Triples

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Puzzle

Valid sequence

(1, 1, 1)
(1, 2, 3)
(1, 4, 4)
(2, 5, 1) *for every two rows, at least two coordinates increase*
(3, 1, 5)
(1, 6, 6)
(2, 7, 8)

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Question

What's the length of longest valid sequence from $\{1, \ldots, L\}$?

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Question

What's the length of longest valid sequence from $\{1, \ldots, L\}$?

Observation

The length is at most L^2.

Theorem (Erdős-Szekeres 1935)

Every permutation of $\{1, \ldots, n\}$ has a monotone subsequence of length at least $\sqrt{n}$.

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Example

1 5 2 **7** 3 **6** **4**

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Example

1 5 2 **7** 3 **6** **4**

Proof. Under each number, write lengths of longest increasing and decreasing subsequences ending there.

	1	**5**	**2**	**7**	**3**	**6**	**4**
inc.	1	2	2	3	3	4	4
dec.	1	1	2	1	2	2	3

Proposition

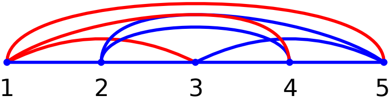
Every 2-coloring of edges of $K_{\{1,\ldots,n\}}$ has a monochromatic forward path of length at least $\sqrt{n}$.

Ordered Ramsey Theory

Proposition

Every 2-coloring of edges of $K_{\{1,\dots,n\}}$ has a monochromatic forward path of length at least $\sqrt{n}$.

Proof. Under each vertex, write lengths of longest red/blue paths ending there.



1 1 2 2 2

1 2 3 4 5

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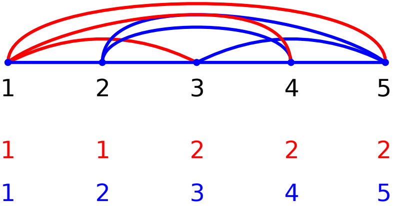
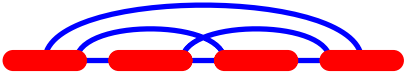


Tight:



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Every 3-coloring of edges of $K_{\{1,\ldots,n\}}$ has a monochromatic forward path of length at least $\sqrt[3]{n}$.

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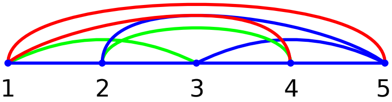

1	1	1	2	2
1	2	3	4	5
1	1	2	2	1

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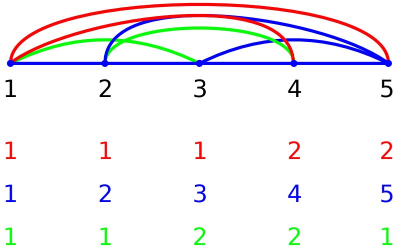

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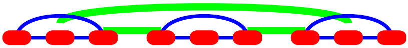

$3 - 1 \neq 1$

Proposition

Every 3-coloring of edges of $K_{\{1,\ldots,n\}}$ has a not-all-colors forward path of length at least $\sqrt[3]{n}$.

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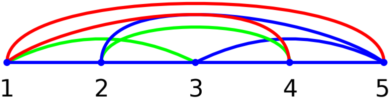

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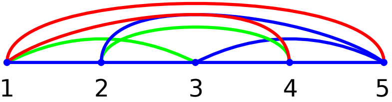

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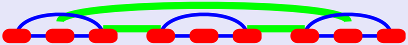
Proposition

There is a 3-coloring of edges of $K_{\{1,\dots,n\}}$ where all not-all-colors forward paths have length at most $n^{2/3}$.



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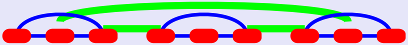

Corollary

There is a valid sequence of triples of length at least $L^{3/2}$.

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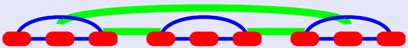

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Observation

Every valid sequence of triples has length at most L^2.

Definition

Given n vertices, and $2k - 1$ preference orderings on them (permutations of $1, \ldots, n$), the k-majority tournament has $\overrightarrow{ij}$ when majority of orderings prefer i over j.

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- social choice theory
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k-MAJORITY TOURNAMENTS

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APPLICATION TO RAMSEY

Potential source of constructions: control size of largest transitive subtournament.

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There is a 2-coloring of the edges of K_n where all monochromatic cliques have order at most $2\log_2 n$.

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CHALLENGE

Discover interesting new Ramsey constructions, esp. explicit.

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- Frankl-Wilson '81: $\leq e^{c\sqrt{\log n \log\log n}}$
- Barak-Rao-Shaltiel-Wigderson '10, '12: $e^{e^{(\log\log n)^{1-\epsilon}}}$
- Cohen / Chattopadhyay-Zuckerman '15, : $\leq e^{(\log\log n)^c}$

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4. etc.

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Special transitive subtournaments

If no tiebreaks necessary, for every two tuples, the later one is greater in at least half the coordinates.

Result

Corollary

There is a valid sequence of triples of length at least $L^{3/2}$.

Observation

Every valid sequence of triples has length at most L^2.

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Theorem (L.)

There is a valid sequence of triples of length at most $L^2/\log^* L$.

