CLASSICAL EXTREMAL GRAPH THEORY

2010-01-08

"Extremal graph theory, in its strictest sense, is a branch of graph theory developed and loved by Hungarians." Bollobás, Extremal Graph Theory preface

Concerns study of relations between various graph parameters, and resulting local graph properties.

Often has questions of form: what is max/min of certain combinatorial parameter over set of combinatorial objects.

PARAMETERS

- # vtxs $\longrightarrow$ Ramsey's Theorem
- # edges
- avg degree / min degree / max degree.
- independence #. / clique #.
- chromatic #.

Such things useful since even in other math, these are most easily accessible.

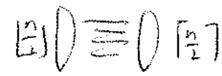
* FOCUS OF COURSE: methods used to get the results

also an appreciation for the broader ramifications of our subject.

(since we need to show everyone else why what we do is important.)

TH. (MANTEL 1907) Max # of edges in n-vtx Δ-free graph is $\lfloor \frac{n^2}{4} \rfloor$.

OFFICER IN FRENCH ARMY

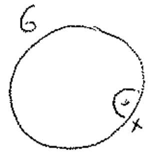
Extremal graphs are precisely 

Equiv: every n-vtx graph with $> \frac{n^2}{4}$ edges contains Δ

PF. (LOWER BOUND) Follows from construction.

PF. (UPPER BOUND)

PF 1. INDUCTION: $E \le \frac{n^2}{4}$. True for $n=1$.

Idea: 

Use $e(G-x) \le \frac{(n-1)^2}{4}$.

Now $e(G) = e(G-x) + d(x)$

So how to find small $d(x)$ to add?

Every edge: $x \text{—} y$ must have $d(x)+d(y) \le n$ since no $\triangle$

(If no edge, then done; else:) $\Rightarrow \exists$ vtx of degree $\le \frac{n}{2}$. Use that for x.

So $e(G) \le \frac{(n-1)^2}{4} + \frac{n}{2} = \left(\frac{n^2}{4} - \frac{n}{2} + \frac{1}{4}\right) + \frac{n}{2}$

$= \frac{n^2}{4} + \frac{1}{4}$.

But $n^2 \equiv 0,1 \pmod 4$, and $e(G) \in \mathbb{Z}$. So $\le \left\lfloor \frac{n^2}{4} \right\rfloor$ $\square$.

NOTE: DIDN'T SHOW EXTREMAL GRAPH.

PF 2. CONVEXITY:

$f''(x) \ge 0$, or all segments above curve.

TH. (JENSEN INEQ) Let $f(x)$ be $\boxed{\text{convex}}$ real function. Then:

$$\text{avg}\{f(x_1), f(x_2), \dots, f(x_t)\} \ge f(\text{avg}\{x_1, \dots, x_t\}).$$

$$n \cdot E \ge \sum_{xy\text{-edges}} [d(x)+d(y)] = \sum_x d(x)^2 \ge n \cdot (\bar{d})^2 = n \cdot \left(\frac{2E}{n}\right)^2 = \frac{4E^2}{n}$$

$$\frac{n^2}{4} \ge E$$

TRACE EQUALITY: for n even.

Jensen $\Rightarrow$ all degrees equal.
$\sum_{\text{edges}} \Rightarrow$ ~~all~~ some $d(x)+d(y) = n$
$\Big\}\Rightarrow$ all degrees $= \frac{n}{2}$.

LEM. $\forall x$, $d(x) = \frac{n}{2}$, and $\forall y \in N(x)$, $N(y) = \overline{N(x)}$ since it is of size $\frac{n}{2}$ and disj. from $N(x)$.

Bounce back once and get $\frac{n}{2}, \frac{n}{2}$ complete bipartite graph. $\square$.

PF 3. ANALYTICAL.

LAGRANGE POLYNOMIAL: $L(x_1, \dots, x_n) = \cancel{\sum} \sum_{\text{edge } ij} x_i x_j$.

Consider COMPACT domain $\{x \geq 0, \ \sum x_i = 1\}$. Let λ be max of $L(x)$.

Note: $(\frac{1}{n}, \dots, \frac{1}{n})$ is there, so $\lambda \geq E (\frac{1}{n})^2 \Rightarrow E \leq \lambda n^2$.

NEED $\lambda \leq \frac{1}{4}$.

VARIATIONAL ARGUMENT. Say we have some $\vec{x}$ vector.

If support has ij not edge, then L is linear in x_i, x_j.

Say $L(x_i, x_j) = \alpha x_i + \beta x_j + \gamma$.

But we are allowed to shift mass between x_i, x_j. Shift to whichever α, β is bigger.

So L increases, and we make another $x_k = 0$.

Keep going (FINITELY LONG!) until support is clique.

Δ-free $\Rightarrow$ only, say, x_1 and x_2, all others 0.

What is max of $x_1 x_2$ over $\{x_1, x_2 \geq 0, \ x_1 + x_2 = 1\}$. $\longrightarrow \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. $\square$.

APPLICATION KATONA PROBABILITY INEQUALITY (TIGHT).

TH. Let ξ, η be IID random variables in $\mathbb{R}^d$, and let x be real constant.

Then $\mathbb{P}[\|\xi + \eta\| \geq x] \geq \frac{1}{2} \mathbb{P}[\|\xi\| \geq x]^2$.

PF. HOMOG $\Rightarrow$ WLOG $x = 1$.

First prove for discrete distribution. Then, we have collection of (possibly repeated) vectors $v_1, \dots, v_n$.

Will select two with replacement. Say m of them are ≥ 1.

LEM. On graph of those m, where edge if $\|v_1 + v_2\| < 1$, it is Δ-free.

So, # edges on those m is $\leq \frac{m^2}{4}$.

Yet if we take 2 with replacement, if both are same vector in ~~the~~ the m, of course already sum ≥ 2. So # ordered pairs sum to ≥ 1 is $\geq m^2 - 2 \cdot \frac{m^2}{4} = \frac{m^2}{2}$. $\square$.

PF of LEM.

Take 3 vectors v_1, v_2, v_3 of norm ≥ 1. Show two sum to ≥ 1.

WLOG one has norm $= 1$, or else scale down.

So WLOG, $v_1 = (+1, 0, 0, \ldots, 0)$

Assume all pairs sum to norm < 1

Then $v_2, v_3 \in$ Ball of radius 1 about $(-1, 0, \ldots, 0)$, excluding boundary

Also know $v_2, v_3 \notin$ Ball of radius 1 about $\vec{0}$.

let $v_2 = (x_1, \vec{y}_1)$ Then we know: $(x_i+1)^2 + \|\vec{y}_i\|_2^2 < 1$

$v_3 = (x_2, \vec{y}_2)$

$$x_i^2 + \|\vec{y}_i\|_2^2 \geq 1.$$

$$\oplus \quad -x_i^2 - \|\vec{y}_i\|_2^2 \leq -1.$$

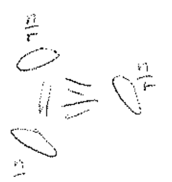
$$2x_i + 1 < 0.$$

$$x_i < -\tfrac{1}{2}.$$

In particular, $v_2 + v_3 = (<-1, *, *, *, \ldots)$

Already norm > 1. $\divideontimes$. $\square$.

TH. (TURÁN 1941)

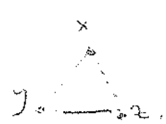
DEF. $T_r(n) =$  With all parts either $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$.

Then the graphs with max # of edges but no K_r are the $T_{r-1}(n)$.

$T(n)$ # edges $\approx \binom{r}{2}\left(\frac{n}{r}\right)^2 = \frac{r(r-1)}{2}\frac{n^2}{r^2} = \left(1-\frac{1}{r}\right)\frac{n^2}{2}$.

PF. (ZYKOV SYMMETRIZATION).

(1). Non-adj vtxs have same degree $x \cdots y$. Else if $d(x) > d(y)$, make y clone x.

(2). Non-adj is equivalence relation.  then $d(x) = d(y) = d(z)$, by above.

But deleting y's, z's edges, and cloning x twice, gain +1 edge

(3). Then graph is complete multipartite, and best is Turán graph.

