bar{b}$ and the set $\{\varphi(\bar{x}; \bar{a}_{\bar{\boldsymbol{\imath}}}) \mid \bar{\boldsymbol{\imath}} \in [\omega]^n\}$ is $[k]^n$-contradictory for some k.

Remarks 2.3.2. It's clear that for $n = 1$ the definition is the same as that of dividing. We now describe the connection between strong n-dividing and dividing for strongly n-simple theories, $n \geq 2$.

The possible parameters from A in the formula φ are assumed to be a part of the variables $\bar{y}_i$, $i < n$.

Recall that the symbol $\text{Ind}(\bar{x}; \bar{y}_0, \ldots, \bar{y}_{n-1})$ denotes the partial type expressing that $\bar{y}_0, \ldots, \bar{y}_{n-1}$ are indiscernible over $\bar{x}$.

It is easy to see (from examples in Section 2) that the property stated in Lemma 2.3.3 may fail outside the n-simple theories. Moreover, we prove in Theorem 2.3.8 that the property is equivalent to strong n-simplicity.

Lemma 2.3.3. *Given $n \geq 2$, suppose that the theory T is strongly n-simple. Let $\varphi(\bar{x}, \bar{a}_0, \ldots, \bar{a}_{n-1})$ be a formula such that the partial type*

$$\varphi(\bar{x}, \bar{a}_0, \ldots, \bar{a}_{n-1}) \cup \text{Ind}(\bar{x}; \bar{a}_0, \ldots, \bar{a}_{n-1})$$

does not divide over A. Then $\varphi(\bar{x}, \bar{a}_0, \ldots, \bar{a}_{n-1})$ does not strongly n-divide over A.

Proof. Suppose not. Then we can find an indiscernible over A sequence $I := \{\bar{a}_i \mid i < \omega\}$ and $\bar{b} \models \varphi(\bar{x}, \bar{a}_0, \ldots, \bar{a}_{n-1})$ such that $\{\bar{a}_0, \ldots, \bar{a}_{n-1}\}$ are indiscernible over $\bar{b}$

and the set $\{\varphi(\bar{x};\bar{a}_{\bar{\imath}}) \mid \bar{\imath} \in [\omega]^n\}$ is $[k]^n$-contradictory for some k. We now build the strong n-tree property with φ.

Let $I_{\langle\rangle} := I$. For $\eta \in ([\omega]^n)^{<\omega}$ of length $k+1$, for all $i < \omega$ let

$$\bar{a}^\eta_i := \bar{a}_{(\max(\eta[k-1])+1)\oplus i}, \quad \text{and} \quad I_\eta := \{\bar{a}^\eta_i \mid i < \omega\},$$

where $\max(\eta[k-1])$ is the maximal (i.e., the last) member of the n-sequence $\eta[k-1]$ (we assume $\max(\langle\rangle) := -1$); and $\oplus$ means that we add the natural number $(\max(\eta[k-1])+1)$ to each entry in the n-sequence $\eta[k]$.

By construction, the set $\{\varphi(\bar{x},\bar{a}^\eta_{\bar{\imath}}) \mid \bar{\imath} \in [\omega]^n\}$ is $[k]^n$-contradictory for every $\eta \in ([\omega]^n)^{<\omega}$. Moreover, for each $\eta \in ([\omega]^n)^\omega$ the sequence $\{\bar{a}^{\eta\restriction l}_{\eta[l]} \mid l < \omega\}$ is indiscernible over A in $\mathrm{tp}(\bar{a}_{\eta[0]}/A) = \mathrm{tp}(\bar{a}_0,\ldots,\bar{a}_{n-1}/A)$. Since the type $\varphi(\bar{x},\bar{y}_0,\ldots,\bar{y}_{n-1}) \cup \mathrm{Ind}(\bar{x};\bar{y}_0,\ldots,\bar{y}_{n-1})$ does not divide over A, there is $\bar{b}_\eta$ realizing

$$\bigcup_{l<\omega} \varphi(\bar{x},\bar{a}^{\eta\restriction l}_{\eta[l]}) \cup \mathrm{Ind}(\bar{x};\bar{a}^{\eta\restriction l}_{\eta[l]}).$$

So the constants $\bar{b}_\eta$ for $\eta \in ([\omega]^n)^\omega\}$, sequences I_η, $\eta \in ([\omega]^n)^{<\omega}\}$, the formula φ, and $k < \omega$ witness the strong n-tree property for T. We get a contradiction by Proposition 2.1.15. $\dashv$

Thus we see that the non-dividing assumption has stronger consequences in strongly n-simple theories.

It is well known that if a type $p(\bar{x},\bar{a})$ does not divide over C, then for any indiscernible sequence I over C containing $\bar{a}$ there is $\bar{b} \models p(\bar{x},\bar{a})$ such that I is indiscernible over $C\bar{b}$. Our next goal is to generalize this statement. To illustrate the significance of statement (2): take $n = 2$. Then the Lemma essentially says: if the Lascar strong types of $\bar{a}_0$, $\bar{a}_1$ over C are the same and $\bar{a}_0$, $\bar{a}_1$ have the same type over $C\bar{b}$, then the Lascar strong types (not just the types) of $\bar{a}_0$, $\bar{a}_1$ over $C\bar{b}$ coincide provided $\bar{b}$ is independent from $\bar{a}_0$, $\bar{a}_1$ over C. The precise statement is in Corollary 2.3.6. This property does actually hold in all the examples from the previous section, but simply because in T_k the notions of Lascar strong type and strong type coincide, and every set is algebraically closed.

The property mentioned in Corollary 2.3.6 also helps to understand the reason why strong n-simplicity is indeed "strong." The property is "too good" for certain simple theories that we would want to treat as 2-simple. Chapter 3 is devoted to this topic.

Lemma 2.3.4. *Suppose that T is strongly n-simple. Let $I := \{\bar{a}_i \mid i < \omega\}$ be an indiscernible sequence over C and $\bar{b} \underset{C}{\perp} \bar{a}_0 \ldots \bar{a}_{n-1}$. Suppose that $\{\bar{a}_0, \ldots, \bar{a}_{n-1}\}$ are indiscernible over $C\bar{b}$.*

Denote the type $\mathrm{tp}(\bar{b}/C\bar{a}_0, \ldots, \bar{a}_{n-1})$ by $p(\bar{x}, \bar{a}_0, \ldots, \bar{a}_{n-1})$. For $\bar{\imath} \in [\omega]^n$, $p(\bar{x}, \bar{a}_{\bar{\imath}})$ stands for the type obtained from $p(\bar{x}, \bar{a}_0, \ldots, \bar{a}_{n-1})$ by replacing $\bar{a}_k$ with $\bar{a}_{\bar{\imath}[k]}$ for $k < n$. Then

(1) the type $q(\bar{x}) := \bigcup_{\bar{\imath} \in [\omega]^n} p(\bar{x}, \bar{a}_{\bar{\imath}})$ is consistent;

(2) there is a sequence I' containing $\bar{a}_0, \ldots, \bar{a}_{n-1}$ that is indiscernible over $C\bar{b}$.

Proof. (1) Otherwise, by compactness and indiscernibility we obtain a formula $\varphi(\bar{x}, \bar{y}_0, \ldots, \bar{y}_{n-1})$, possibly with parameters from C and $k < \omega$ such that $\{\varphi(\bar{x}, \bar{a}_{\bar{\imath}}) \mid \bar{\imath} \in [\omega]^n\}$ is $[k]^n$-contradictory. We may assume that the parameters from C are absorbed in each of the $\bar{a}_i$'s. Since $\bar{b} \models \varphi(\bar{x}, \bar{a}_0, \ldots, \bar{a}_{n-1}) \cup \mathrm{Ind}(\bar{x}; \bar{a}_0, \ldots, \bar{a}_{n-1})$ and $\bar{b} \underset{C}{\perp} \bar{a}_0 \ldots \bar{a}_{n-1}$, the type $\varphi(\bar{x}, \bar{y}_0, \ldots, \bar{y}_{n-1}) \cup \mathrm{Ind}(\bar{x}; \bar{y}_0, \ldots, \bar{y}_{n-1})$ does not divide over C. By the Lemma2.3.3 $\varphi(\bar{x}, \bar{y}_0, \ldots, \bar{y}_{n-1})$ does not strongly n-divide over C, so $\{\varphi(\bar{x}, \bar{a}_{\bar{\imath}}) \mid \bar{\imath} \in [\omega]^n\}$ cannot be $[k]^n$-contradictory.

(2) We first prove that the following set of formulas is consistent:

$$\Gamma(\bar{x}) := q(\bar{x}) \cup \{\varphi(\bar{x}, \bar{a}_0, \ldots, \bar{a}_{k-1}, \bar{c}) \leftrightarrow \varphi(\bar{x}, \bar{a}_{i_0}, \ldots, \bar{a}_{i_{k-1}}, \bar{c}) \mid k < \omega,\ i_0 < \cdots < i_{k-1} < \omega,\ \varphi \in Fml(L(T)),\ \bar{c} \in C\}.$$

By Compactness it's enough to show that for each $i_0 < \cdots < i_{k-1} < \omega$, and every $\varphi(\bar{x}, \bar{y}_0, \ldots, \bar{y}_{k-1}, \bar{z}) \in Fml(L(T))$, and $\psi(\bar{x}, \bar{a}_0, \ldots, \bar{a}_{i_{k-1}}, \bar{c}) \in q(\bar{x})$ we have consistency of

$$\psi(\bar{x}, \bar{a}_0, \ldots, \bar{a}_{i_{k-1}}, \bar{c}) \wedge [\varphi(\bar{x}, \bar{a}_0, \ldots, \bar{a}_{k-1}, \bar{c}) \leftrightarrow \varphi(\bar{x}, \bar{a}_{i_0}, \ldots, \bar{a}_{i_{k-1}}, \bar{c})]. \quad (*)$$

Let $\bar{b}^* \models q(\bar{x})$. By Ramsey's theorem there is an infinite $J \subset I$ such that J is φ-indiscernible over $\bar{c}\bar{b}^*$ (i.e., for any $\bar{a}_{j_1}, \ldots, \bar{a}_{j_{k-1}} \in J$ the truth value of $\varphi(\bar{b}^*, \bar{a}_{j_1}, \ldots, \bar{a}_{j_{k-1}}, \bar{c})$ is fixed). For $\bar{a}'_0, \ldots, a'_{i_{k-1}} \in J$ we have

$$\psi(\bar{d}^*, \bar{a}'_0, \ldots, \bar{a}'_{i_{k-1}}, \bar{c}) \wedge [\varphi(\bar{d}^*, \bar{a}'_0, \ldots, \bar{a}'_{k-1}, \bar{c}) \leftrightarrow \varphi(\bar{d}^*, \bar{a}'_{i_0}, \ldots, \bar{a}'_{i_{k-1}}, \bar{c})].$$

Now indiscernibility of I over $\bar{c}$ gives $(*)$, so $\Gamma(\bar{x})$ is consistent. Let $f \in \mathrm{Aut}_{C\bar{a}_0\ldots\bar{a}_{n-1}}(\mathfrak{C})$ map $\bar{b}^{**} \models \Gamma$ to $\bar{b}$. Then $I' := f(I)$ is as needed. ⊣

A standard argument gives the following:

Corollary 2.3.5. *Suppose that T is strongly n-simple. Let $I := \{\bar{a}_i \mid i < \omega\}$ be an Morley sequence over C and $\bar{b} \underset{C}{\downarrow} \bar{a}_0 \ldots \bar{a}_{n-1}$. Suppose that $\{\bar{a}_0, \ldots, \bar{a}_{n-1}\}$ are indiscernible over $C\bar{b}$.*

Let $p(\bar{x}, \bar{a}_0, \ldots, \bar{a}_{n-1}) := \mathrm{tp}(\bar{b}/C\bar{a}_0 \ldots \bar{a}_{n-1})$. Then the type $\bigcup_{\bar{\imath} \in [\omega]^n} p(\bar{x}, \bar{a}_{\bar{\imath}})$ does not fork over C.

For $n = 2$, Lemma 2.3.4 has a very important corollary.

Corollary 2.3.6. *Let T be strongly 2-simple. Suppose that $\bar{a}_0$ and $\bar{a}_1$ realize the same Lascar strong type over C and $\bar{b}$ is such that $\bar{b} \underset{C}{\downarrow} \bar{a}_i$, $i = 0, 1$ and $\mathrm{tp}(\bar{a}_0/C\bar{b}) = \mathrm{tp}(\bar{a}_1/C\bar{b})$. Then $\mathrm{lstp}(\bar{a}_0/C\bar{b}) = \mathrm{lstp}(\bar{a}_1/C\bar{b})$.*

Proof. First we prove

Subclaim 2.3.7. *In the conditions of the Corollary, if in addition $\bar{b}$ is such that $\bar{b} \underset{C}{\downarrow} \bar{a}_0\bar{a}_1$ and $\bar{a}_0$, $\bar{a}_1$ are independent over C, then $\mathrm{lstp}(\bar{a}_0/C\bar{b}) = \mathrm{lstp}(\bar{a}_1/C\bar{b})$.*

Proof. By (1-)simplicity, there is a Morley sequence $I = \{\bar{a}_i \mid i < \omega\}$ over C that begins with $\bar{a}_0$, $\bar{a}_1$. By Lemma 2.3.4, there is a sequence I' that begins with $\bar{a}_0$, $\bar{a}_1$ and is indiscernible over $C\bar{b}$. Therefore, $\mathrm{lstp}(\bar{a}_0/C\bar{b}) = \mathrm{lstp}(\bar{a}_1/C\bar{b})$. ⊣

By simplicity, we can take a model M of T such that $M \supset C\bar{b}$ and $\bar{a}_0 \underset{C}{\downarrow} M$. By extension, there is $\bar{a}^* \models \mathrm{tp}(\bar{a}_0/M)$ such that $\bar{a}^* \underset{C}{\downarrow} M\bar{a}_0\bar{a}_1$. In particular, $\bar{a}^*$ has the same Lascar strong type as $\bar{a}_0$ over C, $\bar{a}_i \underset{C}{\downarrow} \bar{a}^*$, and $\bar{b} \underset{C}{\downarrow} \bar{a}_i\bar{a}^*$, for $i = 0, 1$. By the

Claim, $\operatorname{lstp}(\bar{a}_0/C\bar{b}) = \operatorname{lstp}(\bar{a}^*/C\bar{b})$ and $\operatorname{lstp}(\bar{a}^*/C\bar{b}) = \operatorname{lstp}(\bar{a}_1/C\bar{b})$, so $\operatorname{lstp}(\bar{a}_0/C\bar{b}) = \operatorname{lstp}(\bar{a}_1/C\bar{b})$. ⊣

We now prove the converse to Lemma 2.3.3.

