21-241 MATRICES AND LINEAR TRANSFORMATIONS
SUMMER 1 2012
COURSE NOTES
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1. Change of basis continued

Yesterday I promised to show you an easy way of computing the representation of a given linear transformation T with respect to an arbitrary basis.

Fact 1. Suppose $T : \mathbb{C}^n \to \mathbb{C}^n$ is a linear transformation, and A is its standard matrix (ie, its representation with respect to the standard basis). If $\mathscr{B} = (b_1, \ldots, b_n)$ is any other (ordered) basis for $\mathbb{C}^n$, then

$$\mathrm{rep}_{\mathscr{B}}(T) = S^{-1}AS$$

where S is the $n \times n$ matrix whose columns are $b_1, \ldots, b_n$.

Proof. Since A is the standard matrix of T,

$$T(b_j) = Ab_j$$

Let $\begin{pmatrix} \lambda_{1j} \\ \vdots \\ \lambda_{nj} \end{pmatrix}$ be the representation of Ab_j in the basis $b_1, \ldots, b_n$, ie the unique sequence of coefficients such that

$$Ab_j = \lambda_{1j}b_1 + \cdots + \lambda_{nj}b_n$$

Then $\mathrm{rep}_{\mathscr{B}}(T)$ is the $n \times n$ matrix with entries λ_{ij}.

Now, note that $Se_j = b_j$, and hence $S^{-1}b_j = e_j$. Then,

$$(S^{-1}AS)e_j = S^{-1}Ab_j = S^{-1}(\lambda_{1j}b_1 + \cdots + \lambda_{nj}b_n) = \lambda_{1j}e_1 + \cdots + \lambda_{nj}e_n = \begin{pmatrix} \lambda_{1j} \\ \vdots \\ \lambda_{nj} \end{pmatrix}$$

The leftmost side of the equation above is just the jth column of $S^{-1}AS$; the rightmost side is the jth column of $\mathrm{rep}_{\mathscr{B}}(T)$. □

Example. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection across the line L which makes an angle of θ with the x-axis. Find the matrix of T with respect to the standard basis, and the basis

$$\mathscr{B} = \left(\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \right)$$

Solution. In the standard basis the matrix is

$$\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

(This was a homework problem from homework 2.) In the alternate basis it's

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Which turns out to be

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

□

2. Diagonalization

Definition. A matrix A is *diagonalizable* if it is similar to some diagonal matrix D. In this case, if $A = SDS^{-1}$, then we say that S *diagonalizes* A.

We like matrices that are diagonalizable.

Fact 2. If D is a diagonal matrix with diagonal entries $d_1, \ldots, d_n$, then

- $\operatorname{spec}(D) = \{d_1, \ldots, d_n\}$,
- $p_D(\lambda) = (\lambda - d_1) \cdots (\lambda - d_n)$, and
- A basis for the eigenspace of λ (with respect to D) is given by

$$\{e_i \mid i \text{ is such that } d_i = \lambda\}$$

Corollary 1. *If A is diagonalizable, and λ is an eigenvalue of A, then the algebraic and geometric multiplicities of λ (with respect to A) are the same.*

Proof. By the above fact, this is true for diagonal matrices, and it's preserved by similarity (by a theorem from yesterday). □

Example. If $n > 1$, then the $n \times n$ shift matrix S is *not* diagonalizable.

Proof. S has only one eigenvalue: 0. Its algebraic multiplicity is n, whereas its geometric multiplicity is 1. Since they're not the same S can't be diagonalizable. □