<u>Pf 2</u> of Turán only quantitative: $\alpha \geq \frac{n}{\bar{d}+1}$. ($\bar{d}$ = avg degree)

Get indep set by randomly permuting vtxs

Keep vtx if it precedes all of its nbrs.

$\mathbb{P}[\text{Keep } \underline{v}] = \frac{1}{d(v)+1}$

$\mathbb{E}[\text{size of indep set}] = \sum_v \frac{1}{d(v)+1} \geq \frac{n}{\bar{d}+1} \Rightarrow \exists$ indep set with that size. $\square$.

Connection to K_{r+1}-freeness

Consider G, a K_{r+1}-free graph. Apply above to complement of G.

Then, $\frac{n}{\bar{d}_c+1} \leq \alpha_c \leq r$ since no K_{r+1}.

of complement

$$\frac{n}{r} - 1 \leq \bar{d}_c$$

So $\bar{d} = (n-1) - \bar{d}_c \geq n - \frac{n}{r} = n\left(1-\frac{1}{r}\right)$.

So $E = \frac{n\bar{d}}{2} \geq \frac{n^2}{2}\left(1-\frac{1}{r}\right)$. $\square$.

RAMSEY THEORY

2010-01-09
(1)

OBS. (SZALAI, HUNGARIAN SOCIOLOGIST) Every group of about 20 children contains:

(1) a set of 4 children, any two of which are friends, OR

(2) ——————— no ———————

Today, this would be called Facebook theorem.

TH. (RAMSEY) For every k, t, $\exists R$ s.t. every k-coloring of edges of K_R contains monochromatic K_t.

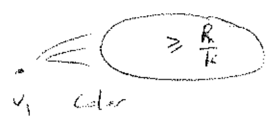
PF. MAJORITY ARGUMENT. See how big R we need to make argument work.

Given k-coloring of edges of K_R.

Pick arbitrary vtx. v_1

Colors of adj. edges are in $[k]$, so one has $\geq \frac{1}{k}$ of the degree.

Focus on those nbr



So need $R \approx k^{kt}$

Repeat...

Majority again here, get t guys monochromatic. $\square$.

DEF. $R(t_1, t_2, \ldots, t_k) = \min R$ s.t. any k-coloring of K_R has some K_{t_i} in color i.

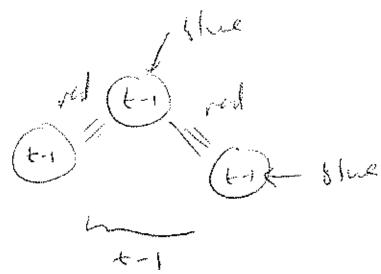
TH. (ERDŐS-SZEKERES, 1935) $R(s,t) \leq \binom{s+t-2}{s-1}$.

PF. Induction: $R(s,t) \leq R(s-1,t) + R(s,t-1)$ $\square$.

This and the above bound give $R(t,t) \leq 4^t$.

~~RK/K...~~

Lower bound constructions. $(t-1)^2$ vtxs. Color:



This was basically the best for over a decade, until...

TH. (ERDÖS, 1947) $R(t,t) \geq 2^{t/2}$, i.e. there is coloring of K_n with $n = 2^{t/2}$ s.t. all mono cliques have size $< t$.

PF. Let blue graph be $G_{n,\frac{1}{2}}$.

For each $|S| = t$, let E_S = event that S gets mono. $\mathbb{P}[E_S] = 2 \cdot \left(\frac{1}{2}\right)^{\binom{t}{2}}$.

So $\mathbb{P}[\text{some } E_S \text{ occurs}] \leq \binom{n}{t} \cdot 2 \cdot \left(\frac{1}{2}\right)^{\binom{t}{2}}$

$$\leq \frac{n^t}{t!} \cdot 2 \cdot \left(\frac{1}{2}\right)^{\left(\frac{t^2 - t}{2}\right)}$$

$$= \frac{2 \cdot 2^{t/2}}{t!} \longrightarrow 0, \text{ is } < 1 \text{ at } t = 3. \qquad \square$$

APPLICATION (SCHUR'S THM, discovered before Ramsey, 15 yrs earlier, while on FLT).

TH. For every k, there is n s.t. if $[n]$ is colored with k colors $\exists$ mono solution to $x + y = z$.

PF. Let $n = R(\overbrace{3, 3, \cdots, 3}^{k \text{ times}}) - 1$

Construct graph with vtx set $\{0, 1, \cdots, n\}$, and color edge ij with $c(|i-j|)$, where $c(\cdot)$ is the given coloring of $[n]$.

Then there is mono $\triangle$ (i, j, k), i.e. $|i-j|$, $|j-k|$, $|k-i|$ are of same color.

TRIANGLE EQUALITY: i j k two sum to the third. $\square$

CONSEQUENCE FOR FERMAT'S LAST THEOREM:

TH. For every d, the equation $x^d + y^d = z^d$ has nontrivial $\mathbb{Z}_p$-soln. for large p.

So cannot hope to prove FLT by going modulo ~~[illegible]~~ those primes.

Needs LEM: $\mathbb{Z}_p^\times$ is cyclic. $\leftarrow$ Finite field since no two nonzero $\mathbb{Z}_p$ multiply to 0, So take fixed $x \in \mathbb{Z}_p$, consider $\{x, 2x, \cdots (p-1)x\}$ Must cycle and hit 1. So $\exists$ inv.

PF. Fund. Thm. of Abelian Groups $\Rightarrow$ $\mathbb{Z}_p^\times \cong \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \cdots \mathbb{Z}_{k_r}$ with $k_1 | \cdots | k_r$.

So not cyclic $\Rightarrow$ $\mathbb{Z}_p^\times$ has subgroup of form $\mathbb{Z}_q \times \mathbb{Z}_q$.

Then $x^q = 1$ has q^2 solutions in ~~$\mathbb{Z}_p$~~ $\mathbb{F}_p$,

$\divideontimes$ $\mathbb{F}_p$ being Unique Factorization Domain. $\square$

PF (Consequence).

Let g be generator of $\mathbb{Z}_p^\times$.

Color $\mathbb{Z}_p^\times$ with d colors, s.t. g^{qd+r} gets color r.

Schur $\Rightarrow \exists\ g^{ad+r} + g^{bd+r} \underset{p}{\equiv} g^{cd+r}$

not even mod p.

So in $\mathbb{Z}_p$, $(g^a)^d + (g^b)^d \equiv (g^c)^d \pmod{p}$. □

TH. (RAMSEY FOR HYPERGRAPHS) For any r, k, t, $\exists R$ s.t. every k-coloring of edges of r-unif hypergraph has monochromatic $K_t^{(r)}$.

PF. $\boxed{\text{EXERCISE}}$

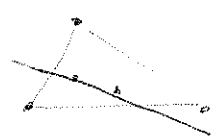
APPLICATION ERDŐS-SZEKERES CONVEX POLYGONS

TH. (ES, 1935) For any $m \geq 4$, there is n s.t. any configuration of n pts in $\mathbb{R}^2$ in general position contains a subset of m pts in convex position.

PF. LEM 1. 5 pts in $\mathbb{R}^2$ in general position $\Rightarrow \exists$ 4 pts in convex position.

PF. Convex hull of pts: if ≥ 4, already done.

Else:



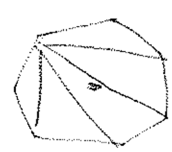
Consider line thru internal pts.

Gen. pos. $\Rightarrow$ intersects interior of some 2 sides.

This gives cvx quad □

LEM 2. m pts in gen. pos. $\mathbb{R}^2$, any 4 in cvx position $\Rightarrow$ all m in cvx position.

PF. If not, $\exists$ pt inside full cvx hull:



Then triangulate cvx hull.

Gen pos $\Rightarrow$ pt inside a Δ $\Rightarrow$ 4 pts NOT in cvx pos. $\divideontimes$. □

PF. Let $n = R^{(4)}(5, m)$.

Color 4-tuple of pts ~~RED~~ BLUE if in cvx position. $\underline{L1} \Rightarrow$ no red $K_5^{(4)}$

$\underline{L2} \Rightarrow$ blue $K_m^{(4)}$ is the answer. □

2010-01-09
~~3~~ 4

GRAPH RAMSEY THEORY

DEF. $R(H_1, H_2)$ = min n s.t. every 2-coloring of K_n has H_1 in color 1 or H_2 in color 2.