Theorem 2.3.8. *Let T be a complete first order theory. T is strongly n-simple if and only if for every $1 \leq k \leq n$, every A, $\bar{a}_0, \ldots, \bar{a}_{k-1}$, and $\varphi(\bar{x}, \bar{y}_0, \ldots, \bar{y}_{k-1})$ the following property holds:*

if there is a Morley sequence $\{\bar{c}_i[0] \ldots \bar{c}_i[k-1] \mid i < \omega\}$ in $\operatorname{tp}(\bar{a}_0 \ldots \bar{a}_{k-1}/A)$ such that

$$\bigcup_{i<\omega} \varphi(\bar{x}, \bar{c}_i[0], \ldots, \bar{c}_i[k-1]) \cup \operatorname{Ind}(\bar{x}; \bar{y}_i[0], \ldots, \bar{y}_i[k-1])$$

is consistent ***then*** *$\varphi(\bar{x}, \bar{y}_0, \ldots, \bar{y}_{k-1})$ does not strongly k-divide over A.*

Proof of Theorem 2.3.8. For $n = 1$, i.e., for simple theories the statement was proved by Kim in [21]. With that in mind, the theorem essentially asserts that strong n-simplicity is equivalent to the situation when non-dividing implies not strong k-dividing for all $1 \leq k \leq n$.

One direction is given by Lemma 2.3.3 and the fact that "strong n-simple" implies "strong k-simple for all $1 \leq k \leq n$." We prove the other direction by induction on n. The base $n = 1$ is given by Kim's result.

Suppose now that $n \geq 2$ and that T is strongly $(n-1)$-simple. Here is the general idea for the rest of the proof. Suppose the theory is not strongly n-simple, and take the corresponding n-dimensional tree. By compactness, we may assume the tree is ω-branching (i.e., every node has ω-many immediate successors) and its height is as big as we like. We show that we can extract a "uniform subtree" from it, the one that will be used to contradict the assumptions of the theorem. We start by proving the technical partition result, but first we need some definitions.

Definition 2.3.9. Given a cardinal κ, let T be the natural tree structure on $\omega^{<\lambda}$, where λ is a regular cardinal, $\lambda > \kappa$. Suppose that the nodes of T are colored in κ-many colors. We will assume that the colors are the members of κ. For a node t of T and $n \in \omega$, we say that *there exists a monochromatic n-tree of color α above t* if

– for $n = 0$: if there is a node t^* of color α above t;

– for $n = k + 1$: there is $t^* \succ t$ of color α such that above *every* immediate successor of t^* there is a monochromatic k-tree of color α.

We say that there is a monochromatic ω-tree of color α above t if there are n-trees of color α above t for all $n < \omega$.

Lemma 2.3.10. *For some $\alpha < \kappa$, there is a monochromatic ω-tree of color α above some $t \in T$.*

Proof. Suppose not. Then for every $\alpha < \kappa$, for every $t \in T$, the height of a monochromatic tree of color α above t is at most $h(t, \alpha) < \omega$. We show that in this case we can find $\{t_\alpha \mid \alpha < \kappa\} \subset T$ such that

(1) $t_\alpha \prec t_\beta$ for $\alpha < \beta < \kappa$;

(2) there is no node of color α above t_α.

This is clearly enough to get a contradiction, since the length of each t_α is less than λ for each $\alpha < \kappa$, so $t^* := \bigcup_{\alpha<\kappa} t_\alpha$ is a sequence whose length has cofinality at most that of κ. Since this is less than λ, $t^* \in T$, and the nodes above t^* would have to be colorless.

Now we do the construction. Let $t_0' := \langle\rangle$. If there are no nodes of color 0 above t_0', then we are done: let $t_0 := t_0'$. Otherwise let $n_0 := h(\langle\rangle, 0)$, the maximal height of the color 0 monochromatic tree above t_0'. Find t_0^0 of color 0 such that $h(t_0^0, 0) = n_0$. By definition, there is $i_0 < \omega$ such that $h(t_0^0{}^\frown\langle i_0\rangle) = n_0 - 1$. Again by the definition, we get that there exists t_0^1 of color 0 and $i_1 < \omega$ such that $h(t_0^1{}^\frown\langle i_1\rangle) = n_0 - 2$. Repeating this for n_0 many steps, we get a node $t_0^{n_0}$ of color 0 and $i_{n_0} < \omega$ such that there are no nodes of color 0 above $t_0 := t_0^{n_0}{}^\frown\langle n_0\rangle$.

Given $t_{\alpha+1}' := t_\alpha$, let $n_{\alpha+1} := h(t_{\alpha+1}', \alpha + 1)$. Similar to the above, construct $t_{\alpha+1}^{n_{\alpha+1}}$ of color $\alpha + 1$ and $i_{n_{\alpha+1}}$ such that there are no nodes of color $\alpha + 1$ above $t_{\alpha+1} := t_{\alpha+1}^{n_{\alpha+1}}{}^\frown\langle i_{n_{\alpha+1}}\rangle$.

For the limit stage, take $\{t_\alpha \mid \alpha < \beta\}$. Let $t'_\beta := \bigcup_{\alpha<\beta} t_\alpha$. By a cofinality argument, $t'_\beta \in T$, and there are no nodes of color α for all $\alpha < \beta$ above t'_β. Now it remains to eliminate the color β: let $t_\beta := t_\beta^{n_\beta \frown}\langle i_{n_\beta}\rangle$, where $n_\beta := h(t'_\beta, \beta)$. $\dashv$

Continuing with the proof of the theorem, suppose the theory is not strongly n-simple, and take a formula $\varphi(\bar{x}, \bar{y}_0, \dots, \bar{y}_{n-1})$, a set of sequences $\{I_\eta \mid \eta \in ([\omega]^n)^{<\omega}\}$, and $k < \omega$ witnessing the strong n-tree property. That is, we have that for every $\eta \in ([\omega]^n)^\omega$, the type $\{\varphi(\bar{x}; \bar{a}_{\eta[l]}^{\eta\restriction l}) \mid l < \omega\}$ is realized by $\bar{b}_\eta$ such that sequences $\bar{a}_{\eta[l]}^{\eta\restriction l}$ are indiscernible over $\bar{b}_\eta$ for each $l < \omega$ and for every $\eta \in ([\omega]^n)^{<\omega}$ the set $\{\varphi(\bar{x}, \bar{a}_{\boldsymbol{\imath}}^{\eta}) \mid \boldsymbol{\imath} \in [\omega]^n\}$ is $[k]^n$-contradictory. By compactness we may assume that the n-dimensional tree has the height $\lambda := (2^{|T|})^+$, i.e., we have sequences I_η for $\eta \in ([\omega]^n)^{<\lambda}$. In addition we may assume that each I_η is indiscernible over $\{\bar{a}_{\eta[l]}^{\eta\restriction l} \mid l < \mathrm{lh}(\eta)\}$.

Since $|[\omega]^n| = \aleph_0$, after enumerating the elements of $[\omega]^n$ by ω in some way we get an ω-branching tree of height λ. Let the color of the node corresponding to the sequence $\eta \in ([\omega]^n)^{<\lambda}$ be the type of first n (or any n) elements over the empty set. Clearly, the number of colors is at most $\kappa = 2^{|T|}$. By Lemma 2.3.10 and compactness, we obtain an n-dimensional tree witnessed by φ, $k < \omega$, and $\{I_\eta \mid \eta \in ([\omega]^n)^{<\omega+\omega}\}$ such that in addition to the usual requirements we have

(1) for every $\eta \in ([\omega]^n)^{<\omega}$ the type of $\bar{a}_{\boldsymbol{n}}^{\eta}$ over the empty set is the same;

(2) I_η is indiscernible over $\{\bar{a}_{\eta[l]}^{\eta\restriction l} \mid l < \mathrm{lh}(\eta)\}$

By Ramsey's theorem and compactness, we may assume that for some fixed $\eta^* \in ([\omega]^n)^{<\omega}$, the sequence $\{\bar{a}_{\eta^*[l]}^{\eta^*\restriction l} \mid l < \omega + \omega\}$ is indiscernible over the empty set. Let $A := \{\bar{a}_{\eta^*[l]}^{\eta^*\restriction l} \mid l < \omega\}$. Then $I := \{\bar{a}_{\eta^*[l]}^{\eta^*\restriction l} \mid \omega \le l < \omega + \omega\}$ is a Morley sequence over A, and by our construction $J := I_{\eta^*\restriction\omega}$ is indiscernible over A.

Then the Morley sequence I and the element $\bar{b}_{\eta^*}$ witness that

$$\varphi(\bar{x}, \bar{y}_0, \dots, \bar{y}_{n-1}) \cup \mathrm{Ind}(\bar{x}; \bar{y}_0, \dots, \bar{y}_{n-1})$$

does not divide over A. However, the sequence J witnesses that the formula $\varphi(\bar{x}, \bar{y}_0, \dots, \bar{y}_{n-1})$ strongly n-divides over A. $\dashv$

2.4 Strong n-dimensional amalgamation

In this section we define the stronger version of n-dimensional amalgamation property and investigate its implications for theories that are at least strongly 2-simple. So for the remainder of this part, "n-simplicity" will refer to the strong n-simplicity and the "n-dimensional amalgamation" will refer to the stronger version of amalgamation property which we define below. Subsequently, we discuss the reasons why this family of amalgamation properties is called strong.

It is convenient to view an n-dimensional cube as the set $\mathcal{P}(n)$ (recall that $n = \{0, 1, \ldots, n-1\}$). Roughly speaking, the n-dimensional amalgamation is about being able to place the "top" element into the n-dimensional cube. We use the notation $\mathcal{P}^-(n)$ for the collection of all the proper subsets of n (i.e., the cube without the top).

Definition 2.4.1. Fix $n < \omega$. A system of types $\{p_w(\bar{x}) \mid w \in \mathcal{P}^-(n)\}$ is said to be an *n-dimensional independent system of types over A* if

(1) $\operatorname{dom}(p_\emptyset) = A$;

(2) if $w_1, w_2 \in \mathcal{P}^-(n)$ and $w_1 \supset w_2$, then p_{w_1} is a non-forking extension of p_{w_2};

(3) if $w_1, w_2 \in \mathcal{P}^-(n)$, then $\operatorname{dom}(p_{w_1}) \underset{\operatorname{dom}(p_{w_1 \cap w_2})}{\perp} \operatorname{dom}(p_{w_2})$.

If in addition for each $w_1, w_2 \in \mathcal{P}^-(n)$ the types p_{w_1}, p_{w_2} extend the same Lascar strong type over $\operatorname{dom}(p_{w_1 \cap w_2})$, we call the system *$n$-dimensional independent system of Lascar strong types over A.*

Remark 2.4.2. We are interested in the generalized amalgamation properties for strongly n-simple theories, $n \geq 2$. If T is strongly 2-simple, then Corollary 2.3.6 gives us the following. Suppose that an n-dimensional independent system of types $\{p_w(\bar{x}) \mid w \in \mathcal{P}^-(n)\}$ over A is such that every p_w extends the same Lascar strong type over A. Then $\{p_w(\bar{x}) \mid w \in \mathcal{P}^-(n)\}$ is an n-dimensional independent system of Lascar strong types over A.

For the remainder of this Chapter, we are dealing with T that is (at least) strongly 2-simple. So we can work with the weaker notion of the n-dimensional independent system of Lascar strong types over A, which generally gives more n-dimensional independent systems. Thus, being able to amalgamate those is a stronger condition.

Definition 2.4.3. We say that an n-dimensional independent system of types over B *can be independently amalgamated* if there is a common non-forking extension $p^*(\bar{x})$ of each $p_w(\bar{x})$, $w \in \mathcal{P}^-(n)$.

A theory T has *the strong n-dimensional amalgamation property over models* if for every $M \models T$ any n-dimensional independent system of types over M can be independently amalgamated.

A theory T has *strong n-dimensional amalgamation for Lascar strong types* if every independent system of Lascar strong types over a set A has a common independent extension.

One of the corollaries of the Independence (or 2-amalgamation) theorem is that for any two independent over A elements with the same Lascar strong type over A there is an indiscernible over A sequence containing them. Also, the Independence theorem allows us to amalgamate over Morley sequences in the sense of Proposition 2.4.6. We prove analogs of these facts for strong n-dimensional amalgamation.

Once again, for the remainder of this Chapter we use the strong n-dimensional amalgamation. Occasionally, we drop the "strong" part, especially in the proofs, not to clutter the text unnecessarily.

Proposition 2.4.4. *Fix $n \geq 2$ and $N \geq n$. Let $\{p_{\bar{\boldsymbol{\imath}}}(\bar{x}) \mid \bar{\boldsymbol{\imath}} \in [N]^{n-1}\}$ be an independent system of Lascar strong types over A. Suppose T has the strong n-dimensional amalgamation property for Lascar strong types. Then the system $\{p_{\bar{\boldsymbol{\imath}}}(\bar{x}) \mid \bar{\boldsymbol{\imath}} \in [N]^{n-1}\}$ can be independently amalgamated.*

Proof. The base $N = n$ is just the n-dimensional amalgamation property. Suppose the statement is true for some $N \geq n$, and fix an independent system $\{p_{\bar{\boldsymbol{\imath}}}(\bar{x}) \mid \bar{\boldsymbol{\imath}} \in [N+1]^{n-1}\}$. Consider the following n types:

– for every $s \subset \{N-(n-2), \ldots, N\}$ of size $n-2$, we take the amalgam $q_s(\bar{x})$ of

$$\{p_{\bar{\imath}}(\bar{x}) \mid \bar{\imath} \in [\underbrace{\{0, \ldots, N-(n-1)\}}_{N-n+2 \text{ elements}} \cup s]^{n-1}\}.$$

It exists by the induction hypothesis.

– the type $p_{\{N-(n-2),\ldots,N\}}(\bar{x})$.

It is easy to see that these n types form an n-dimensional independent system of Lascar strong types over A, so there is an independent amalgam $q(\bar{x})$. It remains to observe q is a non-forking extension of each $p_{\bar{\imath}}$, $\bar{\imath} \in [N+1]^{n-1}$. ⊣

Now we prove that strong n-dimensional amalgamation implies that we can extend a finite Morley sequence $\{\bar{a}_i \mid i < n\}$ over a set A to an infinite Morley sequence assuming that $\{\bar{a}_{\bar{\imath}} \mid \bar{\imath} \in [n]^{n-1}\}$ realize the same Lascar strong type over A. Notice that the assumption of the same Lascar strong types is clearly necessary.

