21-241 MATRICES AND LINEAR TRANSFORMATIONS
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COURSE NOTES
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Definition. A subset S of $\mathbb{R}^n$ is called a *subspace* (of $\mathbb{R}^n$) if:

(1) $0 \in S$.
(2) For all $x \in S$, and $\lambda \in \mathbb{R}$, λx is also in S.
(3) For all $x, y \in S$, $x + y$ is also in S.

Note that the second condition implies the first, *so long as S is nonempty.* Thus the first condition is just there to ensure that a subspace is nonempty.

Fact 1. If S is a subspace of $\mathbb{R}^n$ and $x_1, \ldots, x_k \in S$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, then

$$\lambda_1 x_1 + \cdots + \lambda_k x_k \in S$$

Example. Let A be an $m \times n$ matrix. Then

$$\{x \in \mathbb{R}^n \mid Ax = 0\}$$

is a subspace of $\mathbb{R}^n$. This is called the *null space of A* and is denoted by null(A). On the other hand,

$$\{Ax \mid x \in \mathbb{R}^n\}$$

is a subspace of $\mathbb{R}^m$, and is called the *range space of A*, written ran(A).

Definition. Let S be a subspace of $\mathbb{R}^n$. A subset $X \subseteq S$ of S is said to *span* S if for every $s \in S$, there are some $x_1, \ldots, x_k \in X$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that

$$s = \lambda_1 x_1 + \cdots + \lambda_k x_k$$

If $X \subseteq \mathbb{R}^n$, then the *span of X* is the set

$$\mathrm{span}(X) = \{\lambda_1 x_1 + \cdots + \lambda_k x_k \mid k \in \mathbb{N} \wedge x_1, \ldots, x_k \in X \wedge \lambda_1, \ldots, \lambda_k \in \mathbb{R}\}$$

This last definition provides us with a wealth of examples of subspaces of $\mathbb{R}^n$.

Fact 2. If X is any subset of $\mathbb{R}^n$ then span(X) is a subspace of $\mathbb{R}^n$.

It's easy to see that if $X \subseteq Y \subseteq \mathbb{R}^n$, then span$(X) \subseteq$ span(Y). However, the converse doesn't hold. In fact, we have the following example where $X \not\subseteq Y$ and $Y \not\subseteq X$, but span$(X) =$ span(Y).

Example. Consider the following two finite subsets of $\mathbb{R}^3$.

$$X = \left\{ \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \right\} \qquad Y = \left\{ \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} \right\}$$

Then we have $\operatorname{span}(X) = \operatorname{span}(Y)$.

The proof of the above is made much easier using the following, which we call the "linear combination lemma."

Lemma 1. *Let $x_1, \ldots, x_k \in \mathbb{R}^n$ be vectors in $\mathbb{R}^n$. If each of $y_1, \ldots, y_\ell$ is a linear combination of $x_1, \ldots, x_k$ then so is any linear combination of $y_1, \ldots, y_\ell$.*

Proof. The hardest part of this proof is figuring out what the statement of the lemma is, in formal terms. We have our vectors $x_1, \ldots, x_k \in \mathbb{R}^n$; suppose $y_1, \ldots, y_\ell$ are vectors in $\mathbb{R}^n$, each of which is a linear combination of $x_1, \ldots, x_k$;

$$y_i = \lambda_{i1} x_1 + \cdots + \lambda_{ik} x_k \qquad 1 \le i \le \ell$$

Now suppose $z = \mu_1 y_1 + \cdots + \mu_\ell y_\ell$ is a linear combination of $y_1, \ldots, y_\ell$. Then,

$$\begin{aligned} z &= \sum_{i=1}^{\ell} \mu_i y_i \\ &= \sum_{i=1}^{\ell} \mu_i \sum_{j=1}^{k} \lambda_{ij} x_j \\ &= \sum_{j=1}^{k} \sum_{i=1}^{\ell} \mu_i \lambda_{ij} x_j \\ &= \sum_{j=1}^{k} \left(\sum_{i=1}^{\ell} \mu_i \lambda_{ij} \right) x_j \end{aligned}$$

So z is also a linear combination of $x_1, \ldots, x_k$, namely the one whose coefficient for x_j is

$$\sum_{i=1}^{\ell} \mu_i \lambda_{ij}$$

□

Lemma 2. *Suppose S is a subspace of $\mathbb{R}^n$, and $X \subseteq S$. Then $\operatorname{span}(X) \subseteq S$.*

