21-241 MATRICES AND LINEAR TRANSFORMATIONS SUMMER 1 2012 COURSE NOTES JUNE 27

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Here's some neat facts about diagonalizable matrices.

Fact 1. If $A = S^{-1}DS$, where D is diagonal and S is invertible, then the diagonal values of D are exactly the eigenvalues of A, and the columns of S are eigenvectors of A with associated eigenvalue the corresponding diagonal entry of D.

Fact 2. Suppose A is diagonalizable, and its only eigenvalue is λ. Then $A = \lambda I$.

Proof. We have $A = S^{-1}DS$, where D is diagonal, and its entries are the eigenvalues of A. Since λ is the only one, $D = \lambda I$. But then,

$$A = S^{-1}(\lambda I)S = \lambda(S^{-1}IS) = \lambda S^{-1}S = \lambda I$$

□

Lemma 1. *Suppose $A = S^{-1}DS$, where S is invertible. Then $A^k = S^{-1}D^kS$ for any k.*

The following theorem is known as the Spectral Mapping Theorem, and it holds for all square matrices A. I'll only be able to prove it for the diagonalizable ones.

Fact 3. Let $A \in M_n(\mathbb{C})$, and let $p(z)$ be some polynomial in z. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A, possibly with multiplicity. Then $p(A)$ (the result of substituting A for z, and I for the constant 1, in $p(z)$) has eigenvalues $p(\lambda_1), \ldots, p(\lambda_n)$.

Proof for diagonalizable A. Say $p(z) = c_kz^k + c_{k-1}z^{k-1} + \cdots + c_1z + c_0$. Let $A = S^{-1}DS$, where S is invertible and D diagonal, with diagonal entries $\lambda_1, \ldots, \lambda_n$. Then

$$\begin{aligned} p(A) &= c_kA^k + c_{k-1}A^{k-1} + \cdots + c_1A + c_0I \\ &= c_kS^{-1}D^kS + c_{k-1}S^{-1}D^{k-1}S + \cdots + c_1S^{-1}DS + c_0I \\ &= S^{-1}(c_kD^k + c_{k-1}D^{k-1} + \cdots + c_1D + c_0I)S = S^{-1}p(D)S \end{aligned}$$

Now note that since D is diagonal, D^t is just the diagonal matrix whose entries are those of D, taken to the power t. Hence the matrix

$$c_kD^k + c_{k-1}D^{k-1} + \cdots + c_1D + c_0I$$

is diagonal, and its ith diagonal entry is

$$c_k\lambda_i^k + c_{k-1}\lambda_i^{k-1} + \cdots + c_1\lambda_i + c_0 \cdot 1 = p(\lambda_i)$$

Then these are the eigenvalues of $p(A)$, since $p(A) = S^{-1}p(D)S$. □

This corollary is known as the Cayley-Hamilton theorem. Again, it's known to be true for all square matrices A, but I'll only prove it for the diagonalizable ones.

Corollar. *Let A be a square matrix, and let p_A be its characteristic polynomial. Then $p_A(A)$ is the zero matrix.*

Proof for diagonalizable A. By the spectral mapping theorem, the eigenvalues of $p_A(A)$ are $p_A(\lambda_1), \ldots, p_A(\lambda_n)$ where $\lambda_1, \ldots, \lambda_n$. But $p_A(\lambda_i) = 0$, for all i! So the only eigenvalue of $p_A(A)$ is 0. Now, if A is diagonalizable, then it's easy to see that $p_A(A)$ is, too (the same calculation we did above for the spectral mapping theorem). Hence $p_A(A) = 0I = 0_{n \times n}$. □