Question of asymptotics on $R(K_t, K_t)$ is the famous diagonal Ramsey question.

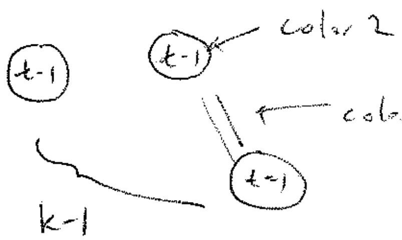
But for changing the second graph to tree, surprisingly can get exact result.

THIS STIMULATED GRAPH RAMSEY THEORY GREATLY.

TH. (CHVÁTAL 1977). For any tree T_t on t vtxs,

$$R(K_k, T_t) = (k-1)(t-1)+1.$$

PF. (LOWER BOUND: CONSTRUCTION)



Trees don't have enough vtxs.
Color-1 set must use ≤ 1 per cluster.

(UPPER BOUND). Induction: Assume have coloring with $(k-1)(t-1)+1$ vtxs and no red K_k. Find blue T_t.

BASE CASE: $t=1$. Trivial

On blue graph, indep number is $\alpha \leq k-1$.

So pull away max indep set.

Max indep set
size $\leq k-1$.

KEY PROPERTIES. $G-S$ still has $\geq (k-1)(t-2)+1$ vtxs
JUST RIGHT FOR INDUCTION $\longrightarrow$ find T_{t-1}
(obtained by deleting leaf of T_t)

Also, every vtx in $G-S$ is adj. to someone in S, by MAX.

So can extend T_{t-1} to T_t.

PF 2. Will follow next week, when we study tree embedding.

2010-01-09

(5)

RAMSEY THEORY: INTEGERS

(Before Ramsey for graphs!)

<u>TH</u>. (VAN DER WAERDEN 1927).

For any $k \geq 2$, $\ell \geq 3$, there is $\underline{n}$ s.t. every k-coloring of $[n]$ contains ℓ-AP.

~~Integers~~ "The set of integers contains arbitrarily long AP's".

Later, Hales/Jewett found this thm which implies many Ramsey-type stmts.

"Tic-tac-toe has no draws EVER in suff. high dimensions".

ASIDE: Strategy-stealing $\Rightarrow$ FIRST PLAYER WINS!

<u>DEF</u>. Consider "board" $[\ell]^d$, i.e. d-dim. vectors of symbols from $[\ell]$.

(TIC TAC TOE is $[3]^2$)

Write: $(1, 2, *, 1, *)$ to mean (in $[3]^5$) the following line:

ℓ

12111
12212
12313.

(it's a winning position in 5-dim $3\times3\times3\times3\times3$ TTT.)

This is a COMBINATORIAL LINE.

So in $[3]^2$, this covers all winning lines except ~~these~~ inverse diagonal.

<u>TH</u>. Given any $\underline{\ell}$ and $\underline{k}$, exists $\underline{d_0}$ st. every k-coloring of $[\ell]^{d \geq d_0}$ contains monochromatic combinatorial line.

<u>PF</u> Can be done in 1 lecture, see Jukna.

<u>PF</u> (VDW $\leftarrow$ HJ) HJ $\to d$ from ℓ, k. Set $n = \ell d$, and color $[\ell]^d$ like this:

$$\text{Color}(x_1, x_2, \ldots, x_d) = \text{color of } x_1 + x_2 + \cdots + x_d \quad (\text{ranges up to } \ell d).$$

Get comb. line, which translates into AP via $(x_1, \ldots, x_d) \mapsto x_1 + \cdots + x_d$.

Common difference is the # of "$*$"s. $\square$

COMPARE DENSITY: (SZEMERÉDI) For any $\delta > 0$ and $\ell \geq 3$, $\exists n_0$ s.t. EVERY subset $|S| \geq \delta n$ of $S \subseteq [n]$ contains ℓ-AP.

Much stronger, much harder. (Consider $\delta = \frac{1}{k}$ to get VDW)

DENSITY HALES-JEWETT: Ergodic Theory $\to$ ineffective bound; Gowers Feb 09, Polymath1 solved in 6 weeks

BIPARTITE TURÁN

2010-01-11
(1)

Can ask Turán-type problems for other non-complete graphs

DEF. $ex(n, H)$ = max # edges in any H-free graph.

Say $H =$  C_4.

Can get n edges with C_n and no C_4. But can do better: $C_4 \not\subset K_3$, so can take $ex(n, K_3)$.

Actually get even better by using $\chi(H)$.

Turán graph with $\chi(H)-1$ parts has no H. $\Rightarrow ex(n, H) \geq t_{\chi-1}(n)$

$$\approx \left(1 - \frac{1}{\chi-1}\right)\frac{n^2}{2}.$$

TH. (ERDŐS–STONE–SIMONOVITS) Given H, $\varepsilon > 0$, then for suff. large n, every graph with $\geq \left(1 - \frac{1}{\chi-1} + \varepsilon\right)\frac{n^2}{2}$ edges contains copy of H.

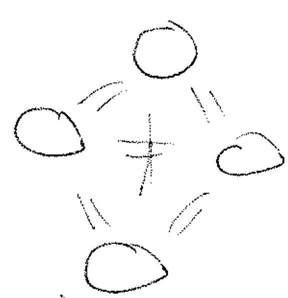
COR. $ex(n, H) = \left(1 - \frac{1}{\chi-1} + o(1)\right)\frac{n^2}{2}$.

NOTE: determines asymptotics for all graphs except bipartite.

Actually very interesting: shows that if you have $\geq \varepsilon n^2$ edges (positive density), then can get any bipartite graph you want.

TIGHT quantitative bounds:

TH. (ERDŐS–STONE–BOLLOBÁS) $\forall \varepsilon > 0$, every graph with $\geq \left(1 - \frac{1}{r-1} + \varepsilon\right)\frac{n^2}{2}$ edges contains:



$c' \log n$ all parts, r-partite.

IMPLIES PREVIOUS.

NOTE: Must be $O(\log n)$ since $G(n, c)$ has largest $K_{t,t}$ with $t = O(\log n)$.

Indeed, $\mathbb{E}[\# K_{t,t}] \leq n^{2t} p^{t^2} = (n^2 c^t)^t$. But if $n^2 c^t \to 0$, then $\mathbb{E} \to 0$.

Just need $t \sim \log n$ since c is const < 1.

One presentation will be new short proof of Nikiforov of this result.
It depends on bipartite Turán $ex(n, H)$ results.

~~TH. (KŐVARI-SÓS-TURÁN) Given any $s \le t$, there is constant $c_{s,t}$ s.t. for large n, every graph with $\ge c_{s,t} n^{2-\frac{1}{s}}$ edges contains copy of $K_{s,t}$.~~

First will be trees.

~~PROP~~. LEM Let T_t be a tree with t vtxs. Then every graph G with min-degree $\delta \ge t-1$ contains copy of T_t. (nice: $\delta \ge e(\text{Tree})$)

PF. Greedy embedding algorithm.

① Arbitrarily place one vtx of T_t.

② Find embedded vtx of T_t which has nbr still not embedded. Say it is $v \in G$.

③ Since in G $d(v) \ge t-1$, we could not have used all nbrs of $\underline{v}$ in embedding yet. (at most $t-1$ vtxs embedded so far — else done, and one of these is $\underline{v}$. So among $t-1$ vtxs in $N(v)$, only $t-2$ occupied.)

④ Arbitrarily choose a non-used nbr of $v \in G$ to be part of copy of T_t.

⑤ Go back to ② until done embedding all T_t. $\square$

Now connect min degree to avg degree.

LEM. ~~Every graph with at least one~~

Every nonempty graph with average degree $\underline{d}$ contains subgraph with minimum degree $\delta > \frac{d}{2}$.

Will give Turán result, via previous Lemma.

PF. Greedy Finding algorithm:

①. While there is vtx of degree $\le \frac{d}{2}$, delete it and all its edges from graph. # vtxs falls by 1.

② At end, <u>if result has ≥ 1 vtx</u>, then this will be the desired subgraph

Why we do not delete entire graph:

As algorithm progresses .....

