SOME CALKIN ALGEBRAS HAVE OUTER AUTOMORPHISMS

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Abstract. We consider various quotients of the C*-algebra of bounded operators on a nonseparable Hilbert space, and prove in some cases that, assuming some restriction of the Generalized Continuum Hypothesis, there are many outer automorphisms.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space. The *Calkin algebra* over $\mathcal{H}$ is the quotient $\mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}$, where $\mathcal{B}(\mathcal{H})$ is the C*-algebra of bounded, linear operators on $\mathcal{H}$, and $\mathcal{K}$ is its ideal of compact operators. Assuming the Continuum Hypothesis, Phillips and Weaver constructed $2^{2^{\aleph_0}}$-many automorphisms of the Calkin algebra on the Hilbert space of dimension $\aleph_0$ ([4]). Since there are only $2^{\aleph_0}$-many automorphisms of $\mathcal{C}(\mathcal{H})$ which are *inner* (that is, implemented by conjugation by a unitary), this implies in particular that there are many more outer automorphisms than there are inner ones, in the presence of CH.

The first author proved in [2] that it is relatively consistent with ZFC that all automorphisms of the Calkin algebra on a separable Hilbert space are inner. This establishes the existence of an outer automorphism as a question independent of ZFC. The assumption made there was *Todorčević's Axiom* (TA), a combinatorial principle also known as the *Open Coloring Axiom*. TA has a number of consequences in other areas of mathematics, and follows from the *Proper Forcing Axiom* (PFA), which is itself well-known for its influence on certain kinds of rigidity in mathematics (see [3]). The first author extended this result to prove that all automorphisms of the Calkin algebra over *any* Hilbert space, separable or not, are inner, assuming PFA ([1]).

The development of these results parallels those in the study of the automorphisms of the Boolean algebra $\mathcal{P}(\omega)/\text{fin}$. Rudin ([5]) discovered early on that, assuming CH, there are many automorphisms of $\mathcal{P}(\omega)/\text{fin}$ that are not *trivial*, i.e. induced by functions $e : \omega \to \omega$; Shelah ([6]) much later proved the consistency of the opposite result, that all automorphisms are trivial. Shelah and Steprans then showed that all automorphisms are trivial

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assuming PFA ([7]), and then Veličković showed using PFA that all automorphisms of $\mathcal{P}(\kappa)/\mathrm{fin}$ are trivial, for every infinite cardinal κ (along with reducing the assumption to $\mathrm{TA} + \mathrm{MA}_{\aleph_1}$ in the original case $\kappa = \omega$).

One might ask for the consistency of outer automorphisms of $\mathcal{C}(\mathcal{H})$ when $\mathcal{H}$ is nonseparable, or nontrivial automorphisms of $\mathcal{P}(\kappa)/\mathrm{fin}$ when κ is uncountable. The latter result is easy, though for trivial reasons, since the automorphisms of $\mathcal{P}(\omega)/\mathrm{fin}$ can all be extended to automorphisms of $\mathcal{P}(\kappa)/\mathrm{fin}$, and any extension of a nontrivial automorphism of $\mathcal{P}(\omega)/\mathrm{fin}$ must also be nontrivial. In the case of $\mathcal{C}(\mathcal{H})$ this is not so clear, and in fact it is not yet known whether the existence of an outer automorphism of $\mathcal{C}(\mathcal{H})$, when $\mathcal{H}$ is nonseparable, is consistent with ZFC. However in the case where $\mathcal{H}$ is nonseparable there is more than one quotient of $\mathcal{B}(\mathcal{H})$ to consider. In this note we study some of these different quotients, and offer some alternatives;

Theorem 1.1. *Let $\mathcal{H}$ be a Hilbert space of some regular, uncountable dimension κ and let $\mathcal{J}$ be the ideal in $\mathcal{B}(\mathcal{H})$ of operators whose range has dimension less than κ. If $2^\kappa = \kappa^+$, then the quotient $\mathcal{B}(\mathcal{H})/\mathcal{J}$ has 2^{κ^+}-many outer automorphisms.*