Lemma 2.4.5. *Fix $n \geq 2$. Suppose T has the strong n-dimensional amalgamation property for Lascar strong types. Suppose $\{\bar{a}_i \mid i < n\}$ is an independent sequence such that for all $\bar{\imath}, \bar{\jmath} \in [n]^{n-1}$ we have $\operatorname{lstp}(\bar{a}_{\bar{\imath}}/A) = \operatorname{lstp}(\bar{a}_{\bar{\jmath}}/A)$. Then $\{\bar{a}_i \mid i < n\}$ can be extended to an infinite Morley sequence over A.*

Proof. We first construct an independent sequence $\{\bar{a}_i \mid i < \omega\}$ such that all the $\bar{a}_{\bar{\imath}}$, $\bar{\imath} \in [\omega]^{n-1}$, realize the same Lascar strong type over A. In particular, we will have $\operatorname{tp}(\bar{a}_0 \ldots \bar{a}_{n-1}/A) = \operatorname{tp}(\bar{a}_{\bar{\imath}}/A)$ for each $\bar{\imath} \in [\omega]^n$

We begin with the original sequence. Suppose we have constructed an independent sequence $\{\bar{a}_i \mid i < N\}$ for some $N \geq n$ such that $\operatorname{lstp}(\bar{a}_i/A) = \operatorname{lstp}(\bar{a}_{\bar{\jmath}}/A)$ for all $\bar{\imath}, \bar{\jmath} \in [N]^{n-1}$. Therefore, for every $\bar{\imath} \in [N]^{n-1}$ we can pick $f_{\bar{\imath}} \in \operatorname{Saut}_A(\mathfrak{C})$ such that $f(\bar{a}_0 \ldots \bar{a}_{n-2}) = \bar{a}_{\bar{\imath}}$. Let $p(\bar{x}) := \operatorname{tp}(\bar{a}_{n-1}/M\bar{a}_0 \ldots \bar{a}_{n-2})$, and consider the family of types $\{f_{\bar{\imath}}(p) \mid \bar{\imath} \in [N]^{n-1}\}$. By Proposition 2.4.4, there is a common non-forking extension q of these types that extends the Lascar strong types as well. Letting $\bar{a}_N \models q(\bar{x})$, we get $\operatorname{lstp}(\bar{a}_{n-1}/A\bar{a}_0 \ldots \bar{a}_{n-2}) = \operatorname{lstp}(\bar{a}_N/A\bar{a}_{\bar{\imath}})$ for each $\bar{\imath} \in [N]^{n-1}$ and $\bar{a}_N \underset{A}{\perp} \{\bar{a}_i \mid i < N\}$.

So we get an infinite independent sequence $\{\bar{a}_i \mid i < \omega\}$ such that for each $\bar{\imath} \in [\omega]^{n-1}$, the sequences $\bar{a}_{\bar{\imath}[0]}, \ldots, \bar{a}_{\bar{\imath}[n-1]}$ have the same type over A. By Ramsey's theorem and compactness, we get an indiscernible independent sequence over A so we are done. ⊣

Proposition 2.4.6. *Suppose that T is strongly 2-simple and has strong n-dimensional amalgamation for Lascar strong types. Let $I := \{\bar{a}_i \mid i < \omega\}$ be a Morley sequence over A and $\bar{b} \underset{A}{\perp} \bar{a}_0 \ldots \bar{a}_{n-1}$. Suppose that $\{\bar{a}_0, \ldots, \bar{a}_{n-1}\}$ are indiscernible over $A\bar{b}$. Denote by $p(\bar{x}, \bar{a}_0, \ldots, \bar{a}_{n-1})$ the type of $\bar{b}$ over $A\bar{a}_0, \ldots, \bar{a}_{n-1}$. For $\bar{\imath} \in [\omega]^n$, $p(\bar{x}, \bar{a}_{\bar{\imath}})$ stands for the type obtained from $p(\bar{x}, \bar{a}_0, \ldots, \bar{a}_{n-1})$ by replacing $\bar{a}_k$ with $\bar{a}_{\bar{\imath}[k]}$ for $k < n$. Then the type $q(\bar{x}) := \bigcup_{\bar{\imath} \in [\omega]^n} p(\bar{x}, \bar{a}_{\bar{\imath}})$ is consistent.*

Proof. By compactness it is enough to prove that for every $N < \omega$ the type $q(\bar{x}) := \bigcup_{\bar{\imath} \in [N]^n} p(\bar{x}, \bar{a}_{\bar{\imath}})$ is consistent. For every $\bar{\imath} \in [N]^n$ pick $f_{\bar{\imath}} \in \mathrm{Saut}_A(\mathfrak{C})$ such that $f(\bar{a}_0 \ldots \bar{a}_{n-1}) = \bar{a}_{\bar{\imath}}$. Consider the family of types $\{f_{\bar{\imath}}(p) \mid \bar{\imath} \in [N]^n\}$.

To apply Proposition 2.4.4, we need to show that this is an independent system of Lascar strong types over A. Independence of the system is clear.

Let $\bar{b}_{\bar{\imath}} := f_{\bar{\imath}}(\bar{b})$. It is enough to show that for $\bar{\imath}, \bar{\jmath} \in [n+1]^n$ we have $\mathrm{lstp}(\bar{b}_{\bar{\imath}}/A(\bar{a}_{\bar{\imath}} \cap \bar{a}_{\bar{\jmath}})) = \mathrm{lstp}(\bar{b}_{\bar{\jmath}}/A(\bar{a}_{\bar{\imath}} \cap \bar{a}_{\bar{\jmath}}))$. We have $\mathrm{lstp}(\bar{b}_{\bar{\imath}}/A) = \mathrm{lstp}(\bar{b}_{\bar{\jmath}}/A)$. Since $\{\bar{a}_0, \ldots, \bar{a}_{n-1}\}$ are indiscernible over $A\bar{b}$, we have $\mathrm{tp}(\bar{b}_{\bar{\imath}}/A(\bar{a}_{\bar{\imath}} \cap \bar{a}_{\bar{\jmath}})) = \mathrm{tp}(\bar{b}_{\bar{\jmath}}/A(\bar{a}_{\bar{\imath}} \cap \bar{a}_{\bar{\jmath}}))$. In addition, we may assume $\bar{b}_{\bar{\imath}} \underset{A}{\perp} \bar{b}_{\bar{\jmath}}$ by extension property. So Corollary 2.3.6 gives

$$\mathrm{lstp}(\bar{b}_{\bar{\imath}}/A(\bar{a}_{\bar{\imath}} \cap \bar{a}_{\bar{\jmath}})) = \mathrm{lstp}(\bar{b}_{\bar{\jmath}}/A(\bar{a}_{\bar{\imath}} \cap \bar{a}_{\bar{\jmath}})).$$

By Proposition 2.4.4, there is a common non-forking extension q of all these types. ⊣

2.5 Toward strong amalgamation in strongly n-simple theories

In this section we prove some auxiliary results that will help in the proof of strong $(n+1)$-dimensional amalgamation.

The following is a standard well-known fact for which I was not able to find a reference.

Fact 2.5.1 (Exchange property). *If* $A \underset{D}{\perp} B \cup C$ *and* $B \underset{D}{\perp} C$*, then* $C \underset{D}{\perp} A \cup B$.

Proof. By monotonicity, $A \underset{D}{\perp} B \cup C$ implies $A \underset{D \cup B}{\perp} C$. By symmetry, $C \underset{D \cup B}{\perp} A$ and $C \underset{D}{\perp} B$. By transitivity, $C \underset{D}{\perp} A \cup B$. ⊣

Lemma 2.5.2. *Let* T *be a theory with strong* n*-dimensional amalgamation property for Lascar strong types. Let* $\{p_w(\bar{x}) \mid w \in \mathcal{P}^-(n+1)\}$ *be an independent system of Lascar strong types over* A *such that* $\mathrm{dom}(p_w) = A \cup \{\bar{a}_i \mid i \in w\}$ *for* $w \in \mathcal{P}^-(n+1)$ *where* $\bar{a}_i \underset{A}{\perp} \{\bar{a}_j \mid j < i\}$ *for all* $1 \leq i < n+1$*. There exists* $\bar{a}'_n$ *such that*

(1) $\bar{a}'_n \models \bigcup_{\substack{v \subset n \\ |v| = n-1}} \mathrm{tp}(\bar{a}_n / A\bar{a}_v)$*;*

(2) the type $\bigcup_{w \in \mathcal{P}^-(n+1)} p'_w(\bar{x})$ *does not fork over* A*, where* $p'_w(\bar{x})$ *is obtained from* $p_w(\bar{x})$ *by replacing* $\bar{a}_n$ *with* $\bar{a}'_n$*.*

Proof. Let $\{p_w(\bar{x}) \mid w \in \mathcal{P}^-(n+1)\}$ be an independent system of Lascar strong types over A. By finite character and monotonicity of forking, we may assume that there are $\{\bar{a}_i \mid i < n+1\}$ such that $\bar{a}_i \underset{A}{\perp} \{\bar{a}_j \mid j < i\}$ for all $1 \leq i < n+1$, and $\mathrm{dom}(p_w) = A \cup \{\bar{a}_i \mid i \in w\}$ for $w \in \mathcal{P}^-(n+1)$.

For $i = 0, \ldots, n$, let $w_i := \{0, \ldots, n\} \setminus \{n-i\}$. Let $\bar{d}_i \models p_{w_i}(\bar{x})$. By Extension we may assume that $\bar{d}_i \underset{A}{\perp} \{b_j \mid j < n+1\}$. By the definition of an independent system of Lascar strong types, $\mathrm{lstp}(\bar{d}_i / A\bar{a}_{w_i \cap w_j}) = \mathrm{lstp}(\bar{d}_j / A\bar{a}_{w_i \cap w_j})$ for all $i, j < n+1$. In particular, letting $v_i := \{0, \ldots, n-1\} \setminus \{n-i\}$ for $i = 1, \ldots, n$, we have $\mathrm{lstp}(\bar{d}_0 / A\bar{a}_{v_i}) = \mathrm{lstp}(\bar{d}_i / A\bar{a}_{v_i})$. Let $f_i \in \mathrm{Saut}_{A\bar{a}_{v_i}}(\mathfrak{C})$ be such that $f_i(\bar{d}_i) = \bar{d}_0$, $i = 1, \ldots, n$. Let $\bar{c}_i := f_i(\bar{a}_n)$.

We plan to amalgamate the types $\mathrm{tp}(\bar{c}_i / A\bar{d}_0\bar{a}_{v_i})$, for $i = 1, \ldots, n$. It is clear that the system of types is independent over $A\bar{d}_0$ and extends the same Lascar strong type over $A\bar{d}_0$. By n-dimensional amalgamation, we get an element $\bar{a}'_n \models \bigcup_{i<n} \mathrm{tp}(\bar{c}_i / A\bar{d}_0\bar{a}_{v_i})$ such that $\mathrm{tp}(\bar{a}'_n / A\bar{a}_{v_i}) = \mathrm{tp}(\bar{a}_n / A\bar{a}_{v_i})$ for $1 \leq i < n$. In addition,

$$\bar{d}_0 \models \bigcup_{w \in \mathcal{P}^-(n+1)} p'_w(\bar{x}),$$

where $p'_w(\bar{x})$ is obtained from $p_w(\bar{x})$ by replacing $\bar{a}_n$ with $\bar{a}'_n$. ⊣

Lemma 2.5.3. *Let T be strongly n-simple and suppose that T has the strong n-dimensional amalgamation property for Lascar strong types. Suppose that the elements $\{\bar{a}_0, \ldots, \bar{a}_{n-1}, \bar{b}_0, \ldots, \bar{b}_{n-1}\}$ are independent over A. Let*

$$\{p_{\bar{\boldsymbol{\imath}}}(\bar{x}, \bar{a}_{\bar{\boldsymbol{\imath}}}, \bar{b}_{\bar{\boldsymbol{\imath}}}) \mid \bar{\boldsymbol{\imath}} \in [n]^{n-1}\} \cup \{q(\bar{x}, \bar{a}_0, \ldots, \bar{a}_{n-1})\}$$

be an independent system of types that extend the same Lascar strong type over A. Then the system can be independently amalgamated.

Proof. For $\bar{\boldsymbol{\imath}} \in [n-1]^{n-2}$, consider the types $r_{\bar{\boldsymbol{\imath}}}$ over $\bar{a}_{\bar{\boldsymbol{\imath}}}$, $\bar{b}_{\bar{\boldsymbol{\imath}}}$, $\bar{a}_{n-1}$. Namely, let $r_{\bar{\boldsymbol{\imath}}}$ be the (2-dimensional) amalgam of

$$p_{\bar{\boldsymbol{\imath}}\frown(n-1)}(\bar{x}, \bar{a}_{\bar{\boldsymbol{\imath}}\frown(n-1)}, \bar{b}_{\bar{\boldsymbol{\imath}}\frown(n-1)}) \upharpoonright (\bar{a}_{\bar{\boldsymbol{\imath}}}, \bar{b}_{\bar{\boldsymbol{\imath}}}, \bar{a}_{n-1}) \text{ and } q(\bar{x}, \bar{a}_0, \ldots, \bar{a}_{n-1})$$

over $A\bar{a}_{\bar{\boldsymbol{\imath}}[0]}, \ldots, \bar{a}_{\bar{\boldsymbol{\imath}}[n-3]}\bar{a}_{n-1}$. It is possible to amalgamate since the types agree over the intersection of their domains and extend the same Lascar strong type over A.

Now note that

$$\{r_{\bar{\boldsymbol{\imath}}}(\bar{x}, \bar{a}_{\bar{\boldsymbol{\imath}}}, \bar{b}_{\bar{\boldsymbol{\imath}}}, \bar{a}_{n-1}) \mid \bar{\boldsymbol{\imath}} \in [n-1]^{n-2}\}, \quad q(\bar{x}, \bar{a}_0, \ldots, \bar{a}_{n-1})$$

form an n-dimensional independent system over $A\bar{a}_0, \ldots, \bar{a}_{n-2}$. They extend the same Lascar strong type over $A\bar{a}_0, \ldots, \bar{a}_{n-2}$ because each type is an independent extension of the same Lascar strong type over A, and all the types agree on $A\bar{a}_0, \ldots, \bar{a}_{n-2}$. The amalgam of this system gives

$$p^*_{\langle 0,\ldots,n-2\rangle}(\bar{x}, \bar{a}_0, \ldots, \bar{a}_{n-2}, \bar{b}_0, \ldots, \bar{b}_{n-2}, \bar{a}_{n-1})$$

that extends both $p_{\langle 0,\ldots,n-2\rangle}$ and q; and agrees with $p_{\bar{\boldsymbol{\imath}}}$ for other $\bar{\boldsymbol{\imath}} \in [n]^{n-1}$. So now we have

$$p^*_{\langle 0,\ldots,n-2\rangle}, \ \{p_{\bar{\boldsymbol{\imath}}} \mid \bar{\boldsymbol{\imath}} \in [n]^{n-1}, \bar{\boldsymbol{\imath}} \neq \langle 0, \ldots, n-2\rangle\}$$

an n-dimensional independent system over $A\bar{a}_{n-1}$. Its independent amalgam gives the needed type. ⊣

Theorem 2.5.4. *Suppose T is strongly n-simple. Then the following conditions are equivalent:*

(1) for all $1 \leq k \leq n$, T has the strong $(k+1)$-dimensional amalgamation property for Lascar strong types;

(2) for all $1 \leq k \leq n$, every $(k+1)$-element Morley sequence $\{\bar{c}_i \mid i < k+1\}$ over A such that $\mathrm{lstp}(\bar{c}_{\bar{\boldsymbol{\imath}}}/A) = \mathrm{lstp}(\bar{c}_{\bar{\boldsymbol{\jmath}}}/A)$ for $\bar{\boldsymbol{\imath}}, \bar{\boldsymbol{\jmath}} \in [k+1]^k$ can be extended to an infinite Morley sequence over A;

(3) for all $1 \leq k \leq n$, every $(k+1)$-element Morley sequence $\{\bar{c}_i \mid i < k+1\}$ over A such that $\mathrm{lstp}(\bar{c}_{\bar{\boldsymbol{\imath}}}/A) = \mathrm{lstp}(\bar{c}_{\bar{\boldsymbol{\jmath}}}/A)$ for $\bar{\boldsymbol{\imath}}, \bar{\boldsymbol{\jmath}} \in [k+1]^k$ can be extended by one more element. Namely, there is an element $\bar{c}_{k+1}$ such that $\bar{c}_{k+1} \underset{A}{\perp} \{\bar{c}_i \mid i < k+1\}$ and $\mathrm{tp}(\bar{c}_{\bar{\boldsymbol{\imath}}}/A) = \mathrm{tp}(\bar{c}_{\bar{\boldsymbol{\jmath}}}/A)$ for $\bar{\boldsymbol{\imath}}, \bar{\boldsymbol{\jmath}} \in [k+2]^{k+1}$.

