TRIPLES

2015-06-16 (A)

TH. (ERDŐS-SZEKERES) Every seq. of n distinct numbers has monotonic subseq of length $\geq \sqrt{n}$.

PF. 9 7 4 6 3

longest incr ending here

decr.

distinct, since either incr or decr so pair must $\nearrow$ in one position.

Can also color red if decr, blue if incr

PROP. Every red/blue coloring of edges of $K_{\{1,2,\dots,n\}}$ has monochromatic forward path of length $\geq \sqrt{n}$.
Same proof. transitive tournament directed

PROP. Every r-coloring of edges of any n-vtx tournament has mono. dir path of length $\geq n^{\frac{1}{r}}$.

~~PF~~ OBS. If $r=1$, Hamilton path, good.

PF. Uses TH. (Gallai-Hasse-Roy-Vitaver) Directed graph with chromatic number $\chi \Rightarrow \exists$ dir path with $\geq \chi$ vtxs.

Now we r-colory of edges of n-vtx tournament to partition K_n into $G_1 \cup \dots \cup G_r$, each a color class

$n = \chi(G_1 \cup \dots \cup G_r) \leq \chi(G_1) \cdots \chi(G_r) \Rightarrow$ some $\chi(G_i) \geq n^{\frac{1}{r}}$.

TIGHT. $n = k^r$, identify as ~~triples~~ r-tuples.
lexicographic transitive tournament
color by first index differing.

Two notions of monochromatic when #colors > 2.

Q. Every 3-colory of edges of transitive tournament has 1-color-avoiding path of length $\geq$ ___?

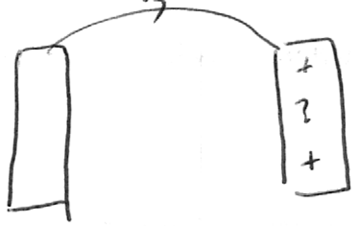
OBS. Can get $\geq \sqrt{n}$. by merging 2 colors.

OBS. Can't beat $n^{\frac{2}{3}}$, since go back to triples construction.
Avoid 1st color: index 1 always same ∴ obvious. $\to k^2$
Avoid 2nd color: for each fixed 1st index, all 2nd indices same $\to$ ~~k~~ $k \times k$
— 3rd — 1st/2nd $\to k^2 \times 1$.

2015-06-16 (B)

6. back to E-S argument

edge colored 2



longest avoiding color 1
2
3

Get a sequence of triples st. $\forall$ pair (not nec. consec.), ≥ 2 positions increase

EQUIV DUAL PROBLEM Seek largest coloring of trans turn that has all ~~2~~ 1-col-avoid paths $\leq n$.
Given any ~~k~~, n, find max N st. $\exists$ seq of triples in $\{1, \ldots, n\}^3$.

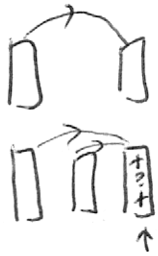
WHY EQUIV Every coloring $\rightarrow$ valid triple sequence.

Every valid triple sequence $\rightarrow$ coloring:

choose a color st. other 2 indices increase

Given

Algo



$\leq$ given, pointwise.

Connection to k-majority tournaments.

DEF. Given n vtxs, and 2k-1 preference orderings on them (permutations of 1, .., n),
define a k-majority tournament where $i \rightarrow j$ if majority of the orderings prefer i over j.

Much work on these, interesting source of constructions.
Interest on the size of largest transitive subtournament.

CONSTRUCTION. $n = m^{2k-1}$

Natural way to have 2k-1 voters: each starts at a different index (major issue), then follows tiebreaks.

Special type of transitive subtournament:
already decidable by major issues: every pair of vtxs has already $\geq k$ indices where there's clear winner.

k=2: triples problem.
This is actually how I came upon the problem... early work with Jacob here

2015-06-16 c

Thus, triples problem is natural in 3 ways: by self, for Ramsey, and for k-majority.

Lower bound on length of longest triple sequence inherits from $n^{2/3}$ upper bound for k-color-avoiding.