Theorem 1.2. *Let $\mathcal{H}$ be a Hilbert space of dimension $\aleph_1$, let $\mathcal{J}$ be the ideal of operators on $\mathcal{H}$ whose range has dimension $< \aleph_1$, and let $\mathcal{K}$ be the ideal of compact operators. If CH holds, then $\mathcal{J}/\mathcal{K}$ has $2^{\aleph_1}$-many outer automorphisms.*

Theorem 1.1 is perhaps most striking in the case $\kappa = \aleph_1$, for in this case its only set-theoretic assumption, $2^{\aleph_1} = \aleph_2$, follows already from PFA. Hence in a model of PFA, there are many outer automorphisms of $\mathcal{B}(\mathcal{H})/\mathcal{J}$, and yet no outer automorphisms of $\mathcal{B}(\mathcal{H})/\mathcal{K}$.

Our notation is mostly standard. All Hilbert spaces considered are complex Hilbert spaces. When $\mathcal{H}$ is a Hilbert space, $\mathcal{B}(\mathcal{H})$ denotes the C*-algebra of bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$, $\mathcal{K}(\mathcal{H})$ denotes the closed $*$-ideal in $\mathcal{B}(\mathcal{H})$ given by the compact operators on $\mathcal{H}$, and $\mathcal{J}(\mathcal{H})$ denotes the $*$-ideal of operators whose range has dimension strictly less than the dimension of $\mathcal{H}$. When the Hilbert space $\mathcal{H}$ is understood we will often drop it in our notation and just use $\mathcal{B}, \mathcal{K}$, and $\mathcal{J}$. Note that when $\mathcal{H}$ is nonseparable, $\mathcal{J}$ is already norm-closed, and

$$\mathcal{K} \subset \mathcal{J} \subset \mathcal{B}$$

If $x \in \mathcal{B}$ then we will use $[x]_\mathcal{K}$ and $[x]_\mathcal{J}$ to denote the quotients of x by $\mathcal{K}(\mathcal{H})$ and $\mathcal{J}(\mathcal{H})$ respectively. When A is a set, we will write $\ell^2(A)$ for the Hilbert space of square-summable functions $\xi : A \to \mathbb{C}$. We will also often write $\mathcal{B}(\mathcal{H}) = \mathcal{B}_A$, $\mathcal{J}(\mathcal{H}) = \mathcal{J}_A$, and $\mathcal{K}(\mathcal{H}) = \mathcal{K}_A$ when $\mathcal{H} = \ell^2(A)$. When $A \subseteq B$ we will identify $\ell^2(A)$ with a closed subspace of $\ell^2(B)$ in the obvious way. Finally, if A is a C*-algebra and x is an element of A then $\mathrm{Ad}\, x : A \to A$ is the map $a \mapsto xax^*$. When A has a multiplicative unit and x is a unitary element of A, i.e. $x^*x = xx^* = 1_A$, then $\mathrm{Ad}\, x$ is an automorphism of A.

2. Large ideals

In this section we prove Theorem 1.1. Before beginning the proof we will need some notation;

Definition 2.1. *If C is club in κ, we define*

$$x \in \mathcal{D}[C] \iff \forall \alpha \in C \quad \ell^2(\alpha) \text{ is an invariant subspace of } x \text{ and } x^*$$

Note that $\mathcal{D}[C]$ is a C*-subalgebra of $\mathcal{B}_\kappa$, and in fact is a von Neumann subalgebra of $\mathcal{B}_\kappa$, though we will not use this latter fact. We also set down some convenient notation for the successor of an ordinal in a club;

Definition 2.2. *If C is club in κ and $\alpha \in C$, then $\mathrm{succ}_C(\alpha)$ denotes the minimal element of C strictly greater than α.*

Note that if C is club in κ, then we have in fact

$$x \in \mathcal{D}[C] \iff \forall \alpha \in C \quad \ell^2([\alpha, \mathrm{succ}_C(\alpha))) \text{ is an invariant subspace of } x$$

Finally, if $A, B \subseteq \kappa$ then we write $A \subseteq^* B$ if and only if $|A \setminus B| < \kappa$.

