On a problem of Erdős and Rothschild on edges in triangles

Jacob Fox[*] Po-Shen Loh[†]

Abstract

Erdős and Rothschild asked to estimate the maximum number, denoted by $h(n, c)$, such that every n-vertex graph with at least cn^2 edges, each of which is contained in at least one triangle, must contain an edge that is in at least $h(n, c)$ triangles. In particular, Erdős asked in 1987 to determine whether for every $c > 0$ there is $\epsilon > 0$ such that $h(n, c) > n^\epsilon$ for all sufficiently large n. We prove that $h(n, c) = n^{O(1/\log\log n)}$ for every fixed $c < 1/4$. This gives a negative answer to the question of Erdős, and is best possible in terms of the range for c, as it is known that every n-vertex graph with more than $n^2/4$ edges contains an edge that is in at least $n/6$ triangles.

1 Introduction

A *book* of size h in a graph is a collection of h triangles that share a common edge. The *booksize* of a graph G is the size of the largest book in G. The study of books in graphs was started by Erdős [5] in 1962, and has since attracted a great deal of attention in extremal graph theory (see, e.g., [2, 9, 10, 13]) and graph Ramsey theory (see, e.g., [11, 14, 15, 16, 17, 18, 20]).

Erdős and Rothschild [6] initiated the study of the booksize of graphs with the property that every edge is in a triangle. Let $h(n, c)$ be the largest integer such that every n-vertex graph with at least cn^2 edges, each of which is contained in at least one triangle, must contain an edge that is in at least $h(n, c)$ triangles. Erdős and Rothschild asked to estimate $h(n, c)$ for fixed $c > 0$. This question has received considerable attention (see, e.g., the Erdős problem papers [6, 7, 8], and the book [3]).

Using his regularity lemma, Szemerédi showed that for every $c > 0$, $h(n, c) \to \infty$ as $n \to \infty$, and we will outline this argument at the end of the introduction. This fact has a number of applications to various problems in extremal combinatorics. Ruzsa and Szemerédi [19] showed that the statement $h(n, c) > 1$ for every fixed $c > 0$ and sufficiently large n implies Roth's theorem: that every subset of the first n positive integers without a 3-term arithmetic progression has size $o(n)$. They also showed that it is equivalent to the $(6, 3)$-theorem: that every 3-uniform hypergraph on n vertices in which the union of any 3 edges contains more than 6 vertices has $o(n^2)$ edges. In the other direction, Alon and Trotter (see [8]) proved that for each $c < 1/4$ there is $c' > 0$ such that $h(n, c) < c'\sqrt{n}$. The condition $c < 1/4$ is best possible, because independent results of Edwards

[*]Department of Mathematics, MIT, Cambridge, MA 02139-4307. Email: fox@math.mit.edu. Research supported by a Simons Fellowship and NSF grant DMS-1069197.

[†]Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213. Email: ploh@cmu.edu. Research supported by an NSA Young Investigators Grant.

[4] and Khadžiivanov and Nikiforov [13] state that any n-vertex graph with more than $n^2/4$ edges contains an edge in at least $n/6$ triangles. In particular, this implies for $c > 1/4$, we must have $h(n,c) \geq n/6$.

For over two decades, there was no improvement on the $O(\sqrt{n})$ upper bound for any fixed $c < 1/4$. Indeed, Erdős even proposed that perhaps the lower bound should be improved to a power of n. Specifically, in 1987 he asked in [6] whether there is a constant $\epsilon > 0$ such that $h(n,c) > n^\epsilon$ for every fixed $c > 0$ and all sufficiently large n. This question was also featured in the book *Erdős on Graphs* [3]. We give a negative answer to this question. In fact, Theorem 1.1 below implies that $h(n,c) = n^{o(1)}$ for every fixed $c < 1/4$. By the above remark that $h(n,c) \geq n/6$ for $c > 1/4$, this gives a best possible range for c with this bound and shows that a sharp transition occurs when c is near $1/4$. All logarithms in this paper are in base $e \approx 2.718$.

