21-241 MATRICES AND LINEAR TRANSFORMATIONS SUMMER 1 2012 COURSE NOTES JUNE 5

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Let's recall the rank-nullity theorem.

Theorem 1. *(Rank-Nullity) Let A be an $m \times n$ matrix, and let R be any row-echelon form of A. Say R has k nonzero rows, $r_1, \dots, r_k$, and the leading entry of r_i appears in column ℓ_i. (Since R is in row-echelon form, this means $\ell_1 < \ell_2 < \cdots < \ell_k$.) Let $a_1, \dots, a_n$ be the columns of A, in that order. Then;*

1. *$\{r_1, \dots, r_k\}$ is a basis for $\mathrm{row}(A)$.*
2. *$\{a_{\ell_1}, \dots, a_{\ell_k}\}$ is a basis for $\mathrm{col}(A)$.*
3. *$\{s_1, \dots, s_{n-k}\}$ is a basis for $\mathrm{null}(A)$, where s_i is the vector with a 1 in the entry corresponding to the ith free variable, and a 0 in every entry corresponding to the other free variables. (Note that since s_i must be in $\mathrm{null}(A)$, this determines the rest of the entries in s_i.)*

Proof for row-echelon A. We assume in this proof that $A = R$. Later we'll deal with the case where A is not already in row-echelon form.

Since the rows of R span $\mathrm{row}(R)$, and only the nonzero ones matter, it follows that $\{r_1, \dots, r_k\}$ spans $\mathrm{row}(R)$. So it suffices to show that this set is linearly independent. R looks like this;

$$\begin{pmatrix} 0 & \cdots & R_{1\ell_1} & \cdots & R_{1\ell_2} & \cdots & R_{1\ell_k} & \cdots & R_{1n} \\ & & & & R_{2\ell_2} & \cdots & R_{2\ell_k} & \cdots & R_{2n} \\ & & & & & & & & \\ & & & & & & R_{k\ell_k} & \cdots & R_{kn} \\ & & & & & & & & \\ 0 & & & & \cdots & & & & 0 \end{pmatrix}$$

Let $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ be given. Fix some $i \leq k$. Then the ℓ_ith entry in the row vector

$$\lambda_1 r_1 + \cdots + \lambda_k r_k$$

is exactly

$$\lambda_1 R_{1\ell_i} + \cdots + \lambda_i R_{i\ell_j} + 0 + \cdots + 0$$

since the ℓ_ith entry of r_j is zero, whenever $j > i$. Now suppose

$$\lambda_1 r_1 + \cdots + \lambda_k r_k = 0$$

Then since

$$\lambda_1 R_{1\ell_1} + 0 + \cdots + 0 = 0$$

and $R_{1\ell_1} \neq 0$, it must be that $\lambda_1 = 0$. Then

$$0 + \lambda_2 R_{2\ell_2} + 0 + \cdots + 0 = 0$$

and similarly it follows that $\lambda_2 = 0$. Continuing in this way we find that $\lambda_1 = \lambda_2 = \cdots = 0$, and so $r_1, \ldots, r_k$ are linearly independent.

Similarly for col(R); if $c_1, \ldots, c_n$ are the columns of R, then the ith entry of the column vector

$$\lambda_1 c_{\ell_1} + \cdots + \lambda_k c_{\ell_k}$$

is

$$0 + \cdots + 0 + \lambda_i R_{i\ell_i} + \lambda_{i+1} R_{i+1,\ell_{i+1}} + \cdots + \lambda_k R_{k\ell_k}$$

If this column vector is the zero vector, then recursively we find that $\lambda_k = 0$, $\lambda_{k-1} = 0$, etc. So $\{c_{\ell_1}, c_{\ell_2}, \ldots, c_{\ell_k}\}$ is linearly independent.

Now for null(R), let f_i be the index of the ith free variable. Then for any $\mu_1, \ldots, \mu_{n-k} \in \mathbb{R}$, the f_ith entry of the linear combination

$$\mu_1 s_1 + \cdots + \mu_{n-k} s_{n-k}$$

is exactly μ_i, since s_i has a 1 in the f_ith entry and s_j has a 0 in the f_ith entry for all $j \neq i$. It follows that $\{s_1, \ldots, s_{n-k}\}$ is linearly independent. It spans null(R) by our back-substitution algorithm. □

Lemma 1. *If A and B are matrices such that AB is defined then* $\text{row}(AB) \subseteq \text{row}(B)$.

Corollary 1. *If A and B are row-equivalent then* $\text{row}(A) = \text{row}(B)$.

Proof. If A and B are row-equivalent then there is an invertible matrix E (a product of elementary matrices) such that $B = EA$. Then $\text{row}(B) \subseteq \text{row}(A)$. But we also have $A = E^{-1}B$, so $\text{row}(A) \subseteq \text{row}(B)$. □

Proof of (1) *in Rank-Nullity.* Since A and R are row-equivalent, $\text{row}(A) = \text{row}(R)$. The nonzero rows of R thus make up a basis of $\text{row}(R) = \text{row}(A)$. □

Lemma 2. *If A and B are row-equivalent then* $\text{null}(A) = \text{null}(B)$.

Proof of (3) *in Rank-Nullity.* Since A and R are row-equivalent, $\text{null}(A) = \text{null}(R)$. Then any basis for null(R) is also a basis for null(A). □

Lemma 3. *Suppose A and B are row-equivalent and $m \times n$. Say their columns are $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ respectively. If $1 \leq j_1 < \cdots < j_k \leq n$ are any column indices, then the following are equivalent;*

(a) $\{a_{j_1}, \ldots, a_{j_k}\}$ is linearly independent.
(b) $\{b_{j_1}, \ldots, b_{j_k}\}$ is linearly independent.

Proof. Let A' and B' be the $m \times k$ *submatrices* of A and B, whose columns are $a_{j_1}, \ldots, a_{j_k}$ and $b_{j_1}, \ldots, b_{j_k}$ respectively. Since A and B are row-equivalent, so are A' and B', by the same row operations. So $\mathrm{null}(A') = \mathrm{null}(B')$. The result follows. (How? Work it out...) □

Proof of (2) *in Rank-Nullity.* Since A and R are row-equivalent and the pivot columns of R are a basis for $\mathrm{col}(R)$, by Lemma 3 it follows that $X = \{a_{\ell_1}, \ldots, a_{\ell_k}\}$ is linearly independent, and adding any other column of A to X would make it linearly dependent. Hence X is a basis for $\mathrm{col}(A)$. □

Definition. Let A be a matrix. The *row rank* of A is $\dim(\mathrm{row}(A))$. The *column rank* of A is $\dim(\mathrm{col}(A))$. The *nullity* of A is $\dim(\mathrm{null}(A))$.

Corollary 2. *The row rank of a matrix is the same as its column rank. We call this common dimension the* rank*. If A is $m \times n$, then* $\mathrm{rank}(A) + \mathrm{nullity}(A) = n$.

Example. Let r be a row vector of length n and c a column vector of height n. Let $A = rc$ and $B = cr$. What is the rank of A? of B? What about the nullity?

Example. If A and B are both $n \times n$, how are the rank of AB and BA related? What about the nullity? Look at a 2×2 example.

Example. Consider the $n \times n$ *shift* matrix;

$$S = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Calculate bases for $\mathrm{null}(S^t)$ and $\mathrm{ran}(S^t)$, where $t \geq 1$.