Lemma 2.3. *For every $x \in \mathcal{B}_\kappa$, there is some club C in κ such that $x \in \mathcal{D}[C]$.*

Proof. Let θ be large and regular, and let M_α, for $\alpha < \kappa$, be a club of elementary substructures of $H(\theta)$, each of size $< \kappa$, and with x and $\ell^2(\kappa)$ in M_0. Then if $\delta = \sup(M_\alpha \cap \kappa)$, we clearly have that $\ell^2(\delta)$ is an invariant subspace of x, and such ordinals δ make up a club in κ. □

Lemma 2.4. *If $C \subseteq^* \widetilde{C}$ are clubs in κ, then $\mathcal{D}[\widetilde{C}] \subseteq_{\mathcal{J}} \mathcal{D}[C]$, by which we mean*

$$\forall x \in \mathcal{D}[\widetilde{C}]\, \exists y \in \mathcal{D}[C] \quad x - y \in \mathcal{J}$$

Proof. If $\gamma < \kappa$ is such that $C \cap [\gamma, \kappa) \subseteq \widetilde{C}$, then for every $\delta \in \widetilde{C}$,

$$\delta \geq \gamma \implies [\delta, \mathrm{succ}_{\widetilde{C}}(\delta)) \subseteq [\delta, \mathrm{succ}_C(\delta))$$

Thus if $x \in \mathcal{D}[\widetilde{C}]$, we see that $PxP \in \mathcal{D}[C]$, where P is the projection onto the subspace $\ell^2([\gamma, \kappa))$. □

Lemma 2.5. *Let C be club in κ and let u and v be unitary operators on $\ell^2(\kappa)$, which are diagonal with respect to the standard basis; say $f, g : \kappa \to \mathbb{T}$ are the diagonal values of u and v respectively. Then $\mathrm{Ad}\,[u]_{\mathcal{J}}$ and $\mathrm{Ad}\,[v]_{\mathcal{J}}$ agree on $\mathcal{D}[C]/\mathcal{J}$ if and only if there is some $\epsilon < \kappa$ such that the map*

$$\xi \mapsto \frac{f(\xi)}{g(\xi)} = f(\xi)\overline{g(\xi)}$$

is constant on each interval of the form $[\delta, \mathrm{succ}_C(\delta))$ with $\delta \in C \cap [\epsilon, \kappa)$.

Proof. Let $h(\xi) = f(\xi)\overline{g(\xi)}$ for each $\xi < \kappa$. We will write $(*)$ for the condition

$$\exists \epsilon\, \forall \delta \in C \quad \delta \geq \epsilon \implies h \text{ is constant on the interval } [\delta, \mathrm{succ}_C(\delta))$$

as in the conclusion of the lemma. Now, note that u and v are trivially in the algebra $\mathcal{D}[C]$. The following are equivalent;

(1) $\mathrm{Ad}\,[u]_{\mathcal{J}}$ and $\mathrm{Ad}\,[v]_{\mathcal{J}}$ agree on $\mathcal{D}[C]/\mathcal{J}$,
(2) for each $x \in \mathcal{D}[C]$, $uxu^* - vxv^*$ is in $\mathcal{J}$,
(3) $[v^*u]_{\mathcal{J}}$ is in the center of the algebra $\mathcal{D}[C]/\mathcal{J}$.