Theorem 1.1. *For all sufficiently large n, there are n-vertex graphs with $\frac{n^2}{4}\left(1 - e^{-(\log n)^{1/6}}\right)$ edges, with the property that every edge is in a triangle, but no edge is in more than $n^{14/\log\log n}$ triangles.*

The study of $h(n,c)$ with c near $1/4$ began in the problem papers of Erdős [7, 8]. Let f be such that $cn^2 = n^2/4 - f(n)n$. Erdős [7] proved if f is constant, then $h(n,c) = \Omega(n)$. Bollobás and Nikiforov [2] further showed that $h(n,c)$ is asymptotically $n/6$ if $f \to 0$. If f tends to infinity with n, but not too quickly, so that $f(n) < n^{2/5}$, they showed that $h(n,c)$ is asymptotically $\frac{n}{2\sqrt{2f(n)}}$. Note that Theorem 1.1 shows that this behavior cannot continue when $f(n)$ approaches linearity in n. In fact, similar constructions, which we omit, show that there are positive absolute constants α, ϵ such that $h(n,c) = O(n^{1/2-\epsilon})$ where $f(n) = n^{1-\alpha}$. This shows that the asymptotic behavior of $h(n,c)$ discovered by Bollobás and Nikiforov with c very near $1/4$ already breaks down when $f(n)$ is some power of n which is less than 1.

We close the introduction by discussing lower bounds on $h(n,c)$ for fixed $c > 0$. The fact that $h(n,c)$ tends to infinity follows from the triangle removal lemma, which is a consequence of Szemerédi's regularity lemma. The triangle removal lemma states that for each $\epsilon > 0$ there is $\delta > 0$ such that every graph on n vertices with at most δn^3 triangles can be made triangle-free by removing at most ϵn^2 edges. Let G be a graph on n vertices, $c'n^2$ edges with $c' \geq c$ such that every edge is in at least one triangle and at most $h(n,c)$ triangles. If the total number of triangles is over δn^3, since each triangle contains three edges, the pigeonhole principle already gives an edge in at least $3 \cdot \frac{\delta n^3}{c'n^2} = \frac{3\delta n}{c'}$ triangles. On the other hand, if the graph has fewer than δn^3 triangles, the triangle removal lemma gives ϵn^2 edges which capture all of the triangles. Since each edge is on a triangle, the total number of triangles is at least $\frac{c'n^2}{3}$. Hence, one of those edges is on at least $\frac{c'n^2}{3}/(\epsilon n^2) = \frac{c'}{3\epsilon}$ triangles. Therefore, $h(n,c) \geq \min\left\{\frac{3\delta n}{c'}, \frac{c'}{3\epsilon}\right\}$.

The regularity proof gives a bound for δ^{-1} in the triangle removal lemma which is a tower of twos of height a power of ϵ^{-1}. Using this, one can set δ^{-1} to be $\sqrt{n}$, say, with ϵ^{-1} of order a power of the iterated logarithm $\log^* n$, implying that $h(n,c)$ is at least a power of the iterated logarithm $\log^* n$. Recently, the first author [12] gave a new proof of the triangle removal lemma which avoids Szemerédi's regularity lemma and gives a better bound. Namely, in the triangle removal lemma, we can take δ^{-1} to be a tower of twos of height logarithmic in ϵ^{-1}. This gives a lower bound for $h(n,c)$ which is exponential in $\log^* n$.

2 Tools

The properties of our construction are essentially derived from the concentration of measure. Say that a random variable $X(\omega)$ on an n-dimensional product space $\Omega = \prod_{i=1}^{n} \Omega_i$ is *C-Lipschitz* if changing ω in any single coordinate affects the value of $X(\omega)$ by at most C. The Hoeffding-Azuma inequality (see, e.g., [1]) provides concentration for these distributions.