Proof. (1) implies (2) was proved in Lemma 2.4.5; (2) implies (3) is obvious.

For the remaining implications, we proceed by induction on n. For $n = 1$, all the three statements are well-known to be true. Now we assume that the statements are equivalent for $k = 1, \ldots, n-1$, it remains to prove the two implications for $k = n$.

(3) $\Rightarrow$ (2). Suppose that for every set A, every Morley sequence $\{\bar{c}_i \mid i < n+1\}$ over A such that $\mathrm{lstp}(\bar{c}_{\bar{\boldsymbol{\imath}}}/A) = \mathrm{lstp}(\bar{c}_{\bar{\boldsymbol{\jmath}}}/A)$ for $\bar{\boldsymbol{\imath}}, \bar{\boldsymbol{\jmath}} \in [n+1]^n$ can be extended by one more element. Take $\{\bar{c}_i \mid i < n+1\}$, we show that it can be extended to an infinite Morley sequence over A. By Ramsey's and compactness theorems it is enough to build an infinite independent over A sequence such that every $(n+1)$-element subsequence of it realizes the type of $\{\bar{c}_0, \ldots, \bar{c}_n\}$ over A.

By induction on $k < \omega$, $k \geq n+1$, construct a sequence $\{\bar{c}_i \mid i < k\}$ such that

($*$) $\{\bar{c}_i \mid i < k\}$ is an independent sequence over A;

($**$) $\mathrm{lstp}(\bar{c}_{\bar{\boldsymbol{\imath}}}/A\bar{c}_0 \ldots \bar{c}_{k-n-2}) = \mathrm{lstp}(\bar{c}_{\bar{\boldsymbol{\jmath}}}/A\bar{c}_0 \ldots \bar{c}_{k-n-2})$ for $\bar{\boldsymbol{\imath}}, \bar{\boldsymbol{\jmath}} \in [k-n-1, \ldots k-1]^n$;

($***$) for all $\bar{\boldsymbol{\imath}} \in [k]^{n+1}$, $\mathrm{tp}(\bar{c}_{\bar{\boldsymbol{\imath}}}/A) = \mathrm{tp}(\bar{c}_0, \ldots, \bar{c}_n/A)$

Suppose we have $\{\bar{c}_i \mid i < k\}$ satisfying ($*$) and ($**$) above; for $k = n+1$ we start with the original sequence.

Then $\{\bar{c}_i \mid k-n-1 \leq i < k\}$ is a Morley sequence over $A\bar{c}_0 \dots \bar{c}_{k-n-2}$ (for $k = n+1$ we take it simply over A). Also the following Lascar strong types are equal: $\mathrm{lstp}(\bar{c}_{\bar{\imath}}/A\bar{c}_0 \dots \bar{c}_{k-n-2}) = \mathrm{lstp}(\bar{c}_{\bar{\jmath}}/A\bar{c}_0 \dots \bar{c}_{k-n-2})$ for $\bar{\imath}, \bar{\jmath} \in [k-n-1, \dots k-1]^n$.

By the assumption, there is an element $\bar{c}_k$ such that

$$\bar{c}_k \underset{A\bar{c}_0 \dots \bar{c}_{k-n-2}}{\perp} \{\bar{c}_i \mid k-n-1 \leq i < k\}$$

and $\mathrm{tp}(\bar{c}_{\bar{\imath}}/A\bar{c}_0 \dots \bar{c}_{k-n-2}) = \mathrm{tp}(\bar{c}_{\bar{\jmath}}/A\bar{c}_0 \dots \bar{c}_{k-n-2})$ for $\bar{\imath}, \bar{\jmath} \in [k-n-1, \dots k-1, k]^{n+1}$.

In particular, $\mathrm{tp}(\bar{c}_k/A\bar{c}_0 \dots \bar{c}_{k-n-2}) = \mathrm{tp}(\bar{c}_{k-n-1}/A\bar{c}_0 \dots \bar{c}_{k-n-2})$, so $\bar{c}_k \underset{A}{\perp} \bar{c}_0 \dots \bar{c}_{k-n-2}$. By transitivity, $\bar{c}_k \underset{A}{\perp} \bar{c}_0 \dots \bar{c}_{k-1}$, so $(*)$ is satisfied for $\{\bar{c}_i \mid i < k+1\}$.

Now we get $(**)$. The equality of types for sequences of length $n+1$

$$\mathrm{tp}(\bar{c}_{\bar{\imath}}/A\bar{c}_0 \dots \bar{c}_{k-n-2}) = \mathrm{tp}(\bar{c}_{\bar{\jmath}}/A\bar{c}_0 \dots \bar{c}_{k-n-2}), \ \bar{\imath}, \bar{\jmath} \in [k-n-1, \dots k-1, k]^{n+1}$$

has the following implications for sequences of length n:

- $\mathrm{tp}(\bar{c}_{\bar{\imath}}/A\bar{c}_0 \dots \bar{c}_{k-n-1}) = \mathrm{tp}(\bar{c}_{\bar{\jmath}}/A\bar{c}_0 \dots \bar{c}_{k-n-1})$ for $\bar{\imath}, \bar{\jmath} \in [k-n, \dots k]^n$;
- $\mathrm{lstp}(\bar{c}_{\bar{\imath}}/A) = \mathrm{lstp}(\bar{c}_{\bar{\jmath}}/A)$ for $\bar{\imath}, \bar{\jmath} \in [k-n, \dots k]^n$.

This easily gives $(**)$ by Lemma 2.3.6.

Finally, for $(***)$ there are two cases to consider:

Case 1. $\bar{\imath}[n] < k$. Then the equality $\mathrm{tp}(\bar{c}_{\bar{\imath}}/A) = \mathrm{tp}(\bar{c}_0, \dots, \bar{c}_n/A)$ holds by induction hypothesis.

Case 2. $\bar{\imath}[n] = k$. We show that there is $\bar{\jmath} \in [k]^{n+1}$ with $\bar{\jmath}[n] < k$ such that $\mathrm{tp}(\bar{c}_{\bar{\imath}}/A) = \mathrm{tp}(\bar{c}_{\bar{\jmath}}/A)$, the rest will follow from Case 1. Let l be the smallest such that $\bar{\imath}[l] > k-n-1$. Define $\bar{\jmath}$ as follows: $\bar{\jmath}[\ell] := \bar{\imath}[\ell]$ for $\ell < l$; and $\bar{\jmath}[l+\ell] := k-n+\ell$. Then $\bar{\jmath}$ is as needed.

This completes the construction of the independent sequence $\{\bar{c}_i \mid i < \omega\}$ such that its every $(n+1)$-element subsequence realizes the type of $\{\bar{c}_0, \dots, \bar{c}_n\}$ over A. By Ramsey's and compactness theorems, the sequence $\{\bar{c}_0, \dots, \bar{c}_n\}$ can be extended to an infinite Morley sequence over A.

(2) $\Rightarrow$ (1). By induction hypothesis, we may assume that T has the amalgamation properties up to the dimension n. Let $\{p_{\bar{\imath}}(\bar{x}, \bar{a}_{\bar{\imath}}) \mid \bar{\imath} \in [n+1]^n\}$ be an independent

$(n+1)$-dimensional system of Lascar strong types over A. (The tuples $\bar{a}_i$ may have different lengths, and realize different types over A.) We want to construct a Morley sequence $\{\bar{c}_i \mid i < n+1\}$ with certain properties such that its extension to an infinite Morley sequence over A together with strong n-simplicity gives the needed strong $(n+1)$-dimensional amalgam.

Claim 2.5.5. *There is a sequence $\{\bar{c}_i \mid i < n\}$ and an element $\bar{b}$ such that*

(1) the length of each $\bar{c}_i$ is $\ell(\bar{a}_0) + \cdots + \ell(\bar{a}_n)$; $\bar{c}_i[j]$ refers to the jth block of $\bar{c}_i$ of length $\ell(\bar{a}_j)$;

(2) $\bar{b}$ realizes

$$\bigcup_{\bar{\imath} \in [n+1]^n} p_{\bar{\imath}}(\bar{x}, \bar{c}_0[\bar{\imath}[0]], \ldots, \bar{c}_{n-1}[\bar{\imath}[n-1]]);$$

(3) $\{\bar{c}_i \mid i < n\}$ is a (finite) Morley sequence over $A\bar{b}$;

(4) $\bar{b} \underset{A}{\perp} \{\bar{c}_i \mid i < n\}$;

(5) every subsequence of $\{\bar{c}_i \mid i < n\}$ with $n-1$ members has the same Lascar strong type over $A\bar{b}$.

Proof of Claim 2.5.5. By Lemma 2.5.2, we obtain $\bar{a}'_n$ such that the system of types $\bar{p}_{\bar{\imath}}$ can be amalgamated over $\bar{a}_0, \ldots, \bar{a}_{n-1}, \bar{a}'_n$. So take the amalgam $\bar{b} \underset{A}{\perp} \bar{a}_0, \ldots, \bar{a}_{n-1}, \bar{a}'_n$.

Let $\bar{c}_0 := \bar{a}_0 \ldots, \bar{a}_{n-1}, \bar{a}'_n$; we use $\bar{c}_0[i]$ to refer to $\bar{a}_i$. Since $\bar{b} \underset{A}{\perp} \bar{c}_0$ and the components of $\bar{c}_0$ are independent, by exchange we have

$$\bar{c}_0[n] \underset{A}{\perp} \bar{b}\bar{c}_0[0] \ldots \bar{c}_0[n-1].$$

Therefore there is a Morley sequence I_n in $\operatorname{tp}(\bar{c}_0[n]/A\bar{b}\bar{c}_0[0] \ldots \bar{c}_0[n-1])$ such that $I_n \underset{A}{\perp} \bar{b}\bar{c}_0[0] \ldots \bar{c}_0[n-1]$. By exchange, we will also have $\bar{b} \underset{A}{\perp} \bar{c}_0[0] \ldots \bar{c}_0[n-1] I_n$.

Proceed by induction to construct sequences I_{n-k}, $k = 0, \ldots, n$ such that

(1) I_{n-k} is a Morley sequence in

$$\operatorname{tp}(\bar{c}_0[n-k]/A\bar{b}\bar{c}_0[0] \ldots \bar{c}_0[n-k-1] I_{n-k+1} \ldots I_n);$$

(2) $I_{n-k} \underset{A}{\smile\kern-0.55em|}\, \bar{b}\bar{c}_0[0]\ldots\bar{c}_0[n-k-1]I_{n-k+1}\ldots I_n$;

(3) $\bar{b} \underset{A}{\smile\kern-0.55em|}\, \bar{c}_0[0]\ldots\bar{c}_0[n-k-1]I_{n-k}\ldots I_n$.

Let $\bar{c}_i[j]$ be the ith member of the sequence I_j. Let $\bar{c}_i := \bar{c}_i[0]\ldots\bar{c}_i[n]$. It is routine to check that $\{\bar{c}_i \mid i < \omega\}$ is an independent sequence over $A\bar{b}$. (Unfortunately, it does not have to be indiscernible, so we have to work.)

By indiscernibility of the sequences I_j, for all $i_0 < \cdots < i_{n-1}$ and every $\bar{\boldsymbol{\imath}} \in [n+1]^n$ we have

$$\mathrm{tp}(\bar{c}_{i_0}[\bar{\boldsymbol{\imath}}[0]],\ldots,\bar{c}_{i_{n-1}}[\bar{\boldsymbol{\imath}}[n-1]]/A\bar{b}) = \mathrm{tp}(\bar{c}_0[0],\ldots,\bar{c}_{n-1}[n-1]/A\bar{b}),$$

so

$$\bar{b} \models \bigcup_{\bar{\boldsymbol{\imath}}\in[n+1]^n} p_{\bar{\boldsymbol{\imath}}}(\bar{x},\bar{c}_{i_0}[\bar{\boldsymbol{\imath}}[0]],\ldots,\bar{c}_{i_{n-1}}[\bar{\boldsymbol{\imath}}[n-1]])$$

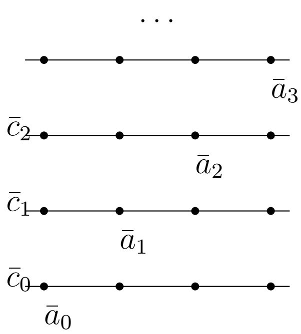
for all $i_0 < \cdots < i_{n-1} < \omega$.

To sum up, we have a independent over $A\bar{b}$ sequence $I := \{\bar{c}_i \mid i < \omega\}$ such that

$$\bar{b} \models \bigcup_{\bar{\boldsymbol{\imath}}\in[n+1]^n} p_{\bar{\boldsymbol{\imath}}}(\bar{x},\bar{c}_{i_0}[\bar{\boldsymbol{\imath}}[0]],\ldots,\bar{c}_{i_{n-1}}[\bar{\boldsymbol{\imath}}[n-1]])$$

for all $i_0 < \cdots < i_{n-1} < \omega$. By Ramsey's and compactness theorems, we may assume that I is indiscernible over $A\bar{b}$. The first n terms of the sequence are as needed. $\dashv$

Our approach to building the sequence $\{\bar{c}_i \mid i < n+1\}$ is this: position the elements $\bar{a}_i$, $i < n+1$ diagonally in the sequence $\bar{c}_i$, $i < n+1$; where the first n elements of sequence $\bar{c}_i$ are "good" in the sense of Claim 2.5.5. Claim 2.5.5 provides the first n elements of the sequence $\bar{c}_i$. The purpose of the construction below is to give the element $\bar{c}_n$.