#V	n	$n-1$	$n-2$	...	1
#E	$\geq \frac{nd}{2}$	$\geq \frac{nd}{2} - \frac{d}{2} = \frac{(n-1)d}{2}$	$\geq \frac{(n-2)d}{2}$	...	$\geq \frac{1d}{2}$

← if ~~#~~ #V left ever hits 1, then #E left is $\geq \frac{d}{2} > 0$.

[BUT CANNOT HAVE GRAPH ON 1 VTX WITH >0 EDGES.]

Therefore, #V never hits 1 in this algorithm, so we are done $\square$.

Implications for finding trees:

PROP. In any graph with avg degree $\geq 2(t-2)$, there is a copy of any T_t.

EQUIV: ~~for~~ any tree T_t on t vtxs, $ex(n, T_t) < n(t-2)$.

PF. avg degree $\geq 2(t-2) \Rightarrow$ find subgraph with $\delta > \cancel{t-}2$.

Since $\delta \in \mathbb{Z}$, actually have $\delta \geq t-1 \Rightarrow$ find T_t by greedy embedding. $\square$

But how tight is this?

Can you come up with a graph that has lots of edges but no trees of ~~size~~ vtxs t?

TRY: K_{t-1} K_{t-1} ... K_{t-1} Not enough space for trees.

$$\# \text{edges is} \approx \frac{n}{t-1} \times \binom{t-1}{2} = \frac{n(t-2)}{2}.$$

THUS: $\frac{n(t-2)}{2} \lesssim ex(n, T_t) < n(t-2)$.

Got up to factor 2. In extr. comb, interested in asymptotics.

First: try to get order of magnitude ← here is $\sim nt$.

Next: get constant up to $(1+o(1))$.

Last: get lower order terms / exact.

CONJ. (ERDŐS-SÓS) $ex(n, T_t) \leq \frac{n(t-2)}{2}$.

Remarks on Erdős-Sós conjecture.

Proof announced 20 (?) yrs ago by Ajtai, Komlós, Simonovits, Szemerédi.
Still not written up.

Miki said he was going to try to write it up last semester.

Very complicated, uses Regularity Lemma

Written proofs for special cases:

1. Approximate soln for large t: see Komlós-Simonovits survey
2. Erdős-Gallai 1959 for paths } → inspired original conjecture
3. trivial for star
4. Special case: diameter ≤ 4 McLennan

See also Univ. Rhode Island dissertation of Gary F. Tiner.

Arxiv paper by UC Davis student Jesse Gilbert, but probably wrong.

How about chromatic number?

PROP $\chi > t \Rightarrow$ graph contains any t-edge tree you want.

TIGHT Disjoint union of K_t's has $\chi = t$ but no t-edge trees.

(Unlike Erdős-Sós)

This ties into two more concepts

DEF degeneracy of a graph $= \max\{\delta(H) : H$ is induced subgraph of $H\}$.

So ~~if~~ "graph has degeneracy $\le x$" equiv to "in every ~~induced~~ subgraph, $\exists$ vtx of degree $\le x$".

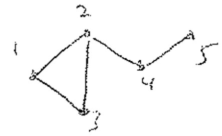
Is that right?

$\Rightarrow$: Consider induced subgraph H, Then $\delta(H) \le$ degeneracy $\le x$.

$\Leftarrow$: $\delta(H) \le x$ for all H, so degeneracy $\le x$.

DEF coloring # of graph = best ~~lowest~~ upper bound on χ that can be obtained by greedy algorithm.

What does this mean?
Consider ordering vtxs of graph:

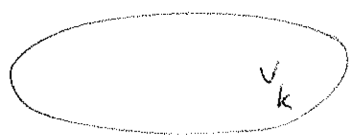


Go from left to right.
Suppose $\underline{d}$ is the max "left-degree" you encounter.
Then can be colored by $d+1$ colors
So each ordering of vtxs $\longleftrightarrow$ max left-degree
gives upper bound on χ.

<u>DEF</u> coloring # $= \min \{ 1 + \text{max left degree}(\pi) : \text{orderings } \pi \text{ of vtxs} \}$
So $\chi \leq col(G)$.

<u>PROP</u> coloring # = 1 + degeneracy

<u>PF</u>. ($\leq$): Consider the particular ordering, $\leftarrow$ "best algorithm" for greedy coloring
gives greedy ordering too.



v_n
min degree vtx of G.

$\uparrow$ ~~next~~
min degree vtx in $G - v_n$, etc.

Say v_m has max left-degree
So left-degree$(v_m) = \delta$(induced subgraph on vtxs up to v_i)
Then
$$\text{Coloring \#} \leq 1 + \text{left degree}(v_m) = \delta(\text{induced subgraph}) \leq \text{degeneracy}$$

($\geq$): col(G) needs to be $\geq 1 + \delta(H)$ for every induced $H \subseteq G$.
But $col(G) \geq col(H)$
and any ordering of H has the final vtx's left-degree $= d_H(v_f)$.
the max includes this, so $col(H) \geq 1 + d_H(v_f)$
$\geq 1 + \delta(H)$. $\square$.

<u>PF PROP X</u>. Since: $t < \chi \leq \text{col} = 1 + \text{degeneracy}$,

$\exists$ induced subgraph H s.t. $\delta(H) \geq t$.

$\Rightarrow$ find any t-edge tree $\square$.

<u>PF 2 OF CHVÁTAL</u> Recall $R(K_k, \cancel{t}\text{-edge tree}) = (k-1)t+1$. Say have this many vtxs.

Consider blue graph

$$\alpha \leq k-1 \Rightarrow \chi \geq \frac{(k-1)t+1}{k-1} > t. \Rightarrow \text{find } t\text{-edge tree}. \quad \square$$

For hypergraphs.

<u>DEF</u> hypertree:  Add new hyperedge on one vtx only

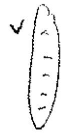
<u>TH</u>. (L.). In r-uniform hypergraphs $\chi > t \Rightarrow$ can find any r-unif. hypertree

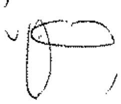
<u>PF</u> of special case (easily extended):

$\underline{*}$ r-uniform hypergraph with $\chi > 2$ contains .

~~State~~ By induction on # of hyperedges, show that -free hyp. have $\chi \leq 2$.

Base case: empty $\longrightarrow$ <u>ok</u>.

Induction step: new hyperedge:  ~~If~~ Can 2-color Graph minus it.
If this 2-coloring has edge 2-col ~~...~~, then done.

So this hyperedge is mono. Pick any vtx $\underline{v}$ of it to color with the other color.
Since no , will not make any other edge mono, so can do it. $\square$

<u>COR</u> (SHAPIRA) Plug this into PF 2 of Chvátal and you get hypergraph Ramsey

$$R_H(K_k^{(r)}, t\text{-edge } r\text{-tree}) = (k-1)t+1.$$

exactly, for case when $(r-1) \mid (k-1)$

2010-01-19

1

ZARANKIEWICZ COUNTING

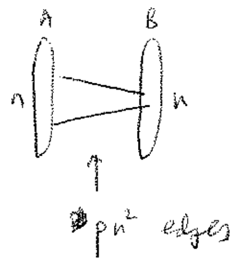
TH. (KÖVARI–SÓS–TURÁN) ~~If #edges $\geq c_{s,t} n^{2-\frac{1}{s}}$~~

Consider $K_{s,t}$, $s \leq t$. Then there is $c_{s,t}$ s.t.:

IF #edges $\geq c_{s,t} n^{2-\frac{1}{s}}$, THEN there is a copy of $K_{s,t}$.

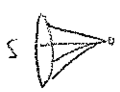
So $ex(n, K_{s,t}) = O(n^{2-\frac{1}{s}})$.

LEM. Bipartite counting: size of largest $s \times t$ submatrix of all-1 in $n \times n$ matrix with pn^2 1's.



$\forall s \leq t$, $\exists c_{s,t}$ s.t.

If $p \geq c_{s,t} n^{-\frac{1}{s}}$, then contains $K_{s,t}$.

PF. Count: #:  $= \sum_{v \in B} \binom{d(v)}{s}$

Convexity of $\binom{x}{s} = \frac{1}{s!}(x)(x-1)\cdots(x-s+1)$.

