Triples

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Puzzle

Valid sequence

(1, 1, 1)
(1, 2, 3)
(1, 4, 4)
(2, 5, 1)   *for every two rows, at least two coordinates increase*
(3, 1, 5)
(1, 6, 6)
(2, 7, 8)

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Question

What's the length of longest valid sequence from $\{1, \ldots, L\}$?

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Question

What's the length of longest valid sequence from $\{1, \ldots, L\}$?

Observation

The length is at most L^2.

Monotone sequences

Theorem (Erdős-Szekeres 1935)

Every permutation of $\{1, \ldots, n\}$ has a monotone subsequence of length at least $\sqrt{n}$.

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Example

$$1 \quad 5 \quad 2 \quad \mathbf{7} \quad 3 \quad \mathbf{6} \quad \mathbf{4}$$

THEOREM (ERDŐS-SZEKERES 1935)

Every permutation of $\{1, \ldots, n\}$ has a monotone subsequence of length at least $\sqrt{n}$.

EXAMPLE

1 5 2 **7** 3 **6** **4**

Proof. Under each number, write lengths of longest increasing and decreasing subsequences ending there.

	1	**5**	**2**	**7**	**3**	**6**	**4**
inc.	1	2	2	3	3	4	4
dec.	1	1	2	1	2	2	3

Proposition

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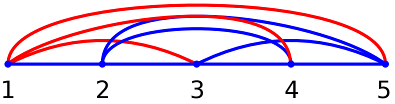
Proof. Under each vertex, write lengths of longest red/blue paths ending there.



1 1 2 2 2

1 2 3 4 5

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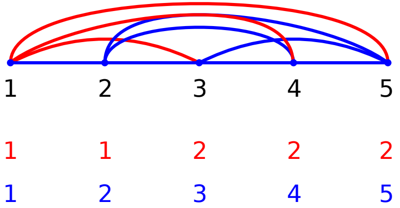
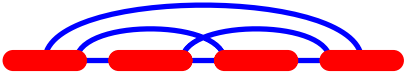


Tight:



Proposition

Every 3-coloring of edges of $K_{\{1,\dots,n\}}$ has a monochromatic forward path of length at least $\sqrt[3]{n}$.

Ordered Ramsey Theory

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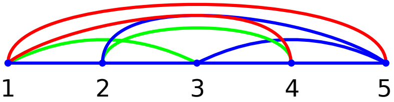
Proof. Under each number, write lengths of longest red/blue/green paths ending there.



1	1	1	2	2
1	2	3	4	5
1	1	2	2	1

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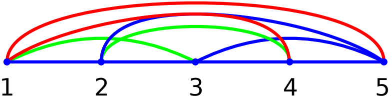

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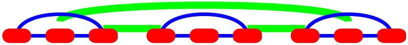

$3 - 1 \neq 1$

Proposition

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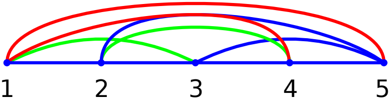

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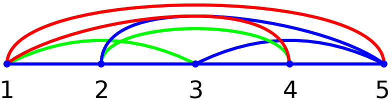

1	2	3	4	5
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1	2	3	4	5

Already Deduced

Every 3-coloring of edges of $K_{\{1,\dots,n\}}$ has a non-rainbow forward path of length at least $\sqrt{n}$.

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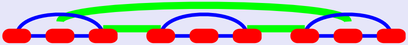
There is a 3-coloring of edges of $K_{\{1,\dots,n\}}$ where all non-rainbow forward paths have length at most $n^{2/3}$.



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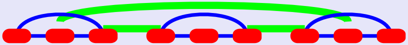

Corollary

There is a valid sequence of triples of length at least $L^{3/2}$.

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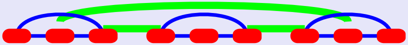

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Observation

Every valid sequence of triples has length at most L^2.

DEFINITION

Given n vertices, and $2k - 1$ preference orderings on them (permutations of $1, \ldots, n$), the k-majority tournament has $\overrightarrow{ij}$ when majority of orderings prefer i over j.

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Application to Ramsey

Potential source of constructions: control size of largest transitive subtournament.

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THEOREM (ERDŐS), PROBABILISTIC METHOD

There is a 2-coloring of the edges of K_n where all monochromatic cliques have order at most $2\log_2 n$.

Theorem (Erdős-Szekeres)

Every 2-coloring of edges of K_n contains a monochromatic clique of order at least $\frac{1}{2}\log_2 n$.

Theorem (Erdős), probabilistic method

There is a 2-coloring of the edges of K_n where all monochromatic cliques have order at most $2\log_2 n$.