By construction of the sequence $\{\bar{c}_i \mid i < n\}$, the diagonal elements $\{\bar{c}_i[i] \mid i < n\}$ realize the same type over A as $\{\bar{a}_i \mid i < n\}$. So we may assume that the diagonal elements are the $\bar{a}_i$'s.

Let $q(\bar{x}, \bar{a}_0, \ldots, \bar{a}_{n-1})$ be the type of $\bar{a}_n$ over $A\bar{a}_0, \ldots, \bar{a}_{n-1}$. Let

$$t_{\langle 0,\ldots,n-2\rangle}(\bar{x}, \bar{c}_0, \ldots, \bar{c}_{n-2}) := \operatorname{tp}(\bar{c}_{n-1}[n]/A\bar{c}_0, \ldots, \bar{c}_{n-2})$$

and for $\bar{\boldsymbol{\imath}} \in [n]^{n-1}$ let $t_{\bar{\boldsymbol{\imath}}}(\bar{x}, \bar{c}_{\bar{\boldsymbol{\imath}}})$ be the corresponding translate of $t_{\langle 0,\ldots,n-2\rangle}$. (We use t because $p_{\bar{\boldsymbol{\imath}}}$ is reserved for the system of types we want to amalgamate.) By the conclusions of Lemma 2.5.2, the types q and $t_{\bar{\boldsymbol{\imath}}}$ are a coherent independent system of types. It is clear that the types extend the same Lascar strong type over A. Thus we can apply Lemma 2.5.3, taking the "off-diagonal" members of $\bar{c}_i$ as $\bar{b}_i$ there. By Lemma 2.5.3, there is a common non-forking realization of the types $p_{\bar{\boldsymbol{\imath}}}$ and q, so there is an element $\bar{c}_n[n]$ such that $\operatorname{tp}(\bar{c}_n[n]/A\bar{a}_0, \ldots, \bar{a}_{n-1}) = \operatorname{tp}(\bar{a}_n/A\bar{a}_0, \ldots, \bar{a}_{n-1})$, and there are strong automorphisms $f_{\bar{\boldsymbol{\imath}}}$, $\bar{\boldsymbol{\imath}} \in [n]^{n-1}$, over A such that $f_{\bar{\boldsymbol{\imath}}} : \bar{c}_0, \ldots, \bar{c}_{n-2}\bar{c}_{n-1}[n] \mapsto \bar{c}_{\bar{\boldsymbol{\imath}}[0]}, \ldots, \bar{c}_{\bar{\boldsymbol{\imath}}[n-2]}\bar{c}_n[n]$. For $\bar{\boldsymbol{\imath}} \in [n]^{n-1}$, let

$$q_{\bar{\boldsymbol{\imath}}}(\bar{y}, \bar{c}_{\bar{\boldsymbol{\imath}}}, \bar{c}_n[n]) := f_{\bar{\boldsymbol{\imath}}}(\operatorname{tp}(\bar{c}_{n-1}[0] \ldots \bar{c}_{n-1}[n-1]/A\bar{c}_0, \ldots, \bar{c}_{n-2}\bar{c}_{n-1}[n]).$$

This is an independent system of Lascar strong types over $A\bar{c}_n[n]$, its amalgam gives the first n members of the tuple $\bar{c}_n$.

To sum up, we have an independent sequence $\{\bar{c}_i \mid i < n+1\}$ such that each n-element subsequence realizes the same Lascar strong type over A. The diagonal elements in $\{\bar{c}_i \mid i < n+1\}$ realize the type of $\bar{a}_0, \ldots, \bar{a}_n$ over A. Also, there is an element $\bar{b} \mathop{\smile\!\!\!\!\!\!|}_A \bar{c}_0 \ldots \bar{c}_{n-1}$ such that

$$\bar{b} \models \bigcup_{\bar{\boldsymbol{\imath}} \in [n+1]^n} p_{\bar{\boldsymbol{\imath}}}(\bar{x}, \bar{c}_0[\bar{\boldsymbol{\imath}}[0]], \ldots, \bar{c}_{n-1}[\bar{\boldsymbol{\imath}}[n-1]])$$

and $\{\bar{c}_i \mid i < n\}$ are indiscernible over $A\bar{b}$. Let $r(\bar{x}, \bar{c}_0, \ldots, \bar{c}_{n-1}) := \operatorname{tp}(\bar{b}/A\bar{c}_0 \ldots \bar{c}_{n-1})$.

Let I be a Morley sequence that extends $\{\bar{c}_i \mid i < n+1\}$. By Lemma 2.3.4, and the corollary after it, there is an independent realization $\bar{b}^*$ of

$$\bigcup_{\bar{\boldsymbol{\imath}} \in [n+1]^n} r(\bar{x}, \bar{c}_{\bar{\boldsymbol{\imath}}}).$$

Then

$$\bar{b}^* \models \bigcup_{\bar{\imath}\in[n+1]^n} p_{\bar{\imath}}(\bar{x}, \bar{c}_{\bar{\imath}[0]}[\bar{\imath}[0]], \ldots, \bar{c}_{\bar{\imath}[n-1]}[\bar{\imath}[n-1]]),$$

this gives the needed amalgam. ⊣

Remark 2.5.6. The equivalence (2) $\iff$ (3) and the implication (1) $\Rightarrow$ (2) in Theorem 2.5.4 hold "level-by-level." However, we did use the strong n-amalgamation property in the proof of (2) $\Rightarrow$ (1). It is not clear if (2) $\Rightarrow$ (1) would also hold without the strong n-amalgamation.

2.6 Strong $(n+1)$-amalgamation in strongly n-simple theories

In the previous section we saw that in strongly n-simple theories strong $(n+1)$-dimensional amalgamation property boils down to extending the $(n+1)$-element Morley sequences to infinite ones. In this section, we prove this property for strongly n-simple theories for all $n \geq 2$ under an additional assumption.

For strongly 2-simple theories, we prove the strong 3-dimensional amalgamation without any extra assumptions.

Definition 2.6.1. Let $\{\bar{a}_i \mid i < n\}$ be a sequence of tuples. We say that $L_n(\bar{a}_0, \ldots, \bar{a}_{n-1}/A)$ holds if there is an element $\bar{a}^* \underset{A}{\perp} \{\bar{a}_i \mid i < n\}$ such that all the n-subsequences of $\{\bar{a}_0, \bar{a}^*, \bar{a}_1, \ldots, \bar{a}_{n-1}\}$ that contain $\bar{a}^*$ realize the same type over A and the sequence $\{\bar{a}^*, \bar{a}_1, \ldots, \bar{a}_{n-1}\}$ can be extended to an infinite Morley sequence over A.

To visualize the property given in the above definition for $n = 3$, picture the elements $\{\bar{a}_0, \bar{a}_1, \bar{a}_2\}$ as the base of a tetrahedron. The property L_3 would hold if there is an element $\bar{a}^*$ (top of the tetrahedron) such that the side faces are contained in infinite Morley sequences. In particular, L_3 would hold if the three elements already belong to a Morley sequence over A.

For simple theories, the property L_2 over A is simply the equality of Lascar strong types over A. Our goal is to find a suitable n-ary generalization. The extra assumption (Assumption $\mathbf{L_n}$ below) that we are making in the proof of $n+1$-dimensional amalgamation property essentially states that the property L_n is determined by the equality of certain Lascar strong types.

Assumption 2.6.2. Let $\{\bar{a}_i \mid i < n\}$ be a finite Morley sequence over A.

$\mathbf{L_n}$ is the following assumption:

> For any such sequence, if all increasing $(n-1)$-element subsequences realize the same Lascar strong type over A, then $L_n(\bar{a}_0, \ldots, \bar{a}_{n-1}/A)$ holds.

Clearly, if $\{\bar{a}_0, \ldots, \bar{a}_{n-1}\}$ belong to an infinite Morley sequence over A, then $L_n(\bar{a}_0, \ldots, \bar{a}_{n-1}/A)$ holds. So certainly $\mathbf{L_n}$ follows from strong n-dimensional amalgamation. However, it is not clear if it follows from strong n-simplicity for $n > 2$.

Intuitively, the type of an n-sequence that can be extended to an infinite indiscernible sequence is "good." So the assumption $\mathbf{L_n}$ says that even if $\{\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{n-1}\}$ is "bad," there exists an n-dimensional tetrahedron with the bad base and upper vertex $\bar{a}^*$ all the sides of which are good.

Theorem 2.6.3. *Let $n \geq 2$, let T be strongly n-simple and let the properties $\mathbf{L_{k+1}}$, $2 \leq k \leq n$ hold. Suppose that $\{\bar{a}_i \mid i < n+1\}$ is a Morley sequence over A such that every n-subsequence realizes the same Lascar strong type over A. Then the sequence can be extended by one more element. Namely, there is an element $\bar{a}_{k+1}$ such that $\bar{a}_{k+1} \underset{A}{\perp} \{\bar{a}_i \mid i < k+1\}$ and $\operatorname{tp}(\bar{a}_{\bar{\imath}}/A) = \operatorname{tp}(\bar{a}_{\bar{\jmath}}/A)$ for $\bar{\imath}, \bar{\jmath} \in [k+2]^{k+1}$.*

Proof. Suppose for contradiction that there is a Morley sequence $\{\bar{a}_i \mid i < n+1\}$ over A such that every n-subsequence realizes the same Lascar strong type over A; and that this sequence cannot be extended by one more element. By the property $\mathbf{L_{n+1}}$, there is an element $\bar{a}^*$ such that all the $(n+1)$-subsequences of $\{\bar{a}_0, \bar{a}^*, \bar{a}_1, \ldots, \bar{a}_n\}$ that contain $\bar{a}^*$ realize the same type over A and the sequence $\{\bar{a}^*, \bar{a}_1, \ldots, \bar{a}_n\}$ can be extended to an infinite Morley sequence over A. Let

$$q_b(\bar{x}_0, \ldots, \bar{x}_n) := \operatorname{tp}(\bar{a}_0, \ldots, \bar{a}_n/A) \text{ and } q_g(\bar{x}_0, \ldots, \bar{x}_n) := \operatorname{tp}(\bar{a}^*, \bar{a}_1, \ldots, \bar{a}_n/A),$$

we think of them as of bad and good types respectively.

Let I be an infinite Morley sequence over A that extends $\{\bar{a}^*, \bar{a}_1, \dots, \bar{a}_n\}$. Since $\{\bar{a}_1, \dots, \bar{a}_n\}$ are indiscernible over $A\bar{a}_0\bar{a}^*$ and $\bar{a}_1, \dots, \bar{a}_n \underset{A}{\perp} \bar{a}_0\bar{a}^*$, by Lemma 2.3.4 we may assume that $\{\bar{a}_1 \dots, \bar{a}_k, \dots\}$ is indiscernible over $A\bar{a}_0\bar{a}^*$. In particular, we have that for all $i_1 < \dots < i_n$ we have $\bar{a}_0\bar{a}_{i_1} \dots \bar{a}_{i_n} \models q_b$ and $\bar{a}_0\bar{a}^*\bar{a}_{i_1} \dots \bar{a}_{i_{n-1}} \models q_g$.

Let $f_i \in \mathrm{Aut}_A(\mathfrak{C})$, $i = 1, \dots,$ be such that $f_i : \bar{a}^*, \bar{a}_1, \dots \mapsto \bar{a}_i, \bar{a}_{i+1}, \dots$. By invariance, $f_i(\bar{a}_0) \underset{A}{\perp} \bar{a}_i, \bar{a}_{i+1}, \dots$ and by extension we may assume that $f_i(\bar{a}_0) \underset{A}{\perp} I\{f_j(\bar{a}_0) \mid j < i\}$

Let $\bar{c}_0 := \bar{a}_0\bar{a}^*$, $\bar{c}_i := f_i(\bar{a}_0)\bar{a}_i$. Then the sequence $\{\bar{c}_i \mid i < \omega\}$ is independent over A and for all $i_0 < \dots < i_n$ we have

$$\bar{c}_{i_0}[0]\bar{c}_{i_1}[1] \dots \bar{c}_{i_n}[1] \models q_b \text{ and } \bar{c}_{i_0}[0]\bar{c}_{i_0}[1]\bar{c}_{i_1}[1] \dots \bar{c}_{i_{n-1}}[1] \models q_g.$$

So by Ramsey's theorem and the compactness theorem we may assume that the sequence $\{\bar{c}_i \mid i < \omega\}$ is a Morley sequence over A satisfying the above property.

Our next step is to show that the types $\{q_b(\bar{x}, \bar{b}_{\bar{\imath}}) \mid \bar{\imath} \in [n+1]^n\}$ can be amalgamated over any $\{\bar{b}_0, \dots, \bar{b}_n\}$ that realize the type q_g over A. Indeed, take $\bar{b}^* \models q_b(\bar{x}, \bar{b}_0, \dots, \bar{b}_{n-1})$. By the definition of q_b, $b^* \underset{A}{\perp} \bar{b}_0, \dots, \bar{b}_{n-1}$, the sequence $\{\bar{b}_0, \dots, \bar{b}_{n-1}\}$ is indiscernible over $A\bar{b}^*$, and by definition of q_g, there is an infinite Morley sequence over A that extends $\{\bar{b}_0, \dots, \bar{b}_n\}$. By Corollary 2.3.5, there is an independent $\bar{b}' \models \bigcup_{\bar{\imath} \in [n+1]^n} q_b(\bar{x}, \bar{b}_{\bar{\imath}})$. This is what we need.

Since $\bar{c}_0[0]\bar{c}_0[1]\bar{c}_1[1] \dots \bar{c}_{n-1}[1] \models q_g$, there is an independent realization $\bar{b}$ of the system of bad types over $\bar{c}_0[0]\bar{c}_0[1]\bar{c}_1[1] \dots \bar{c}_{n-1}[1]$.

Let $q^*(\bar{x}, \bar{c}_0[0], \bar{c}_0[1], \bar{c}_1[1], \dots, \bar{c}_{n-1}[1])$ be the type of $\bar{b}$ over the set $A\bar{c}_0[0]\bar{c}_0[1]\bar{c}_1[1] \dots \bar{c}_{n-1}[1]$, let $t(\bar{x}, \bar{c}_0, \dots, \bar{c}_{n-2})$ be the type of $\bar{b}$ over $A\bar{c}_0 \dots \bar{c}_{n-2}$, and for $\bar{\imath} \in [n-1]^{n-2}$ let $t(\bar{x}, \bar{c}_{\bar{\imath}})$ be the translate of t over A.