We will show that condition (3) holds if and only if $(*)$ holds. First suppose $(*)$ does not hold; then there is an unbounded subset A of C, and sequences σ_δ,τ_δ indexed by $\delta \in A$, such that for each $\delta \in A$, $\delta \leq \sigma_\delta < \tau_\delta < \mathrm{succ}_C(\delta)$ and $h(\sigma_\delta) \neq h(\tau_\delta)$. Let x be the operator defined by

$$x(e_\alpha) = \begin{cases} e_{\sigma_\delta} & \alpha = \tau_\delta \text{ for some } \delta \in A \\ e_{\tau_\delta} & \alpha = \sigma_\delta \text{ for some } \delta \in A \\ 0 & \text{otherwise} \end{cases}$$

Then $x \in \mathcal{D}[C]$, and for each $\delta \in A$,

$$(v^*ux)e_{\sigma_\delta} = h(\tau_\delta)e_{\tau_\delta} \qquad (xv^*u)e_{\sigma_\delta} = h(\sigma_\delta)e_{\tau_\delta}$$

It follows that $v^*ux - xv^*u$ is not in the ideal $\mathcal{J}$, so condition (3) does not hold. Now suppose $(*)$ does hold, and choose ϵ as in this condition. If $x \in \mathcal{D}[C]$, then for all $\alpha \geq \epsilon$, if $\alpha \in [\delta, \mathrm{succ}_C(\delta))$ where $\delta \in C$ then we have

$$(v^*ux)e_\alpha = h(\alpha)xe_\alpha = (xv^*u)e_\alpha$$

and it follows that $P(v^*u)P$ is in the center of $\mathcal{D}[C]$, where P is the projection onto $\ell^2([\epsilon, \kappa))$. □

We are now ready to prove Theorem 1.1.

Proof. Let $\langle E_\alpha \mid \alpha \in \lim(\kappa^+)\rangle$ enumerate the clubs in κ. We will construct a sequence of clubs C_s in κ, and functions $f_s : \kappa \to \mathbb{T}$, indexed by $s \in 2^{<\kappa^+}$, such that

(1) If $s \subset t$, then $C_t \subseteq^* C_s$.
(2) If $s \subset t$, then there is an $\epsilon < \kappa$ such that for every $\delta \in C_s$ with $\delta \geq \epsilon$, the function $f_s\overline{f_t}$ is constant on the interval $[\delta, \mathrm{succ}_{C_s}(\delta))$.
(3) For all s, $C_{s^\frown 0} = C_{s^\frown 1}$, and for unboundedly many $\delta \in C_{s^\frown 0} = C_{s^\frown 1} = C$, the function $f_{s^\frown 0}\overline{f_{s^\frown 1}}$ is not constant on $[\delta, \mathrm{succ}_C(\delta))$.
(4) If s has length some limit ordinal $\alpha < \kappa^+$, then $C_s \subseteq^* E_\alpha$.

Claim 2.6. *This suffices.*

Proof. For each $s \in 2^{<\kappa^+}$, let u_s be the diagonal unitary in $\mathcal{B}_\kappa$ with diagonal elements given by f_s. For each $\zeta \in 2^{\kappa^+}$, and $x \in \mathcal{D}[C_{\zeta\restriction\alpha}]$, define

$$\Phi_\zeta([x]) = [u_{\zeta\restriction\alpha} x u^*_{\zeta\restriction\alpha}]$$

By (1), (2), and Lemma 2.5, Φ_ζ is well-defined on the union of the algebras $\mathcal{D}[C_{\zeta\restriction\alpha}]/\mathcal{J}$, over $\alpha < \kappa^+$; and by (4), and Lemma 2.3, it follows that Φ_ζ is defined on all of $\mathcal{B}_\kappa/\mathcal{J}$. Since on each $\mathcal{D}[C_{\zeta\restriction\alpha}]$, Φ_ζ agrees with $\mathrm{Ad}\,[u_{\zeta\restriction\alpha}]$, Φ_ζ is also an injective homomorphism. Similar arguments show that Φ_ζ^{-1} is defined on all of $\mathcal{B}_\kappa/\mathcal{J}$, and hence Φ_ζ is an automorphism of this quotient algebra. Finally, if ζ and η are distinct members of 2^{κ^+}, then by (3) and Lemma 2.5 we see that Φ_ζ and Φ_η are distinct automorphisms. □