Challenge

Discover interesting new Ramsey constructions, esp. explicit.

Vertex set. $n = r^{2k-1}$, write as $(2k-1)$-tuples from $1, \ldots, r$.

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4. etc.

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Other orderings of less-significant coordinates should be used.

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4. etc.

Other orderings of less-significant coordinates should be used.

Special transitive subtournaments

If no tiebreaks necessary, for every two tuples, the later one is greater in at least half the coordinates.

Result

Corollary

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Observation

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Theorem (L.)

There is a valid sequence of triples of length at most $L^2/\log^* L$.

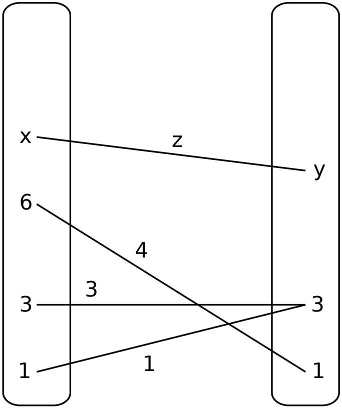
Definition

Tower function $T(n) = 2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}$. Inverse function is $\log^* n$.

Auxiliary graph

(1, 3, 1)
(3, 3, 3)
(6, 1, 4)

(x, y, z)

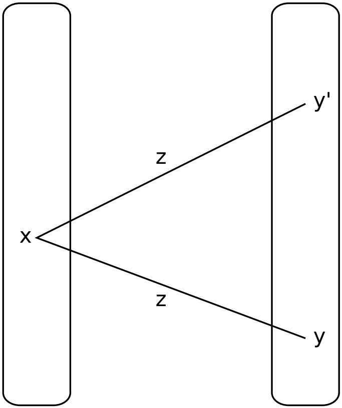


Observation

Same-labeled edges form a matching.

Proof.

(x, y, z)
...
(x, y', z)



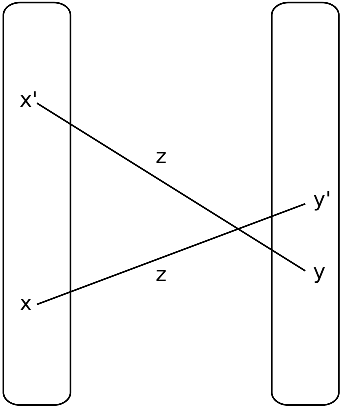
Non-crossing matching

Observation

Same-labeled edges form a non-crossing matching.

Proof.

(x, y', z)
...
(x', y, z)



Induced matching

Observation

Same-labeled edges form an induced matching.

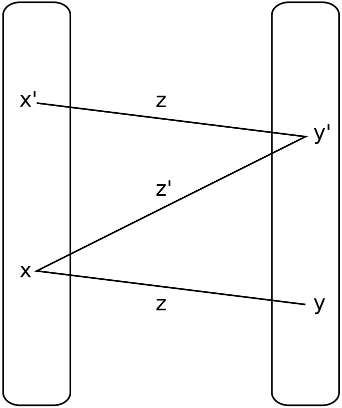
Proof.

(x, y, z)

...

(x, y', z')

$z < z' < z$



Question

Given a graph with $2n$ vertices, where edges can be decomposed into n induced matchings. What is maximum possible number of edges?

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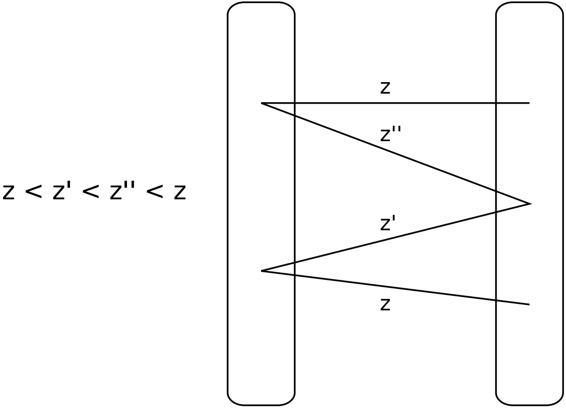
Theorem (Ruzsa-Szémeredi)

It is possible to achieve $n^2/e^{\sqrt{\log n}}$ edges.

OBSERVATION

Same-labeled edges form a Σ-free matching.

Proof.



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PROPOSITION

If also forbid "floppy" Σ, max number of edges is at most $n^{3/2}$.

Ruzsa-Szémeredi generalizations

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Proposition

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Question

Given a graph with $2n$ vertices, where edges can be decomposed into n matchings that are ξ-free. What is maximum possible number of edges?

Any improved bound transfers to triples problem.