# 21-241 MATRICES AND LINEAR TRANSFORMATIONS SUMMER 1 2012 COURSE NOTES JUNE 8

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**Theorem 1.** *Let* $A, B \in M_n(\mathbb{C})$*. Then* $\det(AB) = \det(A)\det(B)$.

*Proof.* By your homework, $AB$ is invertible if and only if both $A$ and $B$ are. Hence if $\det(AB) = 0$ (ie, $AB$ is not invertible) then one of $\det(A)$, $\det(B)$ must be zero as well, so the equation holds.

So let's assume $AB$ is invertible, and hence that $A$ and $B$ are too. Then there are elementary matrices $E_1, \ldots, E_k$ and $F_1, \ldots, F_\ell$ such that

$$A = E_k \cdots E_1 \qquad \text{and} \qquad B = F_\ell \cdots F_1$$

(These are the *reverse* of the row operations used to reduce $A$ and $B$ to $I$.) We've seen that

$$\det(A) = \det(E_k) \cdots \det(E_1) \qquad \det(B) = \det(F_\ell) \cdots \det(F_1)$$

and hence

$$\det(AB) = \det((E_k \cdots E_1)(F_\ell \cdots F_1)) = \det(E_k) \cdots \det(E_1)\det(F_\ell) \cdots \det(F_1) = \det(A)\det(B)$$

□

**Definition.** A *permutation matrix* is an $n \times n$ matrix $P$ such that

(1) every entry in $P$ is either 0 or 1, and
(2) there is exactly one 1 in each row and column of $P$.

If $P$ is a permutation matrix then there is an associated permutation of $[n] = \{1, \ldots, n\}$;

$$\pi(i) = j \iff P_{ij} = 1$$

We write $P = P_\pi$. Note that $\det(P) = \pm 1$, depending on the number of swaps needed to reduce $P$ to $I$. (Which it turns out is the number of transpositions needed to produce $\pi$.)

For convenience we often use a row vector in $\mathbb{C}^n$ to describe a permutation $\pi$ of $[n]$. Specifically, in the $i$th entry of the row vector we write $\pi(i)$; e.g. (312) denotes the permutation $\pi$ such that $\pi(1) = 3$, $\pi(2) = 1$, and $\pi(3) = 2$.

**Definition.** Let $\text{perm}(n) = \{\pi : [n] \to [n] \mid \pi \text{ is a bijection}\}$.

**Fact 1.** If $A$ is an $n \times n$ matrix, then

$$\det(A) = \sum_{\pi \in \text{perm}(n)} \det(P_\pi) \prod_{i=1}^{n} A_{i\pi(i)}$$

*Example.* Say $A$ is the $3 \times 3$ matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

The $3! = 6$ permutations of [3] are listed below, along with the determinant of the associated permutation matrix;

$$\begin{array}{cc} (1 \;\; 2 \;\; 3) & 1 \\ (1 \;\; 3 \;\; 2) & -1 \\ (2 \;\; 1 \;\; 3) & 1 \\ (2 \;\; 3 \;\; 1) & -1 \\ (3 \;\; 1 \;\; 2) & 1 \\ (3 \;\; 2 \;\; 1) & -1 \end{array}$$

Hence we have

$$\det(A) = +aei + bdi + cdh - afh - bfg - ceg$$

*Example.* Say $A$ is the $4 \times 4$ matrix

$$A = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{pmatrix}$$

There are a total of $4! = 24$ permutations of [4], but not very many of them produce a nonzero term in the sum, for this matrix at least. Which ones do?

**Theorem 2.** *Let $A$ be any $n \times n$ complex matrix. Then* $\det(A) = \det(A^\top)$.

*Proof.* Let's look at our formula for $\det(A)$, and the same for $\det(A^\top)$, and compare;

$$\det(A) = \sum_{\pi \in \text{perm}(n)} \det(P_\pi) \prod_{i=1}^{n} A_{i\pi(i)}$$

and

$$\det(A^\top) = \sum_{\pi \in \text{perm}(n)} \det(P_\pi) \prod_{i=1}^{n} A_{\pi(i)i}$$

Now let $\pi \in \text{perm}(n)$ be given. Note that

$$\prod_{i=1}^{n} A_{\pi(i)i} = \prod_{i=1}^{n} A_{i\pi^{-1}(i)}$$

Let's rewrite our formula accordingly and see if it gets us anywhere.

$$\det(A^\top) = \sum_{\pi \in \mathrm{perm}(n)} \det(P_\pi) \prod_{i=1}^{n} A_{i\pi^{-1}(i)}$$

Notice that the map $\pi \mapsto \pi^{-1}$ is a bijection; ie, we may reorder the above sum (by relabeling $\sigma = \pi^{-1}$, so $\sigma^{-1} = \pi$) to get

$$\det(A^\top) = \sum_{\sigma \in \mathrm{perm}(n)} \det(P_{\sigma^{-1}}) \prod_{i=1}^{n} A_{i\sigma(i)}$$

By a problem to be seen on the exam 2 preview, $\det(P_\sigma) = \det(P_{\sigma^{-1}})$ for all $\sigma \in \mathrm{perm}(n)$. This completes the proof. $\square$