We construct C_s and f_s by induction on the length of $s \in 2^{<\kappa^+}$. It is useful to note that all the functions f_s constructed in the following actually have range contained in $\{-1, +1\}$; when proving (2) and (3), then, we will drop all mention of the conjugation. In the base case we simply set $C_{\langle\rangle} = \kappa$ and $f_{\langle\rangle}(\alpha) = 1$ for all $\alpha < \kappa$. For the successor case, let $s \in 2^{<\kappa^+}$ be given. Set $C_{s^\frown 0} = C_{s^\frown 1} = \lim(C_s)$, $f_{s^\frown 0} = f_s$, and

$$f_{s^\frown 1}(\alpha) = \begin{cases} -f_s(\alpha) & \text{if there is } \delta \in \lim(C_s) \text{ such that } \delta \le \alpha < \mathrm{succ}_{C_s}(\delta) \\ +f_s(\alpha) & \text{otherwise} \end{cases}$$

Obviously, the function $f_s f_{s^\frown 0} = f_s^2$ is constant on each interval of C_s (in fact it is constant on all of κ). The same holds for the function $f_s f_{s^\frown 1}$; if $\delta \in \lim(C_s)$ then this function has a constant value of -1 on all of $[\delta, \mathrm{succ}_{C_s}(\delta))$, whereas if $\delta \in C_s \setminus \lim(C_s)$ then it has a constant value of $+1$ on this interval. Hence condition (2) is satisfied in the inductive step. As for condition (3), we note that for every $\delta \in \lim(C_s)$, the function $f_{s^\frown 0} f_{s^\frown 1}$ is not constant on the interval $[\delta, \mathrm{succ}_{\lim(C_s)}(\delta))$, since this function has a value of -1 at δ and a value of $+1$ at $\mathrm{succ}_{C_s}(\delta) < \mathrm{succ}_{\lim(C_s)}(\delta)$. It remains to consider the limit case. Let $s \in 2^{<\kappa^+}$ be given, and let α be the length of s. For $\beta < \alpha$, write $f_\beta = f_{s \restriction \beta}$ and $C_\beta = C_{s \restriction \beta}$. By the inductive hypothesis, for every $\beta < \gamma < \alpha$ there is an $\epsilon < \kappa$ such that

$$\forall \delta \in C_\beta \quad \delta \ge \epsilon \implies f_\beta f_\gamma \text{ is constant on the interval } [\delta, \mathrm{succ}_{C_\beta}(\delta))$$

Let ϵ_β^γ be the minimal $\epsilon \in C_\beta$ satisfying the above. We will define f_s and C_s in two different ways based on the cofinality of α. First, suppose $\theta = \mathrm{cf}\,\alpha < \kappa$, and let α_η, for $\eta < \theta$, be an increasing and continuous sequence of ordinals which is cofinal in α. Define

$$C_s = \left(\bigcap_{\eta < \theta} C_{\alpha_\eta} \right) \cap E_\alpha$$

It remains to define f_s and show that condition 2 holds. Choose a uniform ultrafilter $\widetilde{\mathcal{U}}$ over θ, and let $\mathcal{U}_\alpha$ be the ultrafilter over α defined in the usual way from $\widetilde{\mathcal{U}}$ using the sequence $\langle \alpha_\eta \mid \eta < \theta \rangle$. Now for each $\xi < \kappa$ define

$$f_s(\xi) = \lim_{\beta \in \mathcal{U}_\alpha} f_\beta(\xi)$$

Claim 2.7. *For every $\beta < \alpha$, $f_\beta f_s$ is constant on each interval of a tail of intervals from C_β.*

Proof. Fix $\beta < \alpha$, and let $\epsilon = \sup_{\eta < \theta} \epsilon_\beta^{\alpha_\eta} \in C_\beta$. Let $\delta \in C_\beta$ be given, and suppose $\delta \ge \epsilon$, but that $f_\beta f_s$ is *not* constant on $[\delta, \mathrm{succ}_{C_\beta}(\delta))$; fix witnesses $\sigma < \tau$ in this interval, and say without loss of generality that $f_\beta(\sigma) f_s(\sigma) = +1$ but $f_\beta(\tau) f_s(\tau) = -1$. By the definition of f_s, there are