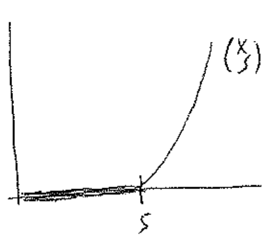
Der 1 $\sim$ ~~pe~~ sum of s products,
each product has $s-1$ factors from $x, \cdots, x-s+1$

Der 2 $\sim$ same, but each product has $s-2$ factors from these.

So if $x > s-1$, then $\binom{x}{s}$ convex.

~~NEED~~

Actually, extend it further since we are using non-integer values.



Define $\binom{x}{s}$ to be 0 on $x \leq s$.
From above, it is convex on all $x \geq 0$.

THUS by convexity,

$$\# = \sum_B \binom{d(v)}{s} \geq n\cdot\binom{\bar{d}_B}{s}$$

$$= n\cdot\binom{pn}{s} \qquad \bar{d}_B = \frac{\#\text{ edges}}{n} = pn.$$

$$\sim n\cdot\frac{(pn)^s}{s!}$$

SINCE: $pn \geq s$, so $\binom{pn}{s}$ is not 0 and s const, $n \to \infty$.

So for every subset $S \subseteq A$ of s guys, let $d(S) = \#$ common nbrs.

Above $\Rightarrow$ $\underset{A}{\text{avg}}\; d(S) \geq \dfrac{n\cdot\frac{(pn)^s}{s!}}{\binom{n}{s}} \approx \dfrac{n\cdot\frac{(pn)^s}{s!}}{\frac{n^s}{s!}} = np^s.$

Now $\# = \sum_{\substack{S\subseteq A \\ \text{size } s}} \binom{d(S)}{t} \underset{\text{cvx}}{\geq} \binom{n}{s}\cdot\binom{np^s}{t}$

$\boxed{\text{Now USE } np^s > t}$ This is why we need $p > cn^{-\frac{1}{s}}$.

Then $\binom{np^s}{t}$ is what we believe it to be.

$$\sim \frac{n^s}{s!}\,\frac{(np^s)^t}{t!}$$

$$= \frac{n^{s+t}\,p^{st}}{s!\,t!}$$

Plug in $p \geq cn^{-\frac{1}{s}}$, get

$$\geq \frac{n^{s+t}\,c^{st}\,n^{-t}}{s!\,t!}$$

$$= n^s\cdot\boxed{\frac{c^{st}}{s!\,t!}}$$ Some constant

$$> 0$$ □

REMARK: In fact, $\# K_{s,t}$ goes to infinity...

<u>COR</u>. (QUANTITATIVE BIPARTITE VERSION).

If # edges is $n^2 p$, then # $K_{s,t}$ in $n \bigcirc = \bigcirc n$ is $\gtrsim \frac{1}{s!t!} n^{s+t} p^{st}$.

<u>NOTE</u>. For random graph, this is expected # $K_{s,t}$:

$$\underbrace{\binom{n}{s}\binom{n-s}{t}}_{\#\text{ sites for } s\bigcirc\ \bigcirc t} p^{st} \longrightarrow \frac{n^s n^t}{s!t!} p^{st}.$$

all edges present

<u>CONJ</u> (SIDORENKO) For any bipartite graph H, in any graph with edge density p, the number of copies of H is minimized by the random graph.

$$\frac{\#\text{of homomorphisms } H \to G}{V(G)^{V(H)}} \geq p^{E(H)}.$$

<u>PF</u> (of K-S-T for non-bipartite host graphs)

Given n-vtx graph with $c_{s,t} n^{2-\frac{1}{s}}$ edges.

Say this means edge density is some $p > cn^{-\frac{1}{s}}$.

How to find $\frac{n}{2}\bigcirc = \bigcirc\frac{n}{2}$ with decent edge density?

<u>2 ways</u>.

① Random partition.

In graph with m edges, E[# edges in random bisection is] ..

$$\mathbb{P}[\underline{e} \text{ crosses } \underline{\text{random cut}}] = \frac{2 \cdot \binom{n-2}{\frac{n}{2}-1}}{\binom{n}{n/2}}$$

choose random subset of $\frac{n}{2}$ guys, cut is it vs complement

$$= 2 \cdot \frac{\frac{(n-2)!}{(\frac{n}{2}-1)!(\frac{n}{2}-1)!}}{\frac{n!}{(\frac{n}{2})!(\frac{n}{2})!}}$$

$$= 2 \cdot \frac{1}{\frac{n(n-1)}{\frac{n}{2}\cdot\frac{n}{2}}}$$

$$= 2 \cdot \frac{\frac{n}{2}\cdot\frac{n}{2}}{n(n-1)} = \frac{1}{2}\frac{n}{n-1} > \frac{1}{2}.$$

<u>So</u>: $E > \frac{1}{2}m$.

$\square$.

REMARK ON FAST ASYMPTOTIC

$$2 \cdot \frac{\binom{n-2}{\frac{n}{2}-1}}{\binom{n}{n/2}} = 2 \cdot \frac{(1+o(1))\,\frac{2^{n-2}}{c\sqrt{n-2}}}{(1+o(1))\,\frac{2^n}{c\sqrt{n}}}$$

$$= (1+o(1)) \cdot 2 \cdot \frac{2^{n-2}}{2^n}$$

$$= (1+o(1))\,\tfrac{1}{2}.$$

RELATED TO
MAX-CUT.

TH. Every m-edge graph contains bipartite subgraph with $\geq \frac{m}{2}$ edges,

AND every vtx has at least as many nbrs across as it does on its own side.

PF. Take extremal cut. If vtx fails greedy property, it should have been moved.

Greedy property $\Rightarrow$ # of edges crossing is:

$$\tfrac{1}{2}\sum_v (\text{cross degree}) \geq \tfrac{1}{2}\sum_v \frac{d(v)}{2} = \tfrac{1}{2}\,\#\,\text{edges}. \qquad \square$$

TH. (EDWARDS '75) Actually, can get cut of size $\geq \frac{m}{2} + \sqrt{\frac{m}{8}+\frac{1}{64}} - \frac{1}{8}$, tight.

OPEN PROBLEM But in our case, we wanted a BISECTION.

CONJ. (BOLLOBÁS, SCOTT). Every graph has a bisection which satisfies local greedy property, up to error ± 1.

PROP. (L.). For regular graphs, can do weighted version:

1. Split vtxs into 2 sides
2. Assign weights.
3. Weighted degrees satisfy greedy.

CONJ. (L.). For even-degree regular graphs can find bisection satisfying greedy with NO error.

But now how about tightness of the K-S-T result?

Need to construct graph with lots of edges that has no $K_{s,t}$.

Since random graph had lots of copies of $K_{s,t}$, suggests it is not best.

Although it has the fewest copies once $p > n^{-\frac{1}{s}}$, this doesn't apply before that.

<u>TEST</u> For what p does $K_{s,t}$ not appear in $G_{n,p}$.

$$\mathbb{E}[\# K_{s,t} \text{ in } G_{n,p}] = \binom{n}{s}\binom{n-s}{t} p^{st}$$

$$\sim \frac{n^s}{s!}\frac{n^t}{t!} p^{st}$$

Just need this < 1.

Will happen when $n^{s+t} p^{st} < \underbrace{s!t!}_{\text{const}}$

i.e. when $p = c\, n^{-\frac{s+t}{st}}$.

How does $\frac{s+t}{st} \gtrless_{vs} \frac{1}{s}$.

$\frac{s+t}{t} \gtrless_{vs} 1$.

So this has p smaller than K-S-T. (fewer edges).

<u>BETTER APPROACH</u> Alterations method

Take $G_{n,p}$ and for every copy of $K_{s,t}$, delete an edge to destroy it.

$\mathbb{E}[\# \text{edges}] = \binom{n}{2}p \sim \frac{n^2}{2}p$

$\mathbb{E}[\# \text{copies of } K_{s,t}] \sim \frac{n^{s+t}}{s!t!} p^{st}$

$\Rightarrow \mathbb{E}[\text{result } \# \text{ of edges}] \sim \frac{n^2 p}{2} - \frac{n^{s+t}}{s!t!} p^{st}$.