The types $t(\bar{x}, \bar{c}_{\bar{\imath}})$, $\bar{\imath} \in [n]^{n-1}$, and $q^*(\bar{x}, \bar{c}_0, \dots, \bar{c}_{n-1})$ form a coherent system, so we can apply Lemma 2.5.3 (using the n-dimensional amalgamation property). By Lemma 2.5.3, there is $\bar{b}^*$ an independent realization of the type q^* such that $\{\bar{c}_i \mid i < n\}$ are indiscernible over $A\bar{b}^*$. By Corollary 2.3.5, there is an independent realization of the union

$$\bigcup_{\bar{\imath}\in[n+1]^n} q^*(\bar{x}, \bar{c}_{\bar{\imath}}).$$

In particular, we get an independent $\bar{b}$ such that $\bigcup_{\bar{\imath}\in[n+1]^n} q_b(\bar{x}, \bar{a}_{\bar{\imath}})$. Thus, we have an extension of the finite Morley sequence by one more element. ⊣

From Theorem 2.6.3 and Theorem 2.5.4 we get:

Corollary 2.6.4. *Let $n \geq 2$, let T be strongly n-simple and let the properties $\mathbf{L_{k+1}}$, $2 \leq k \leq n$ hold. Then T has the strong $(k+1)$-dimensional amalgamation property for all $1 \leq k \leq n$.*

Now we get the strong 3-dimensional amalgamation property without the extra assumption.

Theorem 2.6.5. *Let T be strongly 2-simple. Given $\{\bar{a}_i \mid i < 3\}$, suppose that $\{p_{\bar{\imath}}(\bar{x}, \bar{a}_{\bar{\imath}}) \mid \bar{\imath} \in [3]^2\}$ form an independent 3-dimensional system of Lascar strong types over A. Then the union $\bigcup\{p_{\bar{\imath}}(\bar{x}, \bar{a}_{\bar{\imath}}) \mid \bar{\imath} \in [3]^2\}$ is consistent.*

Proof. We describe the initial construction for any $n \geq 2$, and only at the end of the proof we use the assumption $n = 2$ in an essential way.

Claim 2.6.6. *Let $l_k := \ell(\bar{a}_k)$, $l^* := \sum_{k<n} l_k$. There is a Morley sequence $\{\bar{c}_i \mid i < \omega\}$ over A such that*

(1) for all i, $\bar{c}_i = \bar{c}_i[0] \ldots \bar{c}_i[n]$ and $\ell(c_i[k]) = l_k$ for all $k < n+1$;

(2) $\bar{c}_i[i] = \bar{a}_i$ for $i < n+1$ (i.e., $\bar{a}_i$'s are positioned diagonally in the first $n+1$ elements of the sequence $\bar{c}_i$).

Proof. The construction is similar to the one in Claim 2.5.5. ⊣

Our goal is to realize the types $p_{\bar{\imath}}$ over the $n+1$ diagonal elements of the sequence $\bar{c}_i$ with the aid of non-n-dividing. So first we need to "project" this system of types onto the first n elements of the sequence. By the *projected system of types* we mean

$\{p_{\bar{\imath}}(\bar{x}, \bar{c}_0[\bar{\imath}[0]], \ldots, \bar{c}_{n-1}[\bar{\imath}[n-1]]) \mid \bar{\imath} \in [n+1]^n\}$. (Compare this to the construction in the proof of (2) $\Rightarrow$ (1) in Theorem 2.5.4.)

The reason we are interested in such a system is that finding a solution to it guarantees (with n-simplicity) a solution to the original system of types:

Claim 2.6.7. *Suppose that $\bar{d}$ is a non-forking realization of the system of projected system of types such that $\bar{c}_0, \ldots, \bar{c}_{n-1}$ are indiscernible over $A\bar{d}$. Then there is a $\bar{d}'$ that realizes the original system of types $\{p_{\bar{\imath}}(\bar{x}, \bar{a}_{\bar{\imath}}) \mid \bar{\imath} \in [n+1]^n\}$.*

Proof. By n-simplicity, the type of $\bar{d}$ over $A\bar{c}_0 \ldots \bar{c}_{n-1}$ (denote it by $q(\bar{x}, \bar{c}_0, \ldots, \bar{c}_{n-1})$) does not n-divide over A. So in particular the type $\bigcup_{\bar{\imath} \in [n+1]^n} q(\bar{x}, \bar{c}_{\bar{\imath}})$ is consistent. By construction of q it is clear that $q(\bar{x}, \bar{c}_{\bar{\imath}}) \vdash p_{\bar{\imath}}(\bar{x}, \bar{a}_{\bar{\imath}})$, so there is $\bar{d}' \models \bigcup_{\bar{\imath} \in [n+1]^n} p_{\bar{\imath}}(\bar{x}, \bar{a}_{\bar{\imath}})$. ⊣

So it remains to show that the projected system of types does have a solution as required by the Claim 2.6.7. Amalgamating the projected system of types itself is easy for any n. It is the extra indiscernibility requirement that makes things difficult.

For the remainder of the proof $n = 2$, we have a Morley sequence $\{\bar{c}_i \mid i < \omega\}$ over A such that $\bar{c}_i[i] = \bar{a}_i$ for $i < 3$. The projected system of types in this case is

$$p_{01}(\bar{x}, \bar{c}_0[0], \bar{c}_1[1]), \quad p_{02}(\bar{x}, \bar{c}_0[0], \bar{c}_1[2]), \quad p_{12}(\bar{x}, \bar{c}_0[1], \bar{c}_1[2]).$$

By 2-amalgamation and 2-simplicity (in particular using Corollary 2.3.6) we can find the independent amalgam of the types $p_{01}(\bar{x}, \bar{c}_0[0], \bar{c}_1[1])$ and $p_{02}(\bar{x}, \bar{c}_0[0], \bar{c}_1[2])$ over $A\bar{c}_0[0]$ and then the independent amalgam of $p_{01}(\bar{x}, \bar{c}_0[0], \bar{c}_1[1]) \cup p_{02}(\bar{x}, \bar{c}_0[0], \bar{c}_1[2])$ with $p_{12}(\bar{x}, \bar{c}_0[1], \bar{c}_1[2])$ over $A\bar{c}_1[2]$. Let

$$\bar{d}_0 \models \bigcup_{\bar{\imath} \in [3]^2} p_{\bar{\imath}}(\bar{x}, \bar{c}_0[\bar{\imath}[0]], \bar{c}_1[\bar{\imath}[1]]).$$

Let $q_1(\bar{x}, \bar{c}_0[0], \bar{c}_0[1])$ be the type of $\bar{d}_0$ over $A\bar{c}_0[0], \bar{c}_0[1]$; let $q_2(\bar{x}, \bar{c}_0[1], \bar{c}_0[2])$ be the translate of the type of $\bar{d}_0$ over $A\bar{c}_1[1], \bar{c}_1[2]$ to $A\bar{c}_0[1], \bar{c}_0[2]$ over A. By construction of $\bar{d}_0$ and the definition of the independent system of types, the types q_1 and q_2 agree over $A\bar{c}_0[1]$. So there is $q(\bar{x}, \bar{c}_0)$, an independent amalgam of $q_1(\bar{x}, \bar{c}_0[0], \bar{c}_0[1])$ and $q_2(\bar{x}, \bar{c}_0[1], \bar{c}_0[2])$ over $A\bar{c}_0[1]$. Finally, let

$$\bar{d} \models \mathrm{tp}(\bar{d}_0/A\{\bar{c}_0[0], \bar{c}_0[1], \bar{c}_1[1], \bar{c}_1[2]\}) \cup q(\bar{x}, \bar{c}_0) \cup q(\bar{x}, \bar{c}_1).$$

It is clear that $\bar{d}$ is as needed. ⊣

Corollary 2.6.8. *Let T be a strongly 2-simple theory. Then*

(1) T has (strong) 3-dimensional amalgamation property for Lascar strong types;

(2) In the monster model of T, every 3-element Morley sequence over A such that every 2-subsequence realizes the same Lascar strong type over A can be extended to an infinite Morley sequence;

(3) T has the property $\mathbf{L_3}$.

Question 2.6.9. *Does strong n-simplicity imply the strong $(n+1)$-dimensional amalgamation property? Alternatively, does the property $\mathbf{L_{n+1}}$ hold for strongly n-simple theories?*

Chapter 3

n-simplicity

Introduction

The purpose of this introduction is to describe why a better definitions of n-simplicity (and of n-dimensional amalgamation) are needed. Let me start with two examples where 3-dimensional amalgamation property, as defined in the previous chapter, fails.

An example of a smoothly approximable SU-rank 1 structure in which a certain triple of types cannot be amalgamated was known in the nineties to Bradd Hart, Ambar Chowdhury, Byunghan Kim and others. I would like to thank Bradd Hart for communicating it; however, any mistakes in its presentation are mine.

Example 1. Let V be an infinite vector space over a finite field equipped with a non-degenerate symplectic bilinear form $\langle \cdot, \cdot \rangle$. Let A be the affine part: namely, let A be a set on which V acts regularly and transitively.

Let $c_0, c_1, c_2 \in A$ be distinct and such that $\langle c_2 - c_0, c_1 - c_0 \rangle = 0$. The claim is that the following system of types is inconsistent:

$$p_{01}(x) := \{\langle x, c_1 - c_0 \rangle = 1\}, \quad p_{02}(x) := \{\langle x, c_2 - c_0 \rangle = 1\},$$
$$p_{12}(x) := \{\langle x, c_2 - c_1 \rangle = 1\}.$$

Indeed, whenever x satisfies both $\langle x, c_1 - c_0 \rangle = 1$ and $\langle x, c_2 - c_0 \rangle = 1$ by bilinearity we get $\langle x, c_2 - c_1 \rangle = 0$, contradicting $p_{12}(x)$.

The system of types is clearly coherent, and the required independence conditions hold. What happens in this example is: every vector $v \in V$ defines a finite equivalence relation on A: $E_v(x,y) \iff \langle v, x-y\rangle = 0$. If the field has two elements, the system of types says precisely that c_0 is not x-equivalent to c_1, c_1 is not x-equivalent to c_2, and c_2 not x-equivalent to c_0. But there are only 2 x-equivalence classes!

It can be easily seen that the complete first order theory describing this structure is not strongly 2-simple. Indeed, we have if $v \in V$, and $a, b \in A$ such that $\langle v, a-b\rangle \neq 0$, then from $\mathrm{lstp}(av) = \mathrm{lstp}(bv)$ it does not follow that $\mathrm{lstp}(a/v) = \mathrm{lstp}(b/v)$ (and this cannot happen under strong 2-simplicity).

The strange part about this example is that the 3-amalgamation does not hold for Lascar strong types, but does over models: in fact, if one takes the base of amalgamation to be a model, then there is an A-point in the base. Therefore, when one allows elements of A as parameters, the entire structure will look like two copies of the vector space V.

The idea for the following example was given by Frank Wagner:

Example 2. Let M be a random graph with the edge relation R. Consider $a_0, a_1 \in M$ and $b = \{c, d\} \in M^{eq}$ such that

$$R(a_0, c) \wedge \neg R(a_0, d) \wedge R(a_1, d) \wedge \neg R(a_1, c).$$

Let $p(x, a_0, a_1) := \mathrm{tp}(b/a_0 a_1)$. Let a_2 be another point in M, and consider the system $\{p(x, a_0, a_1), p(x, a_1, a_2), p(x, a_0, a_2)\}$. The types in the system are pairwise consistent (the type will say that a_i, a_j are related to exactly one element of the pair x and that element is different for a_i and a_j), and all the needed independence holds. But again the union is inconsistent, since modulo p we have an x-equivalence relation with two classes, and the 3-system says that there are 3 distinct equivalence classes.

Therefore, for the theory of random graph T_{rg}, the theory T_{rg}^{eq} does not have 3-amalgamation in the sense of last chapter.

It also can easily be seen that T_{rg}^{eq} is not strongly 2-simple (but the “home sort” is strongly ω-simple).

In this chapter, we develop a definition of n-simplicity under which T_{rg}^{eq} remains ω-simple. In fact, we develop the notion of n-simplicity in $\mathfrak{C}^{heq}$. We also present an appropriate definition of n-dimensional amalgamation.

3.1 Preliminary definitions

Let T be a simple complete first order theory. In this Chapter, we work in the structure $\mathfrak{C}^{heq}$. The construction and properties of $\mathfrak{C}^{heq}$ are fully described in [33]. We recall some of the key definitions here.

Definition 3.1.1. (1) If $\bar{x}$ and $\bar{y}$ are tuples of length $\alpha \in \mathrm{On}$, we say that an equivalence relation $E(\bar{x}, \bar{y})$ on $\mathfrak{C}^{\alpha}$ is *type-definable over* A if there is a partial type $p(\bar{x}, \bar{y})$ over A such that $E(\bar{x}, \bar{y}) \iff \mathfrak{C} \models p(\bar{x}, \bar{y})$.

(2) A type-definable equivalence relation E is called *countable* if both α and the partial type p defining E are countable.

(3) Let $E(\bar{x}, \bar{y})$ be a type-definable equivalence relation on $\mathfrak{C}^{\alpha}$. If $\bar{a} \in \mathfrak{C}^{\alpha}$, we denote the equivalence class of $\bar{a}$ modulo E by $\bar{a}_E$. The equivalence class $\bar{a}_E$ is called a *hyperimaginary element of type* E.

(4) A hyperimaginary $\bar{a}_E$ is *countable* if the type definable-equivalence relation E is countable.

The structure $\mathfrak{C}^{heq}$ is the model $\mathfrak{C}$ with the collection of all countable hyperimaginaries modulo type-definable equivalence relations over $\emptyset$. Just as with $\mathfrak{C}^{eq}$, it is easy to see that $\mathfrak{C}^{heq}$ contains all the countable hyperimaginaries over $A \subset \mathfrak{C}$. It is shown in [33] that any type-definable equivalence relation can be described in terms of countable ones. So it is enough to consider only countable hyperimaginaries.

We now describe the closure operators on $\mathfrak{C}^{heq}$.

Definition 3.1.2. (1) The hyperimaginary *definable closure* of A, $\mathrm{dcl}(A)$, is the collection of all countable hyperimaginaries fixed under all automorphisms in $\mathrm{Aut}_A(\mathfrak{C})$.

(2) The hyperimaginary *algebraic closure* of A, $\operatorname{acl}(A)$, is the collection of all countable hyperimaginaries that have finitely many images under automorphisms in $\operatorname{Aut}_A(\mathfrak{C})$.

(3) The hyperimaginary *bounded closure* of A, $\operatorname{bdd}(A)$, is the collection of all countable hyperimaginaries that have boundedly many (i.e., less than $|\mathfrak{C}|$ many) images under all automorphisms in $\operatorname{Aut}_A(\mathfrak{C})$.