$A_0, A_1 \in \mathcal{U}_\alpha$ such that

$$\forall \gamma \in A_0 \quad f_\beta(\sigma) f_\gamma(\sigma) = +1$$
$$\forall \gamma \in A_1 \quad f_\beta(\tau) f_\gamma(\tau) = -1$$

Then if $\gamma \in A_0 \cap A_1$ is larger than β we have $f_\beta(\sigma) f_\gamma(\sigma) = +1$ and $f_\beta(\tau) f_\gamma(\tau) = -1$. By definition of $\mathcal{U}_\alpha$ we may choose such a γ with $\gamma = \alpha_\eta$ for some $\eta < \theta$. But this contradicts the choice of ϵ_β^γ, since $\delta \geq \epsilon > \epsilon_\beta^{\alpha_\eta}$. □

Now consider the case where $\operatorname{cf} \alpha = \kappa$. Let α_η, $\eta < \kappa$, be a continuous, increasing sequence of ordinals which is cofinal in α. Put

$$C_s = \left(\Delta_{\eta < \kappa} C_{\alpha_\eta} \right) \cap E_\alpha$$

Again, it remains only to define f_s and show that condition (2) holds. For this we define, for $\xi < \eta$,

$$\rho_\xi^\eta = \min(C_{\alpha_\xi} \setminus (\xi \cup \epsilon_{\alpha_\xi}^{\alpha_\eta}))$$

and

$$\epsilon(\eta) = \sup_{\xi < \eta} \rho_\xi^\eta$$

Note that $\epsilon(\eta)$ is in C_{α_ξ} for each $\xi < \eta$. Define $f_s(\zeta) = f_{\alpha_\eta}(\zeta)$ whenever $\epsilon(\eta) \leq \zeta < \epsilon(\eta + 1)$ for some $\eta < \kappa$, that is,

$$f_s = \bigcup_{\eta < \kappa} f_{\alpha_\eta} \upharpoonright [\epsilon(\eta), \epsilon(\eta + 1))$$

Claim 2.8. *For every $\beta < \alpha$, $f_\beta f_s$ is constant on a tail of intervals from C_β.*

Proof. We will first prove that $f_{\alpha_\xi} f_s$ is constant on a tail of intervals from C_{α_ξ}, for each $\xi < \kappa$. Let $\epsilon = \epsilon(\xi + 1)$; then if $\delta \in C_{\alpha_\xi}$ and $\delta \geq \epsilon$, we have $\epsilon(\eta) \leq \delta < \operatorname{succ}_{C_{\alpha_\xi}}(\delta) \leq \epsilon(\eta + 1)$ for some $\eta > \xi$. Hence f_s is equal to f_{α_η} on the interval $[\delta, \operatorname{succ}_{C_{\alpha_\xi}}(\delta))$. Since $\delta \geq \epsilon(\eta) \geq \epsilon_{\alpha_\xi}^{\alpha_\eta}$, we see that $f_{\alpha_\xi} f_s$ is constant on this interval, as required.

Now let $\beta < \alpha$ be given, and choose $\xi < \kappa$ such that $\beta < \alpha_\xi$. By the above, there is an ϵ_0 such that $f_{\alpha_\xi} f_s$ is constant on each interval of C_{α_ξ} beyond ϵ_0. Let $\epsilon_1 = \epsilon_\beta^{\alpha_\xi}$, and choose an ϵ_2 such that $C_{\alpha_\xi} \cap [\epsilon_2, \kappa) \subseteq C_\beta$. It follows that with $\epsilon = \max\{\epsilon_0, \epsilon_1, \epsilon_2\}$ we have

$$\forall \delta \in C_\beta \quad \delta \geq \epsilon \implies f_\beta f_s \text{ is constant on the interval } [\delta, \operatorname{succ}_{C_\beta}(\delta))$$

□

Thus we have proven condition (2) in this case, and this finishes the proof of the theorem. □