$\leftarrow$ emphasize that typically cannot count on both $\mathbb{E}$ hold simultaneously

When is $\frac{n^2 p}{4} = \frac{n^{s+t}}{s!t!} p^{st}$?

At $p \sim c \cdot n^{-\frac{s+t-2}{st-1}}$.

And get # edges $\sim \frac{n^2 p}{4}$.

Compare: $\frac{s+t-2}{st-1} \gtrless_{vs} \frac{1}{s}$

$s^2 + st - 2s \gtrless_{vs} st - 1$

$s^2 - 2s \gtrless_{vs} -1$ <u>OK</u>.

THIRD APPROACH H-free process.

~~Randomly pour~~ Randomly add edges that do not make $K_{s,t}$ until exhaustion.
Specifically: generate random permutation of $\binom{n}{2}$ edges
Go through sequence; if can add edge without making $K_{s,t}$ do it.

TH. (BOHMAN-KEEVASH / WOLFOWITZ) Gets more edges for $K_{s,s}$ by factor of $(\log n)^{\frac{1}{s^2-1}}$.

PROOF OF PROP (L.)

Consider adjacency matrix A of graph

Encode partition as vector $x \in \{\pm 1\}$.

A i [0 0 1 0 1 1 0]

Suppose $x_i = +1$.

Consider i^{th} coordinate $(Ax)_i$. This is $+1(\#\text{nbs of } i \text{ on } \oplus) - 1(\#\text{nbs on } \ominus)$

So would like $(Ax)_i \leq 0$.

In general, every coordinate of Ax should have opposite sign of coordinate of x
AND sum of coordinates of x should be 0 (bisection)

LEM. A has eigenvalues $\underline{d}, \ldots, \underline{\lambda}$, where λ is negative.

PF. All-ones vector is eigenvector, since d-regular $\longrightarrow$ gives $\underline{d}$.
$\operatorname{Tr} A = 0$ since diagonal is 0 $\Rightarrow$ have neg. eigenvalue
(All eigs $\mathbb{R}$ since symm matrix)
Also, all eigenvectors orthogonal, so λ's eigenvector $\perp \mathbb{1} \Rightarrow$ sum to 0.

Use y = negative eigenvector. This satisfies everything except ± 1,

BUT ROUNDING TRICK Look for soln in $(\mathbb{R}-0)^n$, so have nonzero weights.

~~Space is defined by inequalities~~

APPLICATIONS OF KÖVARI-SÓS-TURÁN

* PATRICK: move 1:30 meeting.
* Need volunteer speaker for: Mon, Feb 1. (one week from today)

Subject: Nikiforov's proof of Erdős-Stone Bollobás.

Monday will be the speaker first, then I talk about random bipartite graphs, ε-regularity.

RECALL.

TH. (KÖVARI-SÓS-TURÁN) For $s \le t$, $\exists c_{s,t}$ s.t. $ex(n, K_{s,t}) \le c_{s,t} n^{2-\frac{1}{s}}$.

Every graph with $\geqslant c_{s,t} n^{2-\frac{1}{s}}$ edges has $K_{s,t}$.

For $K_{2,2}$, ~~...~~, we had a series of constructions:

$G_{n,p} \to$ for $p \sim n^{-1}$, i.e. $\sim n^1$ edges, no $K_{2,2}$.

Alterations $G_{n,p} - K_{2,2} \longrightarrow$ for $p \sim n^{-\frac{2}{3}}$, i.e. $n^{\frac{4}{3}}$ edges, no $K_{2,2}$.

Bohman-Keevash $\longrightarrow$ gain $(\log n)^{\frac{1}{2}}$ factor.

CONSTRUCTION. (PROJECTIVE PLANE)

DEF PLANE over field $\mathbb{F}$ is just vector space $\mathbb{F}^2$.

DEF. REAL PROJECTIVE SPACE is manifold $(\mathbb{R}^{d+1} - 0)/\sim$, where $v \sim w$ if they are scalar multiples of each other.

$d+1$ since dimension is d after $\sim$.

Interpretation: "points" are lines through 0.

DEF. FINITE PROJECTIVE PLANE: $(\mathbb{F}^3 - 0)/\sim$ for finite field $\mathbb{F}$.

Not manifold since no continuous structure.

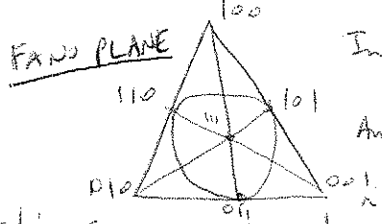
NOTE ~~DEF~~. "POINT" in PROJ. PLANE $\longleftrightarrow$ 1-dimensional vector space over $\mathbb{F}$ (since $\sim$)

DEF "LINE" in PROJ. PLANE $\longleftrightarrow$ 2-dimensional vector space over $\mathbb{F}$.

We could use any def as we want for "straight line". This is convenient.

Note that "line" is 1-dim. set of "points", can see by counting:

2010-01-24



In $\mathbb{F}_2$, know x,y,z in line if $x+y=z$. (2)
And no rescalings, so all 2^3-1 vectors appear.

IN PROJ. PLANE: over $\mathbb{F}_q$

$$\#\text{points} = \frac{q^3-1}{q-1}$$ Each 1-dim space uses $q-1$ nonzero vectors.

$\#\text{lines} = \#\text{points}$ since they are dual: POINT $\leftrightarrow$ ORTHOGONAL COMPLEMENT OF LINE.

Both of these numbers are $\boxed{q^2+q+1}$. Quadratic, so like dimension 2, "plane".

$$\#\text{points in any given line} = \frac{q^2-1}{q-1} = \boxed{q+1}$$ Since line has q^2-1 nonzero vectors.

KEY FACT (Geometry) For every pair of distinct points, ~~determine unique line.~~ there is exactly one line containing both points.

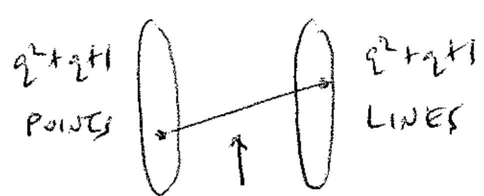
~~PF. Prove that map is bijection: $\#(P_1, P_2) = $ 2-dim subspace $\langle P_1, P_2\rangle$.~~

~~It is a 2-dim subspace since $P_1 \neq P_2$, so they are generated by vectors v_1, v_2 which are NOT scalar multiples, hence independent.~~

~~INJECTIVE~~

PF. Need 2-dim subspace that contains both 1-dim subspaces P_1, P_2, where $P_1 \neq P_2$. So P_1, P_2 generated by v_1, v_2 that are not scalar multiples of each other. Any 2-dim subspace containing both must contain $\langle v_1, v_2\rangle$, already 2-dim, so that is it. $\square$

DEF. INCIDENCE GRAPH of projective plane:



edge if POINT $\in$ LINE.

ANALYSIS

1. CO-DEGREE of any pair of vtxs on LEFT is EXACTLY 1, so no $K_{2,2}$.
2. $\#\text{vtxs} \approx q^2$.
3. $\#\text{edges} = (\#\text{LINES}) \times (\#\text{PTS on LINE}) \approx q^2 \cdot q = q^3. \sim V^{\frac{3}{2}}$. CORRECT.

REMARK. Tracing back KST argument $\Rightarrow$ this is <u>unique</u> extremal graph for $n = 2(q^2+q+1)$.

On Wednesday, will see extremal graphs for $K_{3,3}$ and $K_{s,(s-1)!+1}$.

APPLICATION: ERDŐS UNIT DISTANCE PROBLEM.

Q. Try to position n points in $\mathbb{R}^2$ such that # of pairs of points at Euclidean distance $= 1$ is as large as possible. How many unit distance pairs can you get?

CONJ. (ERDŐS). No matter what, # unit distances $\leq n^{1+o(1)}$.

OBS. # unit distance pairs $\leq \binom{n}{2} = \Theta(n^2)$

PROP. # unit distance pairs $\leq O(n^{3/2})$

PF. Create graph: edge between pair if distance $= 1$.

No $K_{2,3}$, so max # edges $\leq ex(n, K_{2,3}) = O(n^{2-\frac{1}{2}}) = O(n^{3/2})$. $\square$

NOTE DIDN'T USE ALG. GEOMETRY, $x^2+y^2=1$.

