21-241 MATRICES AND LINEAR TRANSFORMATIONS SUMMER 1 2012 COURSE NOTES JUNE 13

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1. Warm-up

What's the determinant of a diagonal matrix $\text{diag}(d_1, \ldots, d_n)$? I can think of three proofs.

(1) Row-ops. [Hard to describe formally.]
(2) Problem 8 on HW4 and induction.
(3) Write the determinant using permutations.

2. Orthonormal Bases

Definition. If $x \in \mathbb{C}^n$ and $\|x\| = 1$, we call x a *unit vector*. Any nonzero vector can be scaled to make a unit vector, and we call this *normalizing* the vector.

If $x_1, \ldots, x_k \in \mathbb{C}^n$ are distinct, pairwise orthogonal, unit vectors then we call the set $\{x_1, \ldots, x_k\}$ *orthonormal*.

Recall that if $x_1, \ldots, x_k$ are nonzero, distinct, and pairwise orthogonal, then $\{x_1, \ldots, x_k\}$ is linearly independent. Hence $\{x_1, \ldots, x_k\}$ makes up a basis for its own span. In this case we call $\{x_1, \ldots, x_k\}$ an *orthogonal basis* for the subspace $S = \text{span}\{x_1, \ldots, x_k\}$. If $\{x_1, \ldots, x_k\}$ is orthonormal then we call it an *orthonormal basis* for S.

Theorem 1. *Suppose S is a subspace of $\mathbb{C}^n$, and $\{x_1, \ldots, x_k\}$ is an orthogonal basis for S. Then for any $y \in S$,*

$$y = \frac{\langle y, x_1 \rangle}{\|x_1\|^2} x_1 + \cdots + \frac{\langle y, x_k \rangle}{\|x_k\|^2} x_k$$

In particular, if $\{x_1, \ldots, x_k\}$ is orthonormal, then for any $y \in S$ we have

$$y = \langle y, x_1 \rangle x_1 + \cdots + \langle y, x_k \rangle x_k$$

Example. Show that $\{e_1, \ldots, e_n\}$ is an orthonormal basis for $\mathbb{C}^n$.

Example. Show that $\{\binom{1}{1}, \binom{-1}{1}\}$ is an orthogonal basis for $\mathbb{C}^2$ but is not orthonormal. What's the "normalized" version? What are the coordinates of e_1 and e_2 in this orthonormal basis?

Definition. Suppose S is a k-dimensional subspace of $\mathbb{C}^n$, and $\{s_1, \ldots, s_k\}$ is an orthonormal basis for S. The *orthogonal projection* of a vector $x \in \mathbb{C}^n$ onto S is the vector

$$\mathbb{P}_S(x) = \sum_{i=1}^{k} \langle x, s_i \rangle \, s_i$$

Example. Let $s = \frac{1}{\sqrt{2}} \binom{1}{1}$, $S = \operatorname{span}\{s\}$, and $X = \operatorname{span}\{e_1\}$. What's $\mathbb{P}_S(e_1)$? $\mathbb{P}_X(s)$? $\mathbb{P}_S(s)$? $\mathbb{P}_X(e_1)$? What about $\mathbb{P}_S(\mathbb{P}_X(s))$?

Example. Find an orthonormal basis for the plane P in $\mathbb{R}^3$ described by $3x - 2y + z = 0$. Find the projections of e_1, e_2, e_3 onto P.

Theorem 2. *(1) $\mathbb{P}_S : \mathbb{C}^n \to \mathbb{C}^n$ is well-defined and a linear transformation.*

(2) If P_S is the $n \times n$ matrix which implements $\mathbb{P}_S$, then P_S is a projection matrix.

(3) For all $x \in \mathbb{C}^n$, $\mathbb{P}_S(x)$ is the unique vector in S whose distance to x is smallest.