3. Small ideals

In this section we work with the Hilbert space $\mathcal{H} = \ell^2(\omega_1)$. Hence the ideals of $\mathcal{B}(\mathcal{H})$ are exactly

$$0 \subset \mathcal{K} \subset \mathcal{J} \subset \mathcal{B}$$

Letting $\mathcal{C}(\mathcal{L})$ denote the usual Calkin algebra over $\mathcal{L}$, i.e. $\mathcal{B}(\mathcal{L})/\mathcal{K}(\mathcal{L})$, it follows that

$$\mathcal{J}/\mathcal{K} = \bigcup_{\alpha<\omega_1} \mathcal{C}(\ell^2(\alpha)) \subset \mathcal{C}(\ell^2(\omega_1))$$

We will shortly prove Theorem 1.2, in a slightly stronger form; namely, assuming CH, there is an automorphism Ψ of the quotient $\mathcal{J}/\mathcal{K}$ whose restriction to each subalgebra $\mathcal{C}(\ell^2(\alpha))$ is an outer automorphism. It follows also that Ψ cannot be the restriction of an inner automorphism of $\mathcal{B}/\mathcal{K}$. Before we start, we will need a special case of Lemma 4.1 from [2]. We include its proof here for completeness.

Lemma 3.1. *Let Φ be an automorphism of $\mathcal{C}(\mathcal{H})$, where $\mathcal{H}$ is any Hilbert space. Then Φ is inner if and only if it is inner on some subspace $\mathcal{L}$ of $\mathcal{H}$ of the same dimension.*

Proof. Let $\mathcal{L}$ be a subspace of $\mathcal{H}$ of the same dimension. Then there is an isometry $U : \mathcal{H} \to \mathcal{L}$; let u be its image in $\mathcal{C}(\mathcal{H})$. Suppose Φ is implemented by conjugation by v on $\mathcal{C}(\mathcal{L})$; then for any $x \in \mathcal{C}(\mathcal{H})$,

$$\Phi(x) = \Phi(u^*uxu^*u) = \Phi(u)^*v\Phi(u)x\Phi(u)^*v^*\Phi(u)$$

and hence Φ is implemented by conjugation by $\Phi(u)^*v\Phi(u)$ on all of $\mathcal{C}(\mathcal{H})$. □

Theorem 3.2. *Assume CH. Then there are $2^{\aleph_1}$-many outer automorphisms of $\mathcal{J}/\mathcal{K}$. Moreover, each of these automorphisms is outer in a strong sense, namely each is outer when restricted to any $\mathcal{C}(\ell^2(\alpha))$, $\alpha < \omega_1$.*

Proof. Let Φ be an automorphism of $\mathcal{C}(\ell^2(\omega))$. Let $f_\alpha : \alpha \to \omega$, $\alpha < \omega_1$, be a sequence of injections satisfying

$$(1) \qquad \forall \alpha < \beta < \omega_1 \quad f_\beta \upharpoonright \alpha =^* f_\alpha$$

for every $\alpha < \beta < \omega_1$. Set $A_\alpha = \operatorname{ran}(f_\alpha)$, let $U_\alpha : \ell^2(\alpha) \to \ell^2(A_\alpha)$ be the unitary operator induced by f_α, and let u_α be its image in $\mathcal{C}(\ell^2(\omega_1))$. Let Ψ be the unique automorphism of $\mathcal{J}/\mathcal{K}$ such that

$$\forall \alpha < \omega_1 \quad (\operatorname{Ad} u_\alpha) \circ \Psi = \Phi \circ (\operatorname{Ad} u_\alpha^*)$$

Condition (1) ensures that such a Ψ exists, and verifying that Ψ is an automorphism is straightforward. Lemma 3.1 implies that if Φ is outer, then Φ is also outer on every $\ell^2(A_\alpha)$, and hence Ψ is outer on every $\ell^2(\alpha)$. By the main result of [4], there are $2^{\aleph_1}$-many outer automorphisms of $\mathcal{C}(\ell^2(\omega))$, and hence $2^{\aleph_1}$-many outer automorphisms of $\mathcal{J}/\mathcal{K}$.

□

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