Theorem 2.1 (Hoeffding-Azuma Inequality)**.** *Let X be a C-Lipschitz random variable on an n-dimensional product space. Then for any $t \geq 0$,*

$$\mathbb{P}\left[|X - \mathbb{E}\left[X\right]| > t\right] \leq 2\exp\left\{-\frac{t^2}{2C^2 n}\right\}.$$

We also need the following well-known formula for the volume of a high-dimensional Euclidean ball. The formula is slightly different for even and odd dimensions. Since our analysis is asymptotic in nature, it suffices to consider only even dimensions (which yield simpler forms).

Theorem 2.2. *For a positive even integer d and a positive real number r, the volume of $B_r^{(d)}$, the d-dimensional Euclidean ball with radius r, is*

$$Vol\left(B_r^{(d)}\right) = \frac{\pi^{d/2} r^d}{(d/2)!}.$$

The following weaker estimate turns out to be more convenient for our analysis.

Corollary 2.3. *For a positive even integer d and a positive real number r,*

$$Vol\left(B_r^{(d)}\right) < (2\pi e)^{d/2} \cdot \frac{r^d}{d^{d/2}}.$$

The desired bound in the corollary follows from the standard estimate $d! > \left(\frac{d}{e}\right)^d$, which is routinely obtained by bounding $\log(d!) = \sum_{i=1}^{d} \log i > \int_1^d \log x \, dx$.

3 Construction

We first describe a graph which almost has the desired properties. Specifically, no edge will be in many triangles, and the number of edges will be quadratic in the number of vertices, but some edges may fail to be in triangles. Throughout this section, we will write $x = y \pm \delta$ or x is in $y \pm \delta$ to denote $y - \delta \leq x \leq y + \delta$.

Pre-Construction. For a positive even integer $r > 2$, let $d = r^5$, let $n = r^d$, and let $\mu = \frac{r^2-1}{6} \cdot d$. Consider the tripartite graph with vertex set $A \cup B \cup C$, where each of A and B are copies of $[r]^d$, and $C = \{0, 1, \ldots, r+1\}^d$. Vertices $a \in A$ and $b \in B$ are joined by an edge if and only if (when considered as lattice points in $[r]^d$) their distance satisfies $\|a - b\|_2^2 = \mu \pm d$. Similarly, vertices $b \in B$ and $c \in C$ are adjacent if and only if $\|b - c\|_2^2 = \frac{\mu}{4} \pm 2d$. Finally, $c \in C$ and $a \in A$ are adjacent if and only if $\|c - a\|_2^2 = \frac{\mu}{4} \pm 2d$.

The following lemma will help us to show that the bipartite graph between A and B is nearly complete.

Lemma 3.1. *Let r and d be given integers, and let U and V be two lattice points sampled independently and uniformly at random from $[r]^d$. Define*

$$\mu = \frac{r^2 - 1}{6} \cdot d\,.$$

Then with probability at least $1 - 2e^{-\frac{d}{2r^4}}$, $\|U - V\|_2^2 = \mu \pm d$.

Proof. Let $U = (U_1, \ldots, U_d)$ and $V = (V_1, \ldots, V_d)$. The squared L_2 distance is precisely $\sum_i (U_i - V_i)^2$, which is a sum of d independent random variables. A simple calculation shows that

$$\mathbb{E}\left[(U_1 - V_1)^2\right] = \mathbb{E}\left[U_1^2\right] - 2\mathbb{E}\left[U_1\right]\mathbb{E}\left[V_1\right] + \mathbb{E}\left[V_1^2\right] = 2\left(\mathbb{E}\left[U_1^2\right] - \mathbb{E}\left[U_1\right]^2\right).$$

