21-241 MATRICES AND LINEAR TRANSFORMATIONS
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COURSE NOTES
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PAUL MCKENNEY

1. Miscellaneous but useful facts

The following theorem is useful in more ways than you might think.

Theorem 1. *If S and T are subspaces of $\mathbb{C}^n$, and $S \subseteq T$, then $\dim(S) \leq \dim(T)$. Moreover if $\dim(S) = \dim(T)$, then $S = T$.*

You should be able to prove the following two facts using just the above theorem.

Fact 1. If S is a subspace of $\mathbb{C}^n$, and $\dim(S) = n$, then $S = \mathbb{C}^n$.

Fact 2. Let A be an $m \times n$ matrix. If $\text{nullity}(A) = n$, then $A = 0$. Equivalently, if $\text{rank}(A) = 0$, then $A = 0$.

The following facts are also easily proven, this time without any reference to anything but the definitions.

Fact 3. Let A and B be $m \times n$ matrices. If $T_A = T_B$, then $A = B$. Equivalently, if $Ax = Bx$ for all $x \in \mathbb{C}^n$, then $A = B$.

Fact 4. Let $x, y \in \mathbb{C}^n$. Then $\{x, y\}$ is linearly dependent if and only if either x is a scalar multiple of y, or vice-versa.

2. Eigenstuff

Definition. Let $A \in M_n(\mathbb{C})$. A complex number λ is an *eigenvalue of* A if there is some nonzero vector $v \in \mathbb{C}^n$ such that $Av = \lambda v$. In this case, v is called an *eigenvector of* A with *associated eigenvalue* λ. If $\lambda \in \mathbb{C}$ then

$$V_\lambda = \{v \in \mathbb{C}^n \mid Av = \lambda v\}$$

is called the *eigenspace of* A *associated to* λ. Note that λ is an eigenvalue of A if and only if $V_\lambda \neq \{0\}$. We write $\text{spec}(A)$ for the set of all eigenvalues of A.

Example. Let $\lambda \in \mathbb{C}$. Then λ is an eigenvalue of λI, and V_λ in this case is $\mathbb{C}^n$. (So every nonzero vector is an eigenvector of λI with associated eigenvalue λ.)

Example. Let $A = \left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)$. Show that $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right)$ are both eigenvectors for A. What are their associated eigenvalues? What does this mean, geometrically, about T_A?

Fact 5. If A is a *real* $n \times n$ matrix and λ is a *real* eigenvalue of A, then there is a *real* eigenvector of A with associated eigenvalue λ.

An eigenvector v of a matrix A gives an important bit of geometric information about A; if λ is its associated eigenvalue, then this tells us that A *stretches* v, in the direction of v by a factor of λ. Of course, this is much more intelligible when $\lambda \in \mathbb{R}$. If $\lambda > 0$, then A stretches v by a factor of λ in the direction of v, whereas if $\lambda < 0$ then A stretches v by a factor of $|\lambda|$, in the opposite direction.

Example. Let $\theta \in [0, 2\pi)$ be given, and let

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

(Recall that this matrix implements a counter-clockwise rotation about the origin, by an angle of θ.) When (ie, for which values of θ) does A have real eigenvalues? What are they? What about complex eigenvalues?

Example. Let $\theta \in [0, 2\pi)$ be given and let

$$A = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

(The reflection across the line which makes an angle of θ with the x-axis.) When does A have real eigenvalues? What are they? What about complex eigenvalues?

Lemma 1. $V_\lambda = \text{null}(\lambda I - A) = \text{null}(A - \lambda I)$.

Theorem 2. *Let $A \in M_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$. Then the following are equivalent.*

(1) λ is an eigenvalue of A.
(2) The nullity of $\lambda I - A$ is nonzero.
(3) $\det(\lambda I - A) = 0$.

