21-241 MATRICES AND LINEAR TRANSFORMATIONS
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COURSE NOTES
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Let $\{x_1, \dots, x_k\}$ be a basis for some subspace V of $\mathbb{C}^n$. We've seen before that any vector v in V can be written uniquely as a linear combination of $x_1, \dots, x_k$;

$$v = \lambda_1 x_1 + \cdots + \lambda_k x_k$$

Definition. Let $\mathscr{B} = \{b_1, \dots, b_n\}$ be a basis for $\mathbb{C}^n$. If $x \in \mathbb{C}^n$, then the *representation of x in the basis $b_1, \dots, b_n$* is the unique vector $\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$ in $\mathbb{C}^n$ such that

$$x = \lambda_1 b_1 + \cdots + \lambda_n b_n$$

We refer to this vector as $\operatorname{rep}_{\mathscr{B}}(x)$.

Definition. Let $\mathscr{B} = \{b_1, \dots, b_n\}$ be a basis for $\mathbb{C}^n$, and let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformation. The *matrix of T with respect to $b_1, \dots, b_n$* is the unique $n \times n$ matrix A such that, for all $x \in \mathbb{C}^n$,

$$\operatorname{rep}_{\mathscr{B}}(T(x)) = A \operatorname{rep}_{\mathscr{B}}(x)$$

Equivalently, A is the matrix with entries

$$A_{ij} = (\operatorname{rep}_{\mathscr{B}}(b_j))_i$$

We refer to this matrix as $\operatorname{rep}_{\mathscr{B}}(T)$.

Example. Let L be the subspace of $\mathbb{R}^2$ spanned by $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$. Compute the matrix of $\mathbb{P}_L$ with respect to the (ordered) standard basis $\mathscr{E} = (e_1, e_2)$, and the basis

$$\mathscr{B} = \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right)$$

Solution. The first is

$$\operatorname{rep}_{\mathscr{E}}(\mathbb{P}_L) = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

The second is simply

$$\operatorname{rep}_{\mathscr{B}}(\mathbb{P}_L) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

□

How are these matrices related? They both represent the same linear transformation $\mathbb{P}_L$, but how does this fact present itself in the algebra of the matrices themselves?

Definition. Two square matrices A and B are *similar*, written $A \sim_s B$, if there is some invertible S such that $A = SBS^{-1}$.

Fact 1. $\sim_s$ is an equivalence relation.

Theorem 1. *Let A and B be square matrices, of the same size. Then the following are equivalent.*

(1) A and B are similar.
(2) A and B represent the same linear transformation.

Example. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y+z \\ x+z \\ x+y \end{pmatrix}$$

Find the matrix of T with respect to the standard basis, and the basis

$$\left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right)$$

Solution. In the standard basis, the matrix of T is

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

whereas in the alternate basis, the matrix of T is

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

□

In physics one often studies “coordinate-free” properties of linear transformations; ie, properties that are *invariant* under a change of basis. The idea is that physical laws should be independent of the observer.

Theorem 2. *If A and B are similar, then* $\operatorname{spec}(A) = \operatorname{spec}(B)$*. Moreover, the (algebraic and geometric) multiplicity of each eigenvalue with respect to A is the same as that with respect to B.*

I'll prove the theorem below, but first I need a lemma.

Lemma 1. *If S is an invertible matrix then* $\det(S^{-1}) = \det(S)^{-1}$.

Proof. We have $\det(S)\det(S^{-1}) = \det(SS^{-1}) = \det(I) = 1$. □

Proof of Theorem 2. Say $A = SBS^{-1}$. Then,

$$\begin{aligned} p_A(\lambda) &= \det(\lambda I - A) = \det(\lambda SIS^{-1} - SBS^{-1}) = \det(S(\lambda I - B)S^{-1}) \\ &= \det(S)\det(\lambda I - B)\det(S)^{-1} = \det(\lambda I - B) = p_B(\lambda) \end{aligned}$$

So A and B have the same characteristic polynomial. This shows that they have the same eigenvalues and their eigenvalues have the same algebraic multiplicities. As for the geometric multiplicities, notice that for all λ,

$$\operatorname{col}(\lambda I - B) = \operatorname{col}(S(\lambda I - B)) \qquad \operatorname{row}(S(\lambda I - B)) = \operatorname{row}(S(\lambda I - B)S^{-1})$$

Hence, $\operatorname{rank}(\lambda I - B) = \operatorname{rank}(\lambda I - A)$. By the rank-nullity theorem, this implies $\operatorname{nullity}(\lambda I - B) = \operatorname{nullity}(\lambda I - A)$. But the nullity of $\lambda I - B$ is simply the geometric multiplicity of λ with respect to B, and analogously for A. □

Of course, the eigenspaces may change under a change of basis. Both of the examples from above are typical in this regard.