**Puzzle.** what's length of longest Sequence where:

(1, 2, 1)
(3, 1, 3)
(4, 4, 2)
(5, 5, 1)

∀ two rows, later row increases in ≥ half coordinates.

all entries 1..L.

Asymptotics in L?

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**TH (ERDŐS–SZEKERES)** foundational result.

Every seq of n distinct #s has a monotone subseq of length ≥ $\sqrt{n}$.

**EX.** 3 1 4 5 9 2 6

← longest incr

← longest decr.

obs. distinct.

**PF** Suppose L is longest monotone subseq. Then $n = \#\square \le L^2$ ✓.

**PROP.** Any 2-coloring of edges of $K_n$ $\{1,2,\dots,n\}$ contains monochromatic fwd. path of length ≥ $\sqrt{n}$ vtxs.

clique is famous central open problem

(PF)

TIGHT.

$\sqrt{n}$

~~Generalize to arbitrary # colors~~

**General flavor of Ramsey Theory.** Find nice structure from pure size.

Generalize to 7 colors, etc. not ~~using~~ using all colors?

**Q** How about ~~non-rainbow path?~~ non-rainbow ← ~~of color-count~~

**PROP.** Any 7-coloring of $e(K_n)$ contains non-rainbow fwd path of length ≥ $\cancel{\sqrt[3]{n}}$ vtxs.

↑ color blind

**PF**

color 1 ← avoid color 1

7 ++ ← color 7.

**OBS.** Construct with all lgth ≤ $n^{2/7}$.

~~Puzzle~~ → best upper is ≤ $L^2$
lower is ≥ $L^{3/2}$

2016-01-27 (B)

~~Col~~ Matrix questions in Ramsey Theory

TH (ERDŐS-SZEKERES 35) Every 2-coloring of edges of $K_n$ contains mono clique of order $\geq \frac{1}{2}\log_2 n$

TH (ERDŐS 47) $\exists$ 2-col of $K_n$ with all mono cliques $\leq 2\log_2 n$.
Random

TH (FRANKL-WILSON 81) linear algebra set system / families $\leq e^{c\sqrt{\log n \log\log n}}$

TH (BARAK ET AL 10, 12) RAO, SHALTIEL, WIGDERSON / Randomness extractor $\leq e^{e^{\log^{1-\alpha}(\log n)}}$ — Construct in polylog # nodes

TH (COHEN / CHATTOPADHYAY - ZUCKERMAN 15) $\leq e^{(\log\log n)^c}$

~~Q FAP. $\exists$ Tournament with all transitive~~ matrix

Q What about ~~transitive~~ tournaments? Then seek transitive subtournament.

State what we can have.

DEF k-majority tournament: n candidates ~~a~~ 2k-1 voters. Each has a preference ordering on the candidates $=(1,3,2,\cancel{4},1)$ ← k=2

$r^{2k-1}$. $c\log n$.
want $r$.
~~k~~ $r^k$ ?

construction. Compare.

Q Size of largest transitive subtournament? Still useful & hard.

Q ______ with no ties also?

$\bigcirc\ \bigcirc\ \bigcirc$
$r^{\frac{5}{3}}$ out of $r^5$.
out $n^{\frac{1}{k}}$. $k=\log n$.
$n^{\frac{1}{\log n}}$ is good of course

TH (L) Best upper is $\leq \frac{L}{e^{\log^* L}}$.

TH (~~SZEMERÉDI~~ ROTH 53) $\forall\epsilon\ \exists N$ s.t. for all $n>N$, every subset of $\{1,\dots,n\}$ with density $\geq\epsilon$ contains a 3AP.

Quantitative Bounds were $N = e^{e^{\frac{c}{\epsilon}}}$

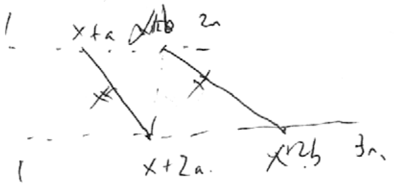
TH (SZEMERÉDI 75) True for any k-AP.

Sanders: $\epsilon = \frac{(\log\log n)^5}{\log n}$.

Related to Szemerédi Regularity Lemma: every graph can be $\epsilon$-approximated by a structure with $< 2^{2^{\cdot^{\cdot^{\cdot}}}}$ } ~~height~~ $\frac{1}{\epsilon^5}$ parts
↑ and necessary — Gowers

→ ~~b-? problem~~ $A\subseteq[n]$

PF Suppose no 3AP. ~~B~~ Upper bound max density of a set.



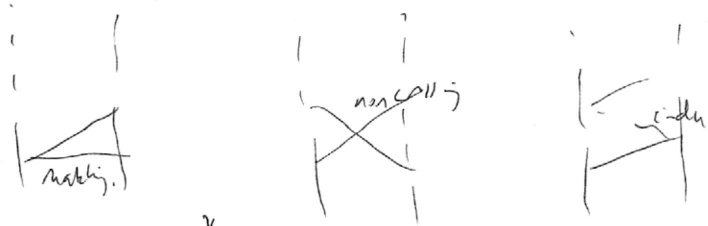
$x+b = x+c$
$x+2a = x+2c$.
$2a-b = \cancel{c}$
$2a = b+c$ 3AP.

TH (RUZSA-SZEMERÉDI 78) System of n induced matchings on n vtxs has max ~~#edges~~ ~~density~~ $\leq \frac{n^2}{e^{\log^* n}}$
(but (1) $\leq \frac{n^2}{e^{\log n}}$)

TH (BEHREND 46) (ELKIN 08) $\exists$ 3AP-free set of size $\geq \frac{n}{e^{\sqrt{\log n}}}$ → Best possible is $\frac{n^2}{e^{\log n}}$

PF( TRIPLES)






→ max. ✓

to drop to $k^{2}$.

OBS. Not enough since Behrend + RS $\Rightarrow$ can get construction to build up to $\frac{L^2}{e^{\sqrt{\log L}}}$ without all of above but can't identify it as a triple set

OBS.


Q. (Generalization of R-S) Max # edges of ^ Σ-free matchings on n sets?

Potentially find deeper construction? Can't use simple NT + Geom.

Q. ≩-free matching?

Q.

Prop. If ≠-free, can show $\le n^{3/2}$ edges. Just keep list of who's touching.

Not tight.

∃ optimal?

1-step — R-S$_2$. — α-ry!

2-step — $n^{3/2}$

3-step — ↓

walks of length 2


$L_1, L_2$ disj

$\sum_{u\sim v} d_u \le n$.

for all green

$\sum_v \sum_{u\sim v} d_u \le 2n^2$.

$2n\cdot(\bar d)^2 \le \sum d_u^2 \le 2n^2$

$(\bar d)^2 \le n$. ✓

So it's deep, and in the ordering

Summary. ① Surprisingly simple problem not easy answer.

②. Deeply connected to: Ordered Ramsey Theory
k-majority tournaments
Szemerédi Regularity
Challenge construction of NT + Geo.

③ Open ordered generalizations of RSz?