Since U_1 is an integer picked uniformly at random from $[r]$, then $\mathbb{E}\left[U_1\right] = \frac{r+1}{2}$ while

$$\mathbb{E}\left[U_1^2\right] = \frac{1}{r} \cdot \frac{r(r+1)(2r+1)}{6} = \frac{(r+1)(2r+1)}{6}\,,$$

so

$$\mathbb{E}\left[(U_1 - V_1)^2\right] = \frac{r^2 - 1}{6}\,,$$

and hence

$$\mathbb{E}\left[\|U - V\|_2^2\right] = \frac{r^2 - 1}{6} \cdot d\,.$$

On the other hand, each $(U_i - V_i)^2$ is less than r^2, so by the Hoeffding-Azuma inequality (Theorem 2.1), the probability that $\|U - V\|_2^2$ deviates from its expectation by more than d is at most

$$2e^{-\frac{d^2}{2r^4 d}} = 2e^{-\frac{d}{2r^4}}\,,$$

as claimed. □

Next, we show that every edge between A and B is in a positive number of triangles, but not too many.

Lemma 3.2. *In the Pre-Construction, the number of edges that join A and B is at least $(1 - 2e^{-\frac{d}{2r^4}})n^2$, and every one of those edges is contained in between 2^{d-1} and 15^d triangles.*

Proof. The first claim is an immediate consequence of the previous lemma. We then move to establish a lower bound on the number of triangles that contain a given edge ab. By definition, we have $\|a - b\|_2^2 = \mu \pm d$. Let $m = (m_1, \ldots, m_d)$ denote the midpoint of $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$ when considered as points in $[r]^d$. Note that although a and b have integer coordinates, m may have half-integer coordinates. Let $x_i = b_i - a_i$; then $m_i - a_i = \frac{x_i}{2}$. For each i, if x_i is odd, define $\delta_i = \frac{1}{2}$, and if x_i is even, define $\delta_i = 1$. Consider lattice points c of the form $c_i = m_i + \delta_i \epsilon_i$, where $\epsilon_i \in \{\pm 1\}$. All such points still lie in C because $C = \{0, \ldots, r+1\}^d$. Then,

$$\|c - a\|_2^2 = \sum_i \left(\frac{x_i}{2} + \delta_i \epsilon_i\right)^2 = \frac{\|b - a\|_2^2}{4} + \sum_i \delta_i^2 + \sum_i x_i \delta_i \epsilon_i$$

$$\|b - c\|_2^2 = \sum_i \left(\frac{x_i}{2} - \delta_i \epsilon_i\right)^2 = \frac{\|b - a\|_2^2}{4} + \sum_i \delta_i^2 - \sum_i x_i \delta_i \epsilon_i\,.$$

Since $\|b-a\|_2^2 = \mu \pm d$ and $\sum_i \delta_i^2 \le d$, every choice of (ϵ_i) satisfying

$$\left|\sum_i x_i \delta_i \epsilon_i\right| \le \frac{3}{4} d$$

will produce a point $c \in C$ which is permissible as the third vertex of a triangle containing ab. (It would make $\|c-a\|_2^2$ and $\|b-c\|_2^2$ both in $\frac{\mu}{4} \pm 2d$.) Now consider the ϵ_i as independent uniform random variables over $\{\pm 1\}$, and define the random variable $Z = \sum_i x_i \delta_i \epsilon_i$. By symmetry, $\mathbb{E}[Z] = 0$, and since $|x_i| \le r$, changing the choice of a particular ϵ_i cannot affect Z by more than $2r$. Therefore, the Hoeffding-Azuma inequality (Theorem 2.1) gives

$$\mathbb{P}\left[|Z| > \frac{3}{4} d\right] < 2 \exp\left\{-\frac{\left(\frac{3}{4} d\right)^2}{2(2r)^2 d}\right\} < 2e^{-\frac{d}{15r^2}} ,$$

which implies that the number of valid points c is at least

$$\left(1 - 2e^{-\frac{d}{15r^2}}\right) \cdot 2^d > 2^{d-1} ,$$

as claimed.