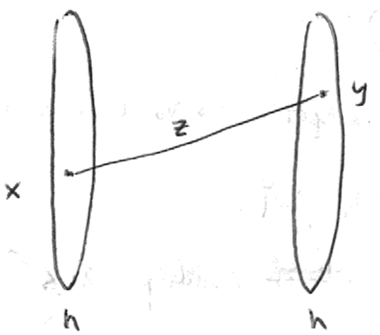
Again, translated here, it's:

$$
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 2 & 2 \\
 & \vdots & \\
1 & \sqrt{n} & \sqrt{n}. \\
2 & 1 & \sqrt{n}+1 \\
2 & 2 & \sqrt{n}+2 \\
 & \vdots & \\
2 & \sqrt{n} & 2\sqrt{n}. \\
3 & 1 & 2\sqrt{n}+1 \\
 & \vdots & \\
3 & \sqrt{n} & 3\sqrt{n} \\
 & \vdots & \\
\sqrt{n} & \sqrt{n} & n,
\end{array}
\longrightarrow
\begin{array}{ccc}
\sqrt{n}+1 & \sqrt{n}+1 & 1 \\
\sqrt{n}+1 & \sqrt{n}+2 & 2 \\
 & \vdots & \\
\sqrt{n}+1 & 2\sqrt{n} & \sqrt{n} \\
\sqrt{n}+2 & \sqrt{n}+1 & \sqrt{n}+1 \\
 & \vdots & \\
\sqrt{n}+2 & 2\sqrt{n} & 2\sqrt{n} \\
 & \vdots & \\
2\sqrt{n} & 2\sqrt{n} & n
\end{array}
\longrightarrow \text{etc.}
$$

$\rightarrow n^{3/2}$ is possible

Upper bound. For any fixed $(x, y, \cdot)$, there's only one choice for 3rd coord $\Rightarrow \leq n^2$.

Dig deeper. Use this obs. to construct a labeled graph:

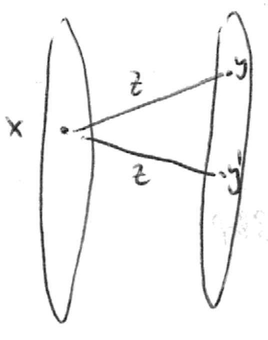


OBS. Each z gives an INDUCED MATCHING.

↑ matching, with no edges from other z's interfering inside,

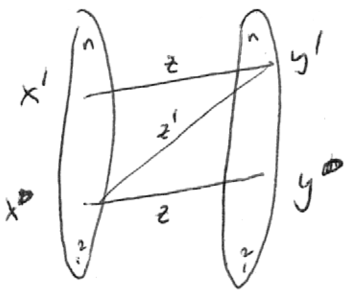
PF. Suppose we have:

~~already~~ know ~~matching~~.



then (x, y, z)
(x, y', z) ✱.

Now suppose we have:



$x < x'$ $\qquad$ $y < y'$

(x, y', z')
~~(A)~~ (x, y, z)
$= \uparrow$ force $\uparrow$ so $z' > z$.

(x', y', z)
(x, y', z')
$\uparrow$ = force so $z > z'$
$\ast$.

Indeed, we must have increase at every pivot.

So, we have a disjoint union of n induced matchings, on a $2n$ vtx graph.

Q. What's the max # of edges in ~~that~~ such a graph? Can't possibly be much! So much space!

~~DEF~~ This turns out to be a really fundamental problem in Combinatorics, with very rich history

TH. (RUZSA-SZEMERÉDI)

Such graph has $o(n^2)$ edges.

TH (FOX).

$\#\text{edges} \le \dfrac{n^2}{e^{\lg^* n}}$ $\qquad$ MADE PROGRESS ON TRIPLES!

~~DEF~~ TH (RUZSA-SZEMERÉDI, USING BEHREND CONSTRUCTION)

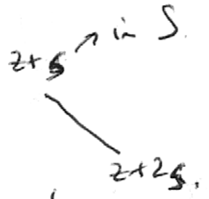
$\exists$ n-vtx graph, union of n induced matchings with $\ge \dfrac{n^2}{e^{\sqrt{\lg n}}}$ edges $\to$ No hope on improving.

Actually, it's based on 3AP-avoiding subset $S \subseteq \{1, 2, \ldots, n\}$:

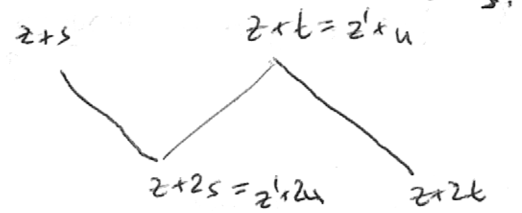
1 2 3 . . . . $2n$

1 2 3 . . . . . . $3n$.