Example. Let A be the rotation matrix from the previous example. Let $\lambda \in \mathbb{C}$ be given. Then,

$$\det(\lambda I - A) = (\lambda - \cos\theta)^2 + \sin^2\theta = \lambda^2 - 2(\cos\theta)\lambda + 1$$

This quadratic is zero if and only if

$$\lambda = \cos\theta \pm \sqrt{\cos^2\theta - 1} = \cos\theta \pm i\sin\theta$$

Hence A has eigenvalues $\cos\theta + i\sin\theta$ and $\cos\theta - i\sin\theta$. The eigenvectors can be found by reducing

$$\begin{pmatrix} (\cos\theta + i\sin\theta) - \cos\theta & +\sin\theta \\ -\sin\theta & (\cos\theta + i\sin\theta) - \cos\theta \end{pmatrix} = \begin{pmatrix} i\sin\theta & +\sin\theta \\ -\sin\theta & i\sin\theta \end{pmatrix}$$

and

$$\begin{pmatrix} (\cos\theta - i\sin\theta) - \cos\theta & +\sin\theta \\ -\sin\theta & (\cos\theta - i\sin\theta) - \cos\theta \end{pmatrix} = \begin{pmatrix} -i\sin\theta & +\sin\theta \\ -\sin\theta & -i\sin\theta \end{pmatrix}$$

If $\sin\theta \neq 0$, then these reduce to, respectively,

$$\begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$$

So the first gives an eigenvector of $\left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right)$, and the second $\left(\begin{smallmatrix} -i \\ 1 \end{smallmatrix}\right)$.

Definition. The *characteristic polynomial* of an $n \times n$ matrix A is the polynomial, in variable z, described by

$$p_A(z) = \det(zI - A)$$

Fact 6. If p is a degree-n polynomial with complex coefficients, then p can be factored into n-many linear terms, with a constant;

$$p(z) = \mu(z - \lambda_1) \cdots (z - \lambda_n)$$

Moreover, μ is simply the coefficient of z^n in p.

Example. Let A be the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

What are the eigenvalues and eigenvectors of A? What does this mean geometrically about T_A?

Theorem 3. *The eigenvalues of a square matrix A are simply the roots to its characteristic polynomial p_A.*

Fact 7. If A is $n \times n$ then p_A has degree n.

Theorem 4. *If $A \in M_n(\mathbb{C})$ then A can have at most n eigenvalues.*

Example. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Solution. The matrix $\lambda I - A$ is

$$\begin{pmatrix} \lambda & -1 & 0 \\ 1 & \lambda & -1 \\ 0 & 1 & \lambda \end{pmatrix}$$

and its determinant is thus

$$\lambda^3 + 0 + 0 - \lambda(1)(-1) - (1)(-1)\lambda - 0 = \lambda^3 + 2\lambda$$

The roots of the polynomial $z^3 + 2z$ are $z = 0$ and $z = \pm\sqrt{2}i$, hence these are the eigenvalues of A. Let's look at the eigenspace associated to eigenvalue 0;

$$0I - A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

This matrix has nullity 1, and its null space is spanned by the single vector $\left(\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right)$.

For eigenvalue $\sqrt{2}i$, we get

$$\sqrt{2}iI - A = \begin{pmatrix} \sqrt{2}i & -1 & 0 \\ 1 & \sqrt{2}i & -1 \\ 0 & 1 & \sqrt{2}i \end{pmatrix}$$

This reduces to

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \sqrt{2}i \\ 0 & 0 & 0 \end{pmatrix}$$

hence the vector $\left(\begin{smallmatrix} -1 \\ -\sqrt{2}i \\ 1 \end{smallmatrix}\right)$ is an eigenvector for this eigenvalue. (And it spans the eigenspace.)

Finally, for eigenvalue $-\sqrt{2}i$, similar work shows that $\left(\begin{smallmatrix} 1 \\ -\sqrt{2}i \\ 1 \end{smallmatrix}\right)$ is an eigenvector. (And, again, it spans the eigenspace.) $\square$