For the upper bound, again assume that we are given a, b such that $\|a-b\|_2^2 = \mu \pm d$, and let $x_i = b_i - a_i$. We will bound the number of half-lattice points c of the form $c_i = a_i + \frac{x_i}{2} + \frac{w_i}{2}$, where $w_i \in \mathbb{Z}$, which satisfy $\|c-a\|_2^2 = \frac{\mu}{4} \pm 2d$ and $\|b-c\|_2^2 = \frac{\mu}{4} \pm 2d$. For this, observe that

$$\|c-a\|_2^2 = \sum_i \left(\frac{x_i}{2} + \frac{w_i}{2}\right)^2 = \frac{\|b-a\|_2^2}{4} + \frac{1}{4}\sum_i w_i^2 + \frac{1}{2}\sum_i x_i w_i$$

$$\|b-c\|_2^2 = \sum_i \left(\frac{x_i}{2} - \frac{w_i}{2}\right)^2 = \frac{\|b-a\|_2^2}{4} + \frac{1}{4}\sum_i w_i^2 - \frac{1}{2}\sum_i x_i w_i ,$$

so we always have

$$\|c-a\|_2^2 + \|b-c\|_2^2 = \frac{\|b-a\|_2^2}{2} + \frac{1}{2}\sum_i w_i^2 .$$

Hence whenever both $\|c-a\|_2^2$ and $\|b-c\|_2^2$ are in $\frac{\mu}{4} \pm 2d$, we also have $\sum_i w_i^2 \le 9d$. It therefore suffices to bound the number of lattice points in $B_{3\sqrt{d}}^{(d)}$, the d-dimensional Euclidean ball of radius $3\sqrt{d}$ centered at the origin. Observe that this is at most the volume of $B_{3.5\sqrt{d}}^{(d)}$, because by placing a unit d-dimensional cube centered at each lattice point in $B_{3\sqrt{d}}$, we obtain a non-overlapping collection of unit cubes all contained in the ball of radius $3\sqrt{d} + \frac{1}{2}\sqrt{d}$ by the triangle inequality (the greatest distance from the center of a unit cube to a point on its boundary is $\frac{1}{2}\sqrt{d}$).

Yet Corollary 2.3 bounds the volume of the d-dimensional Euclidean ball of radius $3.5\sqrt{d}$ by

$$(2\pi e)^{d/2} \cdot \frac{\left(3.5\sqrt{d}\right)^d}{d^{d/2}} < 15^d ,$$

as claimed. $\square$

Lemma 3.3. *In the Pre-Construction, every edge joining B and C, or joining A and C, is contained in at most 15^d triangles.*

Proof. Assume that we are given a, c such that $\|c-a\|_2^2 = \frac{\mu}{4} \pm 2d$, and let $y_i = c_i - a_i$. We will bound the number of lattice points b of the form $b_i = a_i + 2y_i + w_i$, where $w_i \in \mathbb{Z}$, which satisfy $\|b-c\|_2^2 = \frac{\mu}{4} \pm 2d$ and $\|b-a\|_2^2 = \mu \pm d$. For this, observe that

$$\|b-c\|_2^2 = \sum_i (y_i + w_i)^2 = \|c-a\|_2^2 + \sum_i w_i^2 + 2\sum_i y_i w_i$$
$$\|b-a\|_2^2 = \sum_i (2y_i + w_i)^2 = 4\|c-a\|_2^2 + \sum_i w_i^2 + 4\sum_i y_i w_i \,,$$

and hence

$$\|b-a\|_2^2 - 2\|b-c\|_2^2 = 2\|c-a\|_2^2 - \sum_i w_i^2 \,.$$

Therefore, the only way to have both $\|b-c\|_2^2 = \frac{\mu}{4} \pm 2d$ and $\|b-a\|_2^2 = \mu \pm d$ is to also have $\sum_i w_i^2 \leq 9d$. By the same computation as in the proof of the previous lemma, the number of such integral vectors (w_i) is less than 15^d. Hence, every edge between A and C is in at most 15^d triangles. By symmetry, every edge between B and C is also in at most 15^d triangles. □

We are now ready to prove the main theorem.