$\forall z \in [n]$, create ~~induced~~ matching $z+s$ (s in S)



No:



$z + t = z' + u$

$z + 2s = z' + 2u$

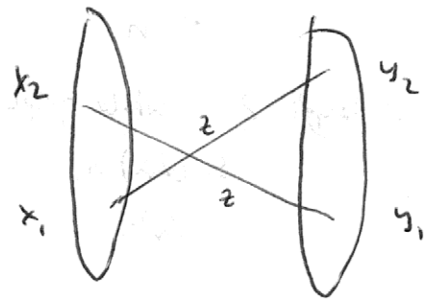
$2s - t = u$

$2s = u + t \Rightarrow 3AP.$

So we need to do more

OBS. The matchings are not only induced. They're also noncrossing.

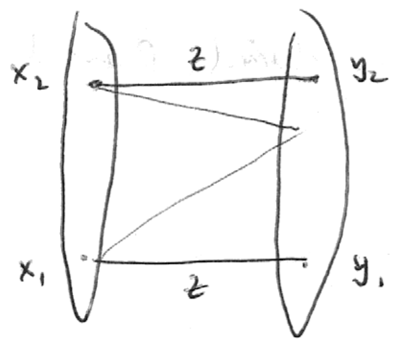
PF.



No way.

Oh, but so are Ruzsa-Szemerédi ones!

OBS. The matchings are ~~2-SEPARATED~~ Σ-FREE. No:



PF Must increase at pivot.

(Actually, also ~~2~~ Σ-free.) ~~and other~~

Q (EXTENSION OF RUZSA-SZEMERÉDI) What is max # of edges in disjoint union of n Σ-free matchings?

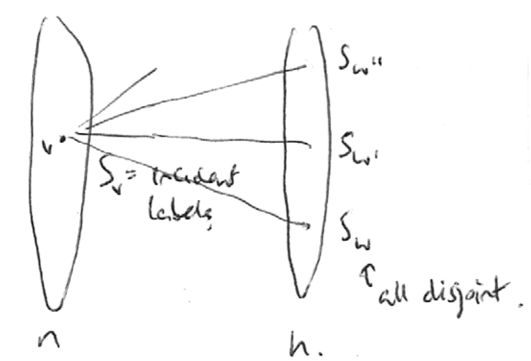
Couldn't answer, seems interesting.

If can show $\leq n^{3/2}$, then done.

TH (6). Disjoint union of Σ-free matchings has $\leq n^{3/2}$ edges.
(2-SEPARATED)

2015-06-16 (F)

PF



So, in particular, from v, # ~~path~~ walks of length 2 $\le n$.

$\Rightarrow$ # walks of length 2 in whole graph $\le (2n)\,n$
$= 2n^2$.

Yet this equals $\sum_v d_v^2 \ge (2n)(\bar{d})^2 \cancel{= (2n)\left(\frac{2m}{n}\right)^2}$

$\Rightarrow (\bar{d})^2 \le n$

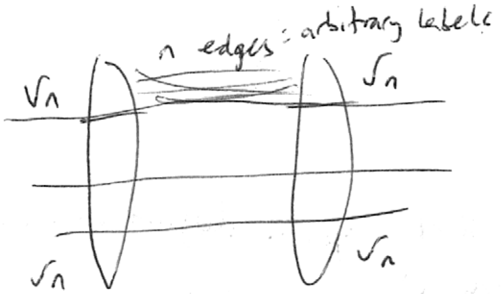
$\Rightarrow \bar{d} \le \sqrt{n}$

$\Rightarrow$ #edges $\le \frac{1}{2}(2n)\sqrt{n} = n^{3/2}$, as claimed. Razor-sharp

Q Does this keep dropping as we go to 3-SEPARATED?

A No.

CONSTRUCTION of ∞-SEPARATED



CONCLUSION 1-SEPARATED: Ruzsa-Szemerédi: $o(n^2)$

2-SEP : $n^{3/2}$

3-SEP

Q. What is answer for triples?

Q. What is answer for Σ-free matchings?

Q. More colors on Ramsey? 2-out-of-3 / 2-out-of-4 / etc?