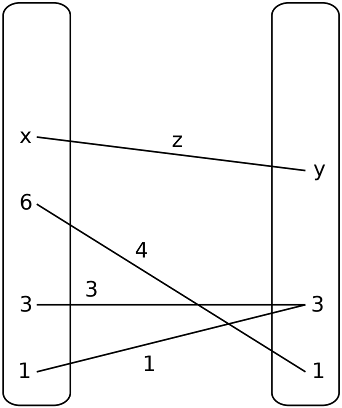
Definition

Tower function $T(n) = 2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}$. Inverse function is $\log^* n$.

Auxiliary graph

(1, 3, 1)
(3, 3, 3)
(6, 1, 4)

(x, y, z)

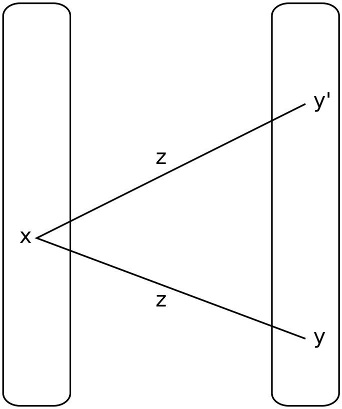


OBSERVATION

Same-labeled edges form a matching.

Proof.

(x, y, z)
...
(x, y', z)

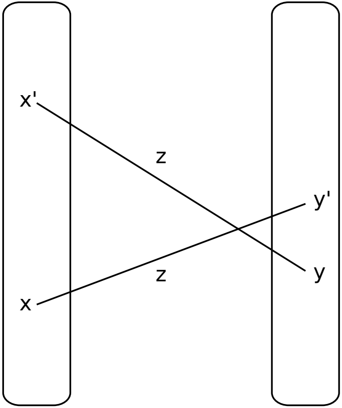


Observation

Same-labeled edges form a non-crossing matching.

Proof.

(x, y', z)
...
(x', y, z)



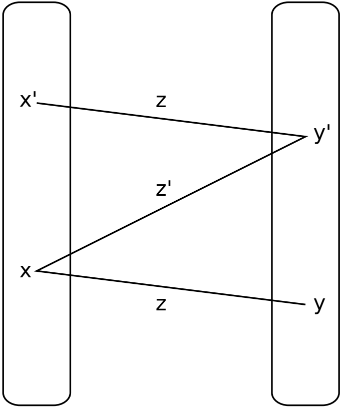
Observation

Same-labeled edges form an induced matching.

Proof.

(x, y, z)
...
(x, y', z')

z < z' < z



Theorem (Roth '53)

For every $\epsilon > 0$, there is an integer N so that every subset of $\{1, \ldots, N\}$ with density $\geq \epsilon$ contains a 3-term arithmetic progression.

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Also true for k-term arithmetic progressions.

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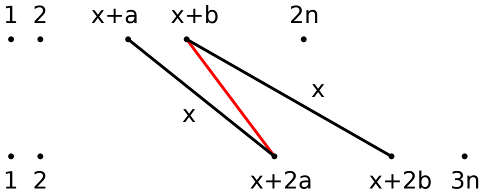
Also true for k-term arithmetic progressions.

Szemerédi Regularity Lemma

For every $\epsilon > 0$, there is an integer N so that every graph is approximately an ϵ-pseudorandom structure with less than N parts.

- Suppose $A \subseteq \{1, \ldots, n\}$ has no 3-term arithmetic progressions.
- For every $x \in \{1, \ldots, n\}$ and $a \in A$, create an x-edge between top $x + a$ and bottom $x + 2a$.



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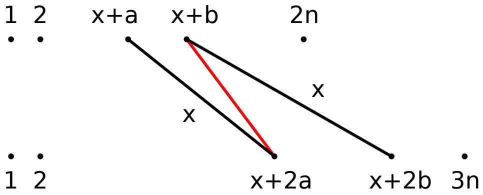

- Show that $n|A|$, the number of edges, is $o(n^2)$.

$$x + b = y + c$$
$$x + 2a = y + 2c$$
$$2a - b = c \qquad \Longrightarrow x\text{-edges induced matching}$$

Ruzsa-Szémeredi

Question

Given a graph with n vertices, where edges can be decomposed into n induced matchings. What is maximum possible number of edges?

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Theorem (Ruzsa-Szémeredi)

Maximum $\leq n^2 / \log^* n$ edges.

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Theorem (Fox)

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Theorem (Ruzsa-Szémeredi)

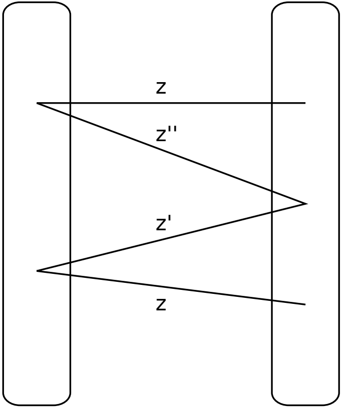
It is possible to achieve $n^2/e^{\sqrt{\log n}}$ edges, using 3-term arithmetic progression free set.

OBSERVATION

Same-labeled edges form a Σ-free matching.

Proof.

$z < z' < z'' < z$



Question

Given a graph with $2n$ vertices, where edges can be decomposed into n matchings that are Σ-free. What is maximum possible number of edges?

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Proposition

If also forbid "floppy" Σ, max number of edges is at most $n^{3/2}$.

Ruzsa-Szémeredi generalizations

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Given a graph with $2n$ vertices, where edges can be decomposed into n matchings that are Σ-free. What is maximum possible number of edges?

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Given a graph with $2n$ vertices, where edges can be decomposed into n matchings that are ξ-free. What is maximum possible number of edges?

Any improved bound transfers to triples problem.