Proof of Theorem 1.1. Start with the Pre-Construction for a (sufficiently large) even integer r, with $d = r^5$ and $n = r^d = |A| = |B|$. Note that $n = d^{d/5}$, so $d = (1 + o(1))\frac{5 \log n}{\log \log n}$. We will take a random subgraph by sparsifying C. Let $C' \subset C$ with $|C'| = 2^{-d/2}|C|$ be picked uniformly at random.

Next, consider an edge ab joining A and B. By Lemma 3.2, in the Pre-Construction the edge ab was in at least 2^{d-1} triangles with vertices in C. Let E_{ab} be the event that the edge ab is not in a triangle with a vertex from C'. This happens precisely when none of the at least 2^{d-1} vertices in C that form a triangle with ab are in C'. Hence,

$$\mathbb{P}\left[E_{ab}\right] \leq \binom{|C| - 2^{d-1}}{\frac{|C|}{2^{d/2}}} \Big/ \binom{|C|}{\frac{|C|}{2^{d/2}}} \leq \left(1 - \frac{2^{d-1}}{|C|}\right)^{|C|/2^{d/2}} \leq e^{-2^{\frac{d}{2}-1}} ,$$

and the expected number of edges ab for which E_{ab} occurs is at most

$$|A||B|e^{-2^{\frac{d}{2}-1}} = n^2 e^{-2^{\frac{d}{2}-1}} \,.$$

Fix a choice of C' with at most $n^2 e^{-2^{\frac{d}{2}-1}}$ edges ab satisfying E_{ab}. Consider the subgraph induced by $A \cup B \cup C'$. The total number of vertices in the graph is only

$$N = 2n + |C'| = \left(2 + 2^{-d/2}\left(\frac{r+2}{r}\right)^d\right) n < \left(2 + 2^{-\frac{d}{2}} \cdot e^{\frac{2d}{r}}\right) n < \left(2 + 2^{-\frac{d}{3}}\right) n \,. \qquad (1)$$

Unfortunately, now some edges are no longer in triangles. We resolve this by deleting all such edges throughout the graph. By Lemma 3.2 with $r = d^{1/5}$, the number of edges between A and B was originally at least $n^2\left(1 - 2e^{-\frac{1}{2}d^{1/5}}\right)$, so since we chose C' such that at most $n^2 e^{-2^{\frac{d}{2}-1}}$ edges between A and B are not in triangles, the number of remaining edges between A and B after deleting those not in triangles is still at least $n^2\big(1 - 3e^{-\frac{1}{2}d^{1/5}}\big)$. Therefore, by (1), the number of remaining edges between A and B is at least

$$\frac{N^2}{\left(2+2^{-\frac{d}{3}}\right)^2}\left(1 - 3e^{-\frac{1}{2}d^{1/5}}\right) > \frac{N^2}{4}\left(1 - 4e^{-\frac{1}{2}d^{1/5}}\right) > \frac{N^2}{4}\left(1 - e^{-(\log N)^{1/6}}\right),$$

and the number of remaining edges between C' and $A \cup B$ is positive, so the remaining graph has the claimed total number of edges. Finally, note that our deletions cannot create any new triangles, so by Lemmas 3.2 and 3.3, every edge is still in at most

$$15^d = 15^{(1+o(1))\frac{5\log n}{\log\log n}} < N^{14/\log\log N}$$

triangles, completing our proof. □

Remark 1. The use of randomness to pick C' in the above construction is not necessary. Indeed, the construction can be made explicit by instead picking C' greedily so that each new vertex added to C' (locally) maximizes the number of edges between A and B that are in triangles with vertices from C'.

Remark 2. After publicizing this result, the authors received the following nice observation from Noga Alon. The objective of sparsifying C to C' was to raise the edge density to approach $1/4$. A simpler way to increase the density is to leave C alone, and instead replace each vertex of $A \cup B$ with exactly 2^d copies of itself, joining two copies of (different) vertices by an edge if their original vertices were initially adjacent, and joining a copy of a vertex in $A \cup B$ to an uncopied vertex $c \in C$ if the corresponding original vertex of $A \cup B$ was adjacent to c. This avoids our final probabilistic arguments altogether, and allows for the further simplification that in the Pre-Construction, all of A, B, C can be taken to be $[r]^d$. Then, it suffices to replace the lower bound in Lemma 3.2 with the observation that for any edge ab between A and B, the integer-rounded midpoint produces at least one point c which completes ab to a triangle.