Fact 3.1.3 ([33]). *For every A, $A \subseteq \operatorname{dcl}(A) \subseteq \operatorname{acl}(A) \subseteq \operatorname{bdd}(A)$.*

Another important fact involves characterization of Lascar strong types over A in terms of types over the bounded closure of A.

Fact 3.1.4 ([33]). *For every A, $\bar{a}$, $\bar{b}$, $\operatorname{lstp}(\bar{a}/A) = \operatorname{lstp}(\bar{b}/A)$ if and only if $\operatorname{tp}(\bar{a}/\operatorname{bdd}(A)) = \operatorname{tp}(\bar{b}/\operatorname{bdd}(A))$.*

We finish by presenting several facts that are probably well-known, but for which we have not found a convenient reference.

Proposition 3.1.5. *Let T be simple, work in $\mathfrak{C}^{heq}$.*

(1) If $\bar{a} \underset{A}{\perp} \bar{b}$, then $\operatorname{bdd}(A\bar{a}) \underset{A}{\perp} \operatorname{bdd}(A\bar{b})$.

(2) If $I = \{\bar{a}_i \mid i < \omega\}$ is a Morley sequence over A, then there is a Morley sequence $I' = \{\bar{a}'_i \mid i < \omega\}$ with $\operatorname{tp}(I/A) = \operatorname{tp}(I'/A)$ such that the sequence $\{\operatorname{bdd}(A\bar{\imath}) \mid i < \omega\}$ is a Morley sequence over A.

(3) Even stronger: If $I = \{\bar{a}_i \mid i < \omega\}$ is a Morley sequence over A, then there is a Morley sequence $I' = \{\bar{a}'_i \mid i < \omega\}$ with $\operatorname{tp}(I/A) = \operatorname{tp}(I'/A)$ such that for any $n < \omega$ and $i_0 < \cdots < i_{n-1} < \omega$ the type $\operatorname{tp}(\operatorname{bdd}(A\bar{a}_{i_0} \ldots \bar{a}_{i_{n-1}}))$ is fixed.

(4) Suppose $\bar{a} \underset{\bar{c}}{\perp} \bar{b}$. Then $\operatorname{bdd}(\bar{a}\bar{c}) \cap \operatorname{bdd}(\bar{b}\bar{c}) = \operatorname{bdd}(\bar{c})$.

Proof. (1) By symmetry, it is enough to prove that $\bar{a} \underset{A}{\perp} \operatorname{bdd}(A\bar{b})$. By transitivity, we just need to prove $\bar{a} \underset{A\bar{b}}{\perp} \operatorname{bdd}(A\bar{b})$. But this is obvious, since for any $\bar{c} \in \operatorname{bdd}(A\bar{b})$ there is no infinite $A\bar{b}$-indiscernible sequence starting with $\bar{c}$.

(2) By (1), the sequence $\{\mathrm{bdd}(A\bar{a}_i) \mid i < \omega\}$ is A-independent. Extending I to a long indiscernible sequence with the same type diagram over A, we apply Morley's method to find a sequence such that $\mathrm{bdd}(A\bar{a}_i)$ are both A-independent and A-indiscernible.

(3) Essentially the same argument as in (2): we use Morley's method, but control not only $\mathrm{tp}(\mathrm{bdd}(A\bar{a}_{i_0}) \ldots \mathrm{bdd}(A\bar{a}_{i_{n-1}}))$, but the type of the entire $\mathrm{bdd}(A\bar{a}_{i_0} \ldots \bar{a}_{i_{n-1}})$.

(4) By (1), we get $\mathrm{bdd}(\bar{a}\bar{c}) \underset{\mathrm{bdd}(\bar{c})}{\perp} \mathrm{bdd}(\bar{b}\bar{c})$. This implies the statement.

$\dashv$

3.2 n-simplicity and $n+1$-dimensional amalgamation

We define n-simplicity by a property similar to the one in Section 2.3. The difference is that the "indiscernibility" assumptions are more restrictive. We are working in the structure $\mathfrak{C}^{heq}$, where $\mathfrak{C} \models T$.

Definition 3.2.1. Given $n \geq 1$, we say that the theory T is *n-simple* if for any $1 \leq k \leq n$, Morley sequence $I = \{a_i \mid i < \omega\}$ over A, and a partial type $p(\bar{x}, \bar{a}_0, \ldots, \bar{a}_{k-1})$, with parameters from $\mathrm{bdd}(A\bar{a}_0 \ldots \bar{a}_{k-1})$, such that

(1) there is $\bar{b} \models p$ such that $\bar{b} = \mathrm{bdd}(A\bar{b})$;

(2) for all $\bar{\boldsymbol{\imath}}, \bar{\boldsymbol{\jmath}} \in [k]^{k-1}$ we have $\mathrm{tp}(\mathrm{bdd}(A\bar{a}_{\bar{\boldsymbol{\imath}}})/\bar{b}) = \mathrm{tp}(\mathrm{bdd}(A\bar{a}_{\bar{\boldsymbol{\jmath}}})/\bar{b})$;

(3) $\bar{b} \underset{A}{\perp} \bar{a}_0 \ldots \bar{a}_{k-1}$

we have the union

$$\bigcup_{\bar{\boldsymbol{\imath}} \in [\omega]^k} p(\bar{x}, \bar{a}_{\bar{\boldsymbol{\imath}}})$$

is consistent.

Lemma 3.2.2. *Let $I = \{a_i \mid i < \omega\}$ be a Morley sequence over A and $p(\bar{x}, \bar{a}_0, \ldots, \bar{a}_{k-1})$ be such that the union*

$$\bigcup_{\bar{\imath} \in [\omega]^k} p(\bar{x}, \bar{a}_{\bar{\imath}})$$

is consistent. Then the union does not fork over A.

Proof. Suppose the union does fork over A. By finite character, there is $N < \omega$ such that

$$\bigcup_{\bar{\imath} \in [N]^k} p(\bar{x}, \bar{a}_{\bar{\imath}})$$

forks over A. Define the sequence $\bar{c}_m := \bar{a}_{mN} \ldots \bar{a}_{mN+N-1}$ of "N-blocks" from the sequence I. It has to be a Morley sequence over A, and

$$q(\bar{x}, \bar{c}_0) := \bigcup_{\bar{\imath} \in [N]^k} p(\bar{x}, \bar{a}_{\bar{\imath}})$$

forks over A by our assumption. On the other hand,

$$\bigcup_{\bar{\imath} \in [\omega]^k} p(\bar{x}, \bar{a}_{\bar{\imath}}) \vdash \bigcup_{m<\omega} \bigcup_{\bar{\imath} \in [mN, mN+N-1]^k} p(\bar{x}, \bar{a}_{\bar{\imath}}) = \bigcup_{m<\omega} q(\bar{x}, \bar{c}_m).$$

so the union $\bigcup_{m<\omega} q(\bar{x}, \bar{c}_m)$ is consistent. Contradiction by Kim's characterization of forking and dividing. ⊣

Similar to Section 2.3, Lemma 2.3.4, by Ramsey's theorem and compactness we obtain

Lemma 3.2.3. *Let T be n-simple. Let $I := \{\bar{a}_i \mid i < \omega\}$ be a Morley sequence over A and $\bar{b} \mathop{\smile}\limits_{A} \bar{a}_0 \ldots \bar{a}_{n-1}$. Suppose that $\bar{b} = \operatorname{bdd}(A\bar{b})$ and for every $\bar{\imath} \in [n]^{n-1}$ the type $\operatorname{tp}(\operatorname{bdd}(A\bar{a}_{\bar{\imath}})/\bar{b})$ is fixed;*

Then (1) there is a sequence I' containing $\bar{a}_0, \ldots, \bar{a}_{n-1}$ that is indiscernible over $A\bar{b}$.

(2) there is $\bar{b}' \models \operatorname{tp}(\bar{b}/\operatorname{bdd}(A\bar{a}_0, \ldots, \bar{a}_{n-1}))$ such that I is $A\bar{b}'$-indiscernible.

Remark 3.2.4. Examples 1 and 2 in the introduction to this Chapter show why assumption (1) in the definition of n-simplicity is necessary. In view of Lemma 3.2.3, Proposition 3.1.5(3) shows that the strong indiscernibility assumption (2) above is necessary as well.

Now we deal with the definition of generalized amalgamation properties. Our approach here is more direct than in Chapter 2 in that we explicitly specify the domains of types involved.

Definition 3.2.5. Fix $n < \omega$. Let $\bar{a}_0, \ldots, \bar{a}_{n-1}$ be independent over A (not necessarily having the same type). A system of types $\{p_{\bar{\imath}}(\bar{x}) \mid \bar{\imath} \in [n]^{n-1}\}$ is called a *strong n-dimensional independent system of types over A* if

(1) $\mathrm{dom}(p_{\bar{\imath}}) = A\bar{a}_{\bar{\imath}}$;

(2) there are $\bar{b}_{\bar{\imath}} \models p_{\bar{\imath}}(\bar{x})$ such that $\bar{b}_{\bar{\imath}} = \mathrm{bdd}(A\bar{b}_{\bar{\imath}})$; $\bar{b}_{\bar{\imath}} \underset{A}{\perp} \bar{a}_{\bar{\imath}}$; and $\mathrm{tp}(\bar{b}_{\bar{\imath}}/\,\mathrm{bdd}(\bar{a}_{\bar{\imath}\cap\bar{\jmath}}A)) = \mathrm{tp}(\bar{b}_{\bar{\jmath}}/\,\mathrm{bdd}(\bar{a}_{\bar{\imath}\cap\bar{\jmath}}A))$ for all $\bar{\imath}, \bar{\jmath} \in [n]^{n-1}$.

Definition 3.2.6. We say that a strong n-dimensional independent system of types over A *can be independently amalgamated* if there is $\bar{b}^* \models \bigcup_{\bar{\imath}\in[n]^{n-1}} \mathrm{tp}(\bar{b}_{\bar{\imath}}/\,\mathrm{bdd}(A\bar{a}_{\bar{\imath}}))$ such that $\bar{b}^* \underset{A}{\perp} \bar{a}_0, \ldots, \bar{a}_{n-1}$.

As in the strong simplicity case, we obtain

Proposition 3.2.7. *Fix $n \geq 2$ and $N \geq n$. Let $\{p_{\bar{\imath}}(\bar{x}) \mid \bar{\imath} \in [N]^{n-1}\}$ be an independent system of Lascar strong types over A. Suppose T has the strong n-dimensional amalgamation property for Lascar strong types. Then the system $\{p_{\bar{\imath}}(\bar{x}) \mid \bar{\imath} \in [N]^{n-1}\}$ can be independently amalgamated.*

Proof. The base $N = n$ is given by the strong n-dimensional amalgamation property. Suppose the statement is true for some $N \geq n$, and fix an independent system $\{p_{\bar{\imath}}(\bar{x}) \mid \bar{\imath} \in [N+1]^{n-1}\}$. Consider the following n types:

– for every $s \subset \{N-(n-2), \dots, N\}$ of size $n-2$, we take the amalgam $q_s(\bar{x})$ of

$$\{p_{\bar{\boldsymbol{\imath}}}(\bar{x}) \mid \bar{\boldsymbol{\imath}} \in [\underbrace{\{0, \dots, N-(n-1)\}}_{N-n+2 \text{ elements}} \cup s]^{n-1}\}.$$

It exists by the induction hypothesis.

– the type $p_{\{N-(n-2),\dots,N\}}(\bar{x})$.

It is easy to see that these n types form a strong n-dimensional independent system over A, so there is an independent amalgam $q(\bar{x})$. It remains to observe q is a non-forking extension of each $p_{\bar{\boldsymbol{\imath}}}$, $\bar{\boldsymbol{\imath}} \in [N+1]^{n-1}$. ⊣

A more refined definition of n-simplicity allows us to bridge the gap between $(n+1)$-dimensional amalgamation and n-simplicity at least in one direction:

Theorem 3.2.8. *Suppose T is simple and has k-dimensional amalgamation properties for all $2 \leq k \leq n+1$. Then T is n-simple.*

Proof. Using induction, it suffices to prove that if T has $(n+1)$-dimensional amalgamation property, then for any A, Morley sequence $I = \{a_i \mid i < \omega\}$ over A, and a partial type $p(\bar{x}, \bar{a}_0, \dots, \bar{a}_{k-1})$, with parameters from $\operatorname{bdd}(A\bar{a}_0 \dots \bar{a}_{k-1})$, such that

(1) there is $\bar{b} \models p$ such that $\bar{b} = \operatorname{bdd}(A\bar{b})$;

(2) for every $\bar{\boldsymbol{\imath}} \in [k]^{k-1}$ we have $\operatorname{tp}(\operatorname{bdd}(A\bar{a}_{\bar{\boldsymbol{\imath}}})/\bar{b})$ is fixed;

(3) $\bar{b} \underset{A}{\perp} \bar{a}_0 \dots \bar{a}_{k-1}$

we have the union

$$\bigcup_{\bar{\boldsymbol{\imath}} \in [\omega]^k} p(\bar{x}, \bar{a}_{\bar{\boldsymbol{\imath}}})$$

is consistent.

By compactness it is enough to prove that for every $N < \omega$ the type $q(\bar{x}) := \bigcup_{\bar{\boldsymbol{\imath}} \in [N]^n} p(\bar{x}, \bar{a}_{\bar{\boldsymbol{\imath}}})$ is consistent. For every $\bar{\boldsymbol{\imath}} \in [N]^n$ pick $f_{\bar{\boldsymbol{\imath}}} \in \operatorname{Saut}_A(\mathfrak{C})$ such that $f(\operatorname{bdd}(A\bar{a}_0 \dots \bar{a}_{n-1})) = \operatorname{bdd}(A\bar{a}_{\bar{\boldsymbol{\imath}}})$. Consider the family of types $\{f_{\bar{\boldsymbol{\imath}}}(p) \mid \bar{\boldsymbol{\imath}} \in [N]^n\}$.

To apply Proposition 3.2.7, we need to show that this is a strong independent system over A. Independence of the system is clear.

Let $\bar{b}_{\bar{\imath}} := f_{\bar{\imath}}(\bar{b})$. It is enough to show that for $\bar{\imath}, \bar{\jmath} \in [n+1]^n$ we have $\operatorname{tp}(\bar{b}_{\bar{\imath}}/\operatorname{bdd}(A(\bar{a}_{\bar{\imath}} \cap \bar{a}_{\bar{\jmath}}))) = \operatorname{tp}(\bar{b}_{\bar{\jmath}}/\operatorname{bdd}(A(\bar{a}_{\bar{\imath}} \cap \bar{a}_{\bar{\jmath}})))$. But this is easy since $\operatorname{tp}(\operatorname{bdd}(A\bar{a}_{\bar{\imath}})/\bar{b})$ is fixed.