Proof. Suppose $y \in \operatorname{span}(X)$; then there are $x_1, \ldots, x_k \in X$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that $y = \lambda_1 x_1 + \cdots + \lambda_k x_k$. But $x_1, \ldots, x_k \in S$, and so

$$y = \lambda_1 x_1 + \cdots + \lambda_k x_k \in S$$

□

Definition. If A is an $m \times n$ matrix with columns $c_1, \dots, c_n \in \mathbb{R}^m$, and rows $r_1, \dots, r_m \in \mathbb{R}^n$, then we write

$$\operatorname{col}(A) = \operatorname{span}\{c_1, \dots, c_n\} \qquad \operatorname{row}(A) = \operatorname{span}\{r_1, \dots, r_m\}$$

We call $\operatorname{col}(A)$ and $\operatorname{row}(A)$ the *column space* and *row space* of A, respectively.

Fact 3. If A is any matrix, then $\operatorname{ran}(A) = \operatorname{col}(A)$.

Proof. Suppose A is $m \times n$ and $c_1 \dots, c_n \in \mathbb{R}^m$ are the column vectors of A. We've seen before that if $x \in \mathbb{R}^n$, then

$$Ax = x_1 c_1 + \cdots + x_n c_n$$

The right-hand side of the above equation is a member of $\operatorname{col}(A)$, since it's a linear combination of the columns of A. Since $x \in \mathbb{R}^n$ was arbitrary, this shows $\operatorname{ran}(A) \subseteq \operatorname{col}(A)$. Now if $y \in \operatorname{col}(A)$, then y is a linear combination of the columns of A, and hence for some $\lambda_1, \dots, \lambda_n \in \mathbb{R}$,

$$y = \lambda_1 c_1 + \cdots + \lambda_n c_n$$

But then by the same fact,

$$y = \lambda_1 c_1 + \cdots + \lambda_n c_n = A \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

and the right-hand side of this equation is in $\operatorname{ran}(A)$. This shows $\operatorname{col}(A) \subseteq \operatorname{ran}(A)$. $\square$

You might guess now, since $\operatorname{row}(A)$ and $\operatorname{null}(A)$ are both subspaces of $\mathbb{R}^n$, that they are equal; but you'd be wrong! They *are* related, but we won't see how for a while yet. For now, let's see how we can phrase a problem related to spanning sets in terms of Gaussian elimination.

Example. Let

$$X = \left\{ \begin{pmatrix} 0 \\ 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \\ 7 \\ -1 \end{pmatrix} \right\}$$

Is the vector $b = \left(\begin{smallmatrix} 2 \\ 3 \\ 5 \\ 7 \end{smallmatrix}\right)$ in the span of X?

To solve this, let

$$A = \begin{pmatrix} 0 & 2 & 3 \\ 1 & 4 & -5 \\ -2 & 6 & 7 \\ 3 & 0 & -1 \end{pmatrix}$$

Then b is in the span of X if and only if there is some $x \in \mathbb{R}^3$ such that $Ax = b$.

Example. Let X be the following set.

$$\left\{ \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \right\}$$

Prove that $\operatorname{span}(X) = \mathbb{R}^3$.

To prove this it suffices to prove that the following matrix is right-invertible;

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 1 & 2 \\ 1 & -1 & -1 \end{pmatrix}$$

for which it suffices to prove that A is fully invertible;

$$\begin{pmatrix} 0 & 1 & 0 \\ 2 & 1 & 2 \\ 1 & -1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} \rho_2 \to \rho_2 - \rho_1 \\ \rho_3 \to \rho_3 + \rho_1 \end{matrix} \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 1 & 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\rho_2 \to \rho_2 - 2\rho_3 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 4 \\ 1 & 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & -2 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} \rho_1 \leftrightarrow \rho_2 \\ \rho_1 \leftrightarrow \rho_3 \end{matrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ -3 & 1 & -2 \end{pmatrix}$$

$$\begin{matrix} \rho_3 \to \rho_3/4 \\ \rho_1 \to \rho_1 + \rho_3 \end{matrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1/4 & 1/4 & 1/2 \\ 1 & 0 & 0 \\ -3/4 & 1/4 & -1/2 \end{pmatrix}$$

(I computed A^{-1} above, but this is not necessary to prove that A is invertible; it just suffices to show, since A is square, that A reduces to I.)