Acknowledgment

We would like to thank Noga Alon for helpful discussions.

References

[1] N. Alon and J. Spencer, **The Probabilistic Method**, 3rd ed., Wiley, New York, 2007.

[2] B. Bollobás and V. Nikiforov, Books in graphs, *European J. Combin.* **26** (2005), 259–270.

[3] F. Chung and R. Graham, **Erdős on graphs. His legacy of unsolved problems.** A K Peters, Ltd., Wellesley, MA, 1998.

[4] C.S. Edwards, A lower bound for the largest number of triangles with a common edge, 1977 (unpublished manuscript).

[5] P. Erdős, On a theorem of Rademacher-Turán, *Illinois J. Math.* **6** (1962), 122–127.

[6] P. Erdős, Some problems on finite and infinite graphs. Logic and combinatorics (Arcata, Calif., 1985), 223–228, Contemp. Math., 65, Amer. Math. Soc., Providence, RI, 1987.

[7] P. Erdős, Problems and results in combinatorial analysis and graph theory, in: Proceedings of the First Japan Conference on Graph Theory and Applications (Hakone, 1986), *Discrete Math.* **72** (1988), 81–92.

[8] P. Erdős, Some of my favourite problems in various branches of combinatorics. Combinatorics 92 (Catania, 1992). Matematiche (Catania) 47 (1992), no. 2, 231–240 (1993).

[9] P. Erdős, R. Faudree and E. Györi, On the book size of graphs with large minimum degree, *Studia Sci. Math. Hungar.* **30** (1995), 25–46.

[10] P. Erdős, R. Faudree, C. Rousseau, Extremal problems and generalized degrees, Graph Theory and Applications (Hakone, 1990), *Discrete Math.* **127** (1994), 139–152.

[11] R.J. Faudree, C.C. Rousseau, J. Sheehan, More from the good book, in: Proceedings of the Ninth Southeastern Conference on Combinatorics, Graph Theory, and Computing, Florida Atlantic Univ., Boca Raton, Fla., 1978, pp. 289–299. Congress. Numer., XXI, Utilitas Math., Winnipeg, Man., 1978.

[12] J. Fox, A new proof of the graph removal lemma, *Annals of Mathematics* **174** (2011), 561–579.

[13] N. Khadžiivanov and V. Nikiforov, Solution of a problem of P. Erdős about the maximum number of triangles with a common edge in a graph, *C. R. Acad. Bulgare Sci.* **32** (1979) 1315–1318.

[14] V. Nikiforov and C. C. Rousseau, Large generalized books are p-good, *J. Combin. Theory Ser. B* **92** (2004), 85–97.

[15] V. Nikiforov and C. C. Rousseau, A note on Ramsey numbers for books, *J. Graph Theory* **49** (2005), 168–176.

[16] V. Nikiforov and C. C. Rousseau, Book Ramsey numbers, I. *Random Structures Algorithms* **27** (2005), 379–400.

[17] V. Nikiforov, C. C. Rousseau, and R. H. Schelp, Book Ramsey numbers and quasi-randomness, *Combin. Probab. Comput.* **14** (2005), 851–860.

[18] C. C. Rousseau and J. Sheehan, On Ramsey numbers for books, *J. Graph Theory* **2** (1978), 77–87.

[19] I. Z. Ruzsa and E. Szemerédi, Triple systems with no six points carrying three triangles. Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. II, pp. 939–945, Colloq. Math. Soc. János Bolyai, 18, North-Holland, Amsterdam-New York, 1978.

[20] B. Sudakov, Large K_r-free subgraphs in K_s-free graphs and some other Ramsey-type problems, *Random Structures Algorithms* **26** (2005), 253–265.