By Proposition 3.2.7, there is a common non-forking extension q of all these types.

$\dashv$

3.3 Examples

In this section we show that the theories T_k, $k \geq 3$, that were presented as examples of theories in the "levels" of strong n-simplicity are also examples of the theories in the "levels" of n-simplicity.

We recall the key properties of these theories:

For all $k \geq 3$:

(1) T_k is $\aleph_0$-categorical; so for every $1 \leq n < \omega$ there are finitely many types in n variables over the empty set and every such type is isolated by a *basic formula*.

(2) In T_k dividing is trivial, i.e., the only formulas that witness dividing are finite disjunctions of equalities. In particular for $A, B \supset C$ we have $A \underset{C}{\perp} B$ if and only if $A \cap B \subset C$.

(3) For every set A in the home sort, $\operatorname{dcl}(A)$ is algebraically (and boundedly) closed.

For our purposes, (1) and (2) imply that T_k has elimination of hyperimaginaries (as any type-definable equivalence relation in finitely many variables has to be definable).

(2) and (3) imply weak elimination of imaginaries. Namely, for any $\bar{a}_E$ we have $\bar{a} \in \operatorname{acl}(\bar{a}_E)$.

Proposition 3.3.1. *Suppose that T is such that for any hyperimaginary $\bar{a}_E$ there is a real tuple $\bar{a}'$ such that $\operatorname{bdd}(\bar{a}_E) = \operatorname{bdd}(\bar{a}')$ and for all $\bar{a}$, $\bar{b}$ we have $\operatorname{bdd}(\bar{a}\bar{b}) = \operatorname{bdd}(\bar{a}) \cup \operatorname{bdd}(\bar{b})$. If T is strongly n-simple, then T is n-simple.*

Remark 3.3.2. These assumptions on T would hold if T has elimination of hyperimaginaries and weak elimination of imaginaries. The theories T_{rg} and T_k, $k \geq 3$, satisfy these assumptions.

Proof. The indiscernibility requirement in n-simplicity is clearly stronger for the "home sort" than the corresponding requirement for strong n-simplicity. The main issue is to establish consistency of the union

$$\bigcup_{\bar{\imath}\in[\omega]^k} p(\bar{x},\bar{a}_{\bar{\imath}})$$

when $\bar{a}_i$ and $\bar{x}$ can be (hyper)imaginary elements.

Suppose not. Take $\bar{b}$ such that $\bar{b} \underset{A}{\perp} \bar{a}_0 \dots \bar{a}_{k-1}$, and the indiscernibility assumptions hold. Passing to the real tuples, we get an A-indiscernible sequence $\{\bar{a}'_i \mid i<\omega\}$, and the strong indiscernibility assumptions guarantee that $\mathrm{Ind}(\bar{b}'; A\bar{a}'_0, \dots, A\bar{a}'_{n-1})$ holds. By strong n-simplicity, we get the union

$$\bigcup_{\bar{\imath}\in[\omega]^k} p(\bar{x},\bar{a}'_{\bar{\imath}})$$

is consistent, hence so is the union

$$\bigcup_{\bar{\imath}\in[\omega]^k} p(\bar{x},\mathrm{bdd}(\bar{a}'_{\bar{\imath}})).$$

This finishes the proof. ⊣

So the theories T_k, $k \geq 3$, are examples of $(k-2)$-simple, not $(k-1)$-simple theories. The $(k-2)$-simplicity part follows from the above proposition. It is easy to see that the examples are not $(k-1)$-simple. Indeed, fix $k \geq 3$ and let $\{a_i \mid i<\omega\}$ be an indiscernible sequence of (real) elements such that $\models R(\bar{a}_0,\dots,\bar{a}_{k-1})$. Let

$$p(x, a_0 \dots, a_{k-2}) := (a_0,\dots,a_{k-2})\, S\, x.$$

Since every set in the home sort is boundedly closed, the strong indiscernibility assumptions hold. However, the union

$$\bigcup_{\bar{\imath}\in[\omega]^{k-1}} p(\bar{x},\bar{a}_{\bar{\imath}})$$

is inconsistent by the structure of T_k.

Thus, it follows that T_{rg} is ω-simple. Another example of an ω-simple theory is ACFA. By [8] (see also [16]) ACFA has n-dimensional amalgamation for all $2 \leq n < \omega$. By Theorem 3.2.8, it follows that ACFA is ω-simple.

3.4 2-simple theories have 3-dimensional amalgamation

In this section, we present a proof of 3-dimensional amalgamation for strong systems of types under 2-simplicity. The main idea is very similar to the strongly 2-simple case, but we need to be more careful because of extra indiscernibility restrictions. Generally, we expect that most facts true for strong n-simplicity will have appropriate counterparts in the n-simplicity context.

Theorem 3.4.1. *Let T be 2-simple. Given $\{\bar{a}_i \mid i < 3\}$, suppose that $\{p_{\bar{\imath}}(\bar{x}, \bar{a}_{\bar{\imath}}) \mid \bar{\imath} \in [3]^2\}$ form a strong independent 3-dimensional system over A. Then the system can be amalgamated.*

Proof. Using the construction from Chapter 2, we get a Morley sequence $\{\bar{c}_i \mid i < \omega\}$ over A such that $\bar{c}_i[i] = \bar{a}_i$ for $i < 3$. The projected system of types in this case is

$$p_{01}(\bar{x}, \bar{c}_0[0], \bar{c}_1[1]), \quad p_{02}(\bar{x}, \bar{c}_0[0], \bar{c}_1[2]), \quad p_{12}(\bar{x}, \bar{c}_0[1], \bar{c}_1[2]).$$

Here, we use $p(\bar{x}, \bar{c}_0[0], \bar{c}_1[1])$ to denote the complete type over $\text{bdd}(A\bar{c}_0[0]\bar{c}_1[1])$.

By 2-amalgamation we can find the independent amalgam of the types

$$p_{01}(\bar{x}, \bar{c}_0[0], \bar{c}_1[1]) \text{ and } p_{02}(\bar{x}, \bar{c}_0[0], \bar{c}_1[2])$$

over $\text{bdd}(A\bar{c}_0[0])$ and then the independent amalgam of

$$p_{01}(\bar{x}, \bar{c}_0[0], \bar{c}_1[1]) \cup p_{02}(\bar{x}, \bar{c}_0[0], \bar{c}_1[2]) \text{ and } p_{12}(\bar{x}, \bar{c}_0[1], \bar{c}_1[2])$$

over $\text{bdd}(A\bar{c}_1[2])$. Let

$$\bar{d}_0 \models \bigcup_{\bar{\imath} \in [3]^2} p_{\bar{\imath}}(\bar{x}, \bar{c}_0[\bar{\imath}[0]], \bar{c}_1[\bar{\imath}[1]]).$$

Let $q_1(\bar{x}, \bar{c}_0[0], \bar{c}_0[1])$ be the type of $\bar{d}_0$ over $\mathrm{bdd}(A\bar{c}_0[0], \bar{c}_0[1])$; let $q_2(\bar{x}, \bar{c}_0[1], \bar{c}_0[2])$ be the translate of the type of $\bar{d}_0$ over $\mathrm{bdd}(A\bar{c}_1[1], \bar{c}_1[2])$ to $\mathrm{bdd}(A\bar{c}_0[1], \bar{c}_0[2])$ over A. By construction of $\bar{d}_0$ and the definition of the strong independent system of types, the types q_1 and q_2 agree over the intersection: $\mathrm{bdd}(A\bar{c}_0[1])$. So there is $q(\bar{x}, \bar{c}_0)$, an independent amalgam of $q_1(\bar{x}, \bar{c}_0[0], \bar{c}_0[1])$ and $q_2(\bar{x}, \bar{c}_0[1], \bar{c}_0[2])$ over $A\bar{c}_0[1]$. Finally, let

$$\bar{d} \models \bigcup_{\bar{\boldsymbol{\imath}} \in [3]^2} p_{\bar{\boldsymbol{\imath}}}(\bar{x}, \bar{c}_0[\bar{\boldsymbol{\imath}}[0]], \bar{c}_1[\bar{\boldsymbol{\imath}}[1]]) \cup q(\bar{x}, \bar{c}_0) \cup q(\bar{x}, \bar{c}_1).$$

It is clear that $\bar{d}$ is as needed to apply the definition of n-simplicity.

$\dashv$

3.5 Remarks on 4-dimensional amalgamation

The goal of this section is to illustrate some of the notions that arise in our proof of n-dimensional amalgamation as well as the difficulties in generalizing the fact from dimension three to dimension 4.

For the purpose of this illustration, let us work under the assumption of strong 3-simplicity, and consider a nice simple theory T, such that every set is algebraically closed, and is an amalgamation base.

The dimension of system is 4, and we have the types

$$p_{abc}(x, abc),\ p_{abd}(x, abd),\ p_{acd}(x, acd),\ p_{bcd}(x, bcd),$$

where the necessary independence holds, and they extend the same Lascar strong type over A.

Suppose that the four elements $abcd$ are "bad" in the sense that the diagram cannot be amalgamated. To get a contradiction, we want to construct a Morley sequence $\{\bar{f}_i \mid i < \omega\}$ over A such that the two conditions below hold.

(1) the elements a, b, c, and d (or some four elements $\hat{a}\hat{b}\hat{c}\hat{d}$ that have the same type as $abcd$ over A) are on the "diagonal" of the sequence $\bar{f}_i$.

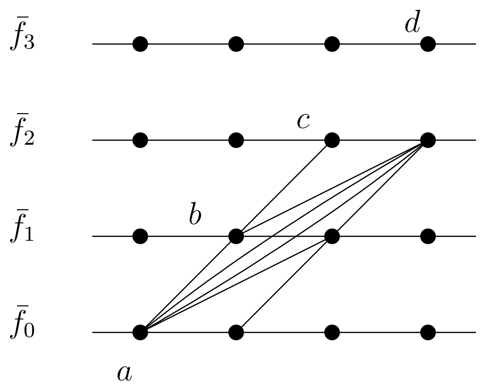
(2) there is a complete type $q(x, \bar{f}_0, \bar{f}_1, \bar{f}_2)$ that does not fork over A; says that $\{\bar{f}_0, \bar{f}_1, \bar{f}_2\}$ are Ax-indiscernible; and such that

$$q(x, \bar{f}_0, \bar{f}_1, \bar{f}_2) \vdash p_{abc}(x, a, b, c),\ q(x, \bar{f}_0, \bar{f}_1, \bar{f}_3) \vdash p_{abd}(x, a, b, d),$$
$$q(x, \bar{f}_0, \bar{f}_2, \bar{f}_3) \vdash p_{acd}(x, a, c, d), \text{ and } q(x, \bar{f}_1, \bar{f}_2, \bar{f}_3) \vdash p_{bcd}(x, b, c, d).$$

For the second condition to hold, it is clear that q should realize the "reflections" of the types p_{abc}, p_{abe}, and so on, on the first three entries of the sequence f_i. In Chapter 2, we call the union of these "reflections" the *projected system of types*.

In the picture below, every triangle represents the type that we want to realize:

$\ldots$

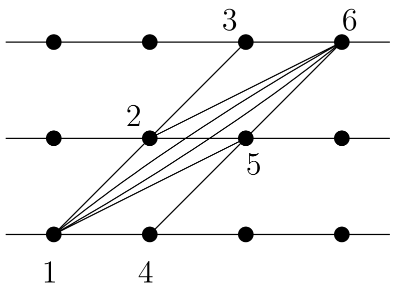


If the conditions (1) and (2) hold, we can use 3-simplicity to get that

$$q(x, \bar{f}_{012}) \cup q(x, \bar{f}_{013}) \cup q(x, \bar{f}_{023}) \cup q(x, \bar{f}_{123})$$

does not fork, and this union implies the union of the original types. This gives us the amalgamation of the system, a contradiction.

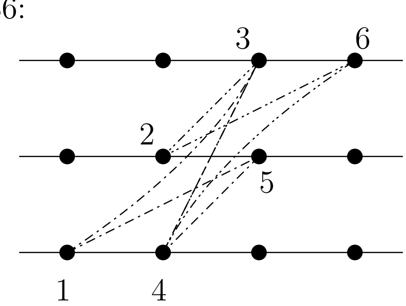
What is difficult in the construction? Suppose we build a Morley sequence $\{\bar{f}_i \mid i < \omega\}$, with the "bad" four elements on the diagonal. Now we want to realize the projected system of types on the first three entries. This is no problem, in fact, we don't even need the 3-dimensional amalgamation for that. Let e^* be the realization, and now we want to make sure that $\{\bar{f}_0, \bar{f}_1, \bar{f}_2\}$ are Ae^* indiscernible. Let me number the relevant elements in the first three entries.



So far we need to realize the types

$$123, \quad 126, \quad 156, \quad 456.$$

Since the pairs of f_i's need to have the same type over Ae^*, it must be that the type of 145 over Ae^* must be the same as the type of 143 over Ae^*. Same with 436 and 236:



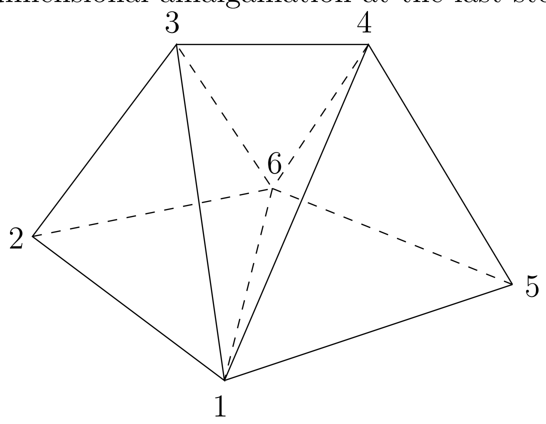
There is no reason why should those types be the same when we simply amalgamate the projected system of types, so we need to amalgamate these conditions in (for 3-dimensional system, this is precisely the approach). In total, we get the following system of types that needs to be amalgamated:

$$123, \ 126, \ 156, \ 456, \ 145, \ 143, \ 436, \ 236.$$

So now the question is: what is the dimension of this system?

One of the possible definitions of the dimension was given by Shelah, in [31]. In our case, the definition translates into a simple condition: if there is a four-element set of indices such that the system lists all its 3-subsets, then the dimension of the system is 4. A quick look at the system confirms that the dimension is less than 4, according to this definition.

However, it is not possible to amalgamate the types using just the 3-dimensional amalgamation property. It is easy to show this graphically. When we put these triangles together, they form a closed surface. This means that when we actually do the 3-dimensional amalgamation, no matter where we start, we will need to do 4-dimensional amalgamation at the last step:



We believe that it is both an interesting and a challenging problem to give the precise definition of the dimension of the system of types that would work in the context of n-simplicity.

The situation becomes worse for larger n, since we have to deal with more indiscernibility conditions (at least 6 for $n = 5$, and they involve more elements).

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