TH. (SPENCER-SZEMEREDI-TROTTER 84) # unit distances $\leq O(n^{4/3})$

SZÉKELY 97 Proof also uses a graph, by drawing unit circles around each point, and geometric edges.

PERFECT EDGE PACKING Another application of projective plane.

PROBLEM (r, t given). Would like to partition edges of K_r into sets, each of which is the edge set of some K_t.

Obviously, need $\binom{t}{2} \mid \binom{r}{2}$, $(t-1) \mid (r-1) \leftarrow$ degrees

TH (WILSON). For every graph H, $\exists\, r_0$ s.t. for any $r \geq r_0$, satisfying:

$$\left\{\begin{array}{l} ①.\ e(H) \mid \binom{r}{2} \\ ②.\ \gcd(\text{degrees of } H) \mid (r-1). \end{array}\right\} \Rightarrow e(K_r) \text{ can be decomposed into copies of } H.$$

But sometimes need a smaller bound on r_0.

The following was used in our recent paper with Bohman, Frieze, Krivelevich, Sudakov:

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(4)

PROP. Let $r = q^2+q+1$ for prime power q. Then can decompose edges $e(K_r)$ into copies of K_{q+1}.

PF. Identify vtxs of K_r with points in proj. plane of order r.
For each line in proj. plane, ~~make a copy~~ take the copy of K_{q+1} corresponding to the $q+1$ points on that line.

WHY IS IT DECOMPOSITION?

Every edge of K_r is covered, since the 2 endpoints are in exactly one line, and that is also why it is covered exactly once $\square$

$ex(n, C_{2k})$

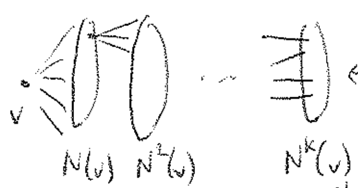
TH. ~~ex~~ (BONDY–SIMONOVITS). $ex(n, C_{2k}) \le c_k n^{1+\frac{1}{k}}$

PF is difficult. Instead, will prove the following which gives intuition for exponent

TH. Graph with no $\{C_3, C_4, \dots, C_{2k}\}$ has $< n^{1+\frac{1}{k}}$ edges.

PF. How does one usually use HIGH GIRTH?

Take arbitrary vertex



← still unique entry, but maybe have edges inside $N^k(v)$ since that can make C_{2k+1}.

no edges inside

← no collisions, or else have short cycle

So size of $N^t(v) \ge \delta^t$. In particular, #vtxs in graph $\ge 1+\delta+\delta^2+\dots+\delta^k$.

THUS If $\delta \ge n^{\frac{1}{k}}$, then we have contradiction, i.e.

PROP. Every graph with $\delta \ge n^{\frac{1}{k}}$ contains cycle of length 3, 4, …, or 2k.

Connect back to extremal number.

LEM Every graph with average degree d contains subgraph ~~of~~ with minimum degree $\delta > \frac{d}{2}$.

So, if #edges $= n(n^{\frac{1}{k}})$, then avg degree $= 2n^{\frac{1}{k}}$

$\Rightarrow$ have subgraph with $\delta > n^{\frac{1}{k}}$.

$\underset{\text{(PROP)}}{\Rightarrow}$ done. $\square$

REMARK For $C_4 = K_{2,2}$, tight via Projective plane
Will see constructions for C_6, C_{10} ~~th~~ Wednesday
C_8 still open! And all other even cycles still unknown order of magnitude
All constructions are ALGEBRAIC

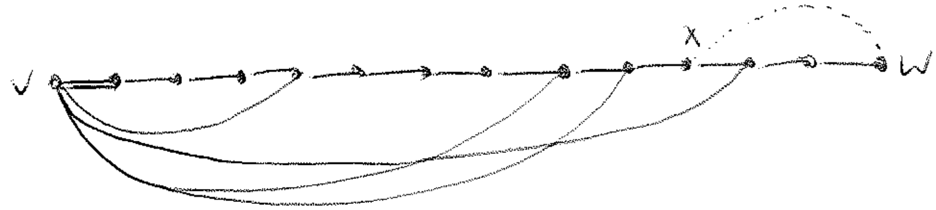
DIRAC'S THM ON HAMILTON CYCLES

We just did cycles of fixed size, $n \to \infty$.
How about spanning cycles?

TH. (DIRAC) Every graph with $\delta \geq \frac{n}{2}$ contains a Hamiltonian cycle

PF. CLAIM 1 If longest path has t vtxs, then $\exists$ cycle with t vtxs.

PF. Consider longest path:



Look at nbrs of v.
Each one blocks the vtx before it, since if w connects there, then get cycle
(Even true for nbr of v which is next vtx on path)
So out of $n-1$ vtxs in graph $G-w$, $\geq \frac{n}{2}$ blocked.
Not enough space for $d(w) \geq \frac{n}{2}$. $\ast$

CLAIM 2. G is connected.

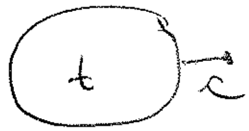
PF. Otherwise, some conn. cpt. of G has $\le \frac{n}{2}$ vtxs, and there $\delta \le \frac{n}{2} - 1$ $\divideontimes$.

FINAL. So take longest path, of ~~the~~ t vtxs

Get cycle

If $t = n$, done

If $t < n$, then:



since connected

$\Rightarrow (t+1)$-vtx path $\divideontimes$. $\square$

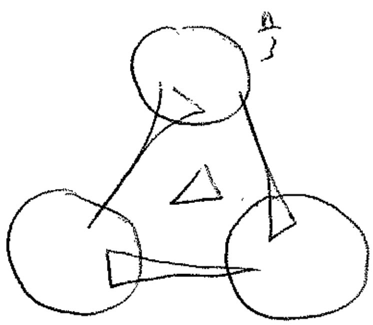
HYPERGRAPH TURÁN PROBLEMS

Can also ask analogous questions for r-uniform hypergraphs

Q. $ex(n, K_4^{(3)})$ is the first open problem.

This is max # of 3-edges in a 3-uniform hypergraph where no set of 4 vtxs spans all $\binom{4}{3}$ edges.

LOWER BOUND. Pack in $\frac{5}{9}\binom{n}{3}$ edges:



$$\# \text{ edges} = \left(\frac{n}{3}\right)^3 + 3 \cdot \frac{n}{3} \cdot \binom{n/3}{2}$$

$$= n^3\left(\frac{1}{27} + \frac{1}{9 \cdot 2}\right)$$

$$= \frac{n^3}{9 \cdot 6}(2+3) = \frac{5}{9}\binom{n}{3}$$

KOSTOCHKA has ∞ family of examples with this many edges $\longrightarrow$ May be a [illegible]

CONJ. This is asymptotic answer.

TH. r-unif hypergraph with avg degree d has $\alpha \geq c \cdot \frac{n}{d^{\frac{1}{r-1}}}$.

PF. Standard prob. method gives $\gtrsim \frac{n}{1+(r-1)d}$. We do better via Alterations

$$\left.\begin{array}{l} \# \text{vtxs} = np \\ \# \text{edges} = \frac{nd}{r}p^r \end{array}\right\} \to \text{Use } \frac{nd}{r}p^r = \frac{1}{2}np \quad \text{Need } p = \left(\frac{2r}{d}\right)^{\frac{1}{r-1}} \Big\} \to \mathbb{E}[\# \text{vtxs left}] = \frac{1}{2}n\left(\frac{2r}{d}\right)^{\frac{1}{r-1}}. \quad \square$$

REVERSING COMPLEMENT $\Rightarrow \exists\, c_t > 0$ s.t. $ex(n, K_t^{(r)}) < \binom{n}{t} - c_t n^r$

TIGHT TO CONST: bunch vtxs into groups of $d^{\frac{1}{r-1}}$. Each complete r-graph $\to$ degree $\sim d$.

Yet $\alpha \le (r-1) \times \#$groups. $\square$

ε-REGULARITY

2010-01-31

①

In research, often try to solve problems in $G_{n,p}$ when they are hard in "general graphs".

DEF. $G_{n,p}$ = random graph, n vtxs, all edges with probability p.

If something holds in $G_{n,\frac{1}{2}}$ WHP, then it actually holds "for almost all graphs"

since $G_{n,\frac{1}{2}}$ is uniform distribution over all labeled graphs: $\mathbb{P}[\text{result is given labeled graph}] = p^{E}(1-p)^{\binom{n}{2}-E}$.

EXAMPLE of problem I was working on last week:

(like to show real problems that we actually work on, to get taste for research).

DEF. Cop number of a graph = min # of cops needed to catch robber with rules:

1. Cops position on board first.
2. Robber picks position.
3. All cops either move 1 or stay still (can have some of each)
4. Robber either moves or stays still.
5. repeat ③

(Not a positional game like Michael Krivelevich last week)

ALSO: GO TO SEMINAR! Learn slowly by watching. Takes long time, must start early.

CONJ. Cop number of any n vtx graph is $\leq O(\sqrt{n})$.

Only results are $\leq \dfrac{n}{2^{\sqrt{\log n}}}$ ← any improvement would be great.

PLANAR → 3

TH (~~LUCZAK, PRALAT~~)

(BOLLOBÁS-KUN-LEADER) Cop number of $G_{n,p}$ is $\leq \sqrt{n}\log n$ for $p \geq \frac{2.1\log n}{n}$

Why are $G_{n,p}$ nice to work with? Since we know a lot about the structure.

TH (CHERNOFF BOUND) For any $\epsilon > 0$, there is $c_\epsilon > 0$ s.t. for any n, p,

$$\left.\begin{array}{l} \mathbb{P}\left[\mathrm{Bin}(n,p) > (1+\epsilon)np\right] \\ \mathbb{P}\left[\mathrm{Bin}(n,p) < (1-\epsilon)np\right] \end{array}\right\} \leq e^{-c_\epsilon np}$$

$$\mathbb{P}\left[\mathrm{Bin}(n,p) \notin (1\pm\epsilon)np\right]$$

Gives nice properties for $G_{n,p}$

<u>PROP</u>. ~~Suppose p~~ Given $\epsilon > 0$. Then $\exists C$ s.t. if $p > \frac{C \log n}{n}$, then WHP ALL vtxs in $G_{n,p}$ have degree in $(1 \pm \epsilon) np$.

<u>PF</u>. ~~For each vtx v~~

Let $B_1, \dots, B_n$ be the events that vtx i gets degree not in $(1 \pm \epsilon) np$.

Each $P[B_i] < e^{-c_\epsilon np}$. Let $C = \frac{2}{c_\epsilon}$.

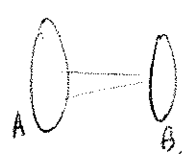
~~So $\exists C$ s.t. $c_\epsilon np > 2 \log n$.~~

Then $< e^{-2 \log n} < \frac{1}{n^2}$.

Union bound over n bad events $\Rightarrow$ failure prob $\frac{1}{n}$, i.e. WHP. $\square$

Slightly more general:

<u>PROP</u>. Given $\epsilon, p > 0$ (constants). Then $\exists C$ s.t. WHP <u>EVERY</u> pair of disjoint sets A, B of ~~sizes~~ $> C\sqrt{n}$ vtxs have # edges in $(1 \pm \epsilon) |A| \cdot |B| \cdot p$.



<u>PF</u>. Union bound: For each pair A, B, define event $E_{A,B}$ that it fails.

$$\mathbb{P}[E_{A,B}] < e^{-c_\epsilon ABp}$$

Choose C^2 s.t. exponent is $-2n$.

$$< e^{-c_\epsilon C^2 pn}$$

$$= e^{-2n}$$

But # of (A, B) is $\le 2^n \cdot 2^n$ (# of subsets!$)^2$.

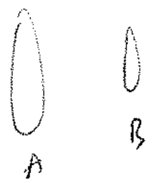
So union bound: Failure prob: $2^{2n} e^{-2n} = o(1)$. $\square$

Regularity lemma lets us find random-like objects in general graphs.

First property is too strong: general graphs can easily have tiny local perturbations, especially in Turán context where we only know about # edges $\sim Cn^2$.

Can drastically change local properties ~~by~~ without using quadratic # edges

DEF. In graph, the pair (A, B) (disjoint subsets of vtxs) is <u>ε-REGULAR</u> if:



For every choice: $A' \subseteq A$, $|A'| \geq \epsilon |A|$
$B' \subseteq B$, $|B'| \geq \epsilon |B|$

Why same ε? To reduce quantifiers.

~~get~~ $\cancel{\left|\frac{e(A',B')}{e(A,B)}\right|}$ have $|d(A',B') - d(A,B)| \leq \epsilon$.

Here, $d(X,Y) = \dfrac{e(X,Y)}{|X|\cdot|Y|}$

<u>NOTE</u>: if had random edges between A, B with $p = d(A,B)$, then we would have ε-regularity for arbitrarily small ε: $\mathbb{E}[\#\text{ edges between 2 sets}] = abp$, and since p is constant (dense graph), we consider just $\mathbb{P}[\mathrm{Bin}(ab, p) \notin (1\pm\epsilon)abp] < e^{-c_\epsilon abp}$. Union bound is over 2^{A+B} choices of the subsets, so if a, b are commensurate, then $a, b > \epsilon A$ is sufficient for union bound.

<u>CONSEQUENCES</u>:

<u>PROP</u>. Almost all vtxs have the right degree.

1. # of vtxs in A with degree $< (d-\epsilon)B$ is $< \epsilon A$
2. ________ $>$ ________

<u>PF</u>. Let A' be set of vtxs in A with degree $< (d-\epsilon)B$.
Let $B' = B$
If $A' > \epsilon A$, then $d(A', B') < d - \epsilon$, contradiction.
Similar for ②. □

<u>PROP</u> For any subset $B' \subseteq B$ of size $> \epsilon B$,

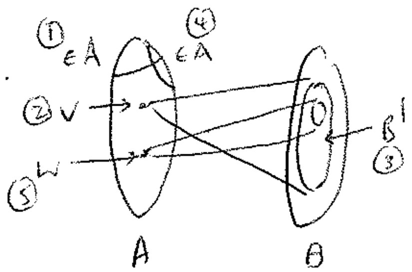
1. # of vtxs in A with degree $< (d-\epsilon)B'$ is $< \epsilon A$.
2. ________ $>$ ________

<u>PF</u>. Same proof. □

This simple property lets us find subgraphs.

<u>PROP</u> Given $\epsilon, d > 0$, constants. Then for all suff. large n, every ϵ-regular pair with $A = B = n$ and density $\geq d$ contains $K_{2,2}$.

<u>PF</u>.



① Delete $< \epsilon A$ vtxs of A so that all vtxs in A have degree $> (d-\epsilon)B$.

② $\boxed{\text{Choose}}$ vtx v from remainder.

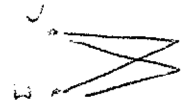
③ Let $B' = N(v)$, of size $> (d-\epsilon)B$.

④ Delete $< \epsilon A$ vtxs of A so that all vtxs in A have degree $> (d-\epsilon)B'$ into B'. ← Needs $B' > \epsilon B$, true since ^ large.

⑤ $\boxed{\text{Choose}}$ vtx w from remainder.

⑥ Inside B', w has $> (d-\epsilon)^2 B$ nbrs.

⑦ $\boxed{\text{Choose}}$ 2 vtxs in there, and get $K_{2,2}$:



$\square$

Observe that just as in Kövari-Sós-Turán, we actually have a quantitative result.

First choice has $(1-\epsilon)n$

Second choice has $(1-2\epsilon)n$.

Third 2 choices have $\sim \frac{1}{2}\left[(d-\epsilon)^2 n\right]^2$

$\Big\} \to \geq (1-\epsilon)(1-2\epsilon) \times \frac{1}{2}(d-\epsilon)^4 n^4$ choices.

And double-count $\times 2$ since $v \leftrightarrow w$.

$\Rightarrow \geq (1-\epsilon^*) \frac{1}{4} d^4 n^4$ copies of $K_{2,2}$.

If edges were random with prob. $p = d$, then $\# K_{2,2}$ would be $\binom{n}{2} \cdot \binom{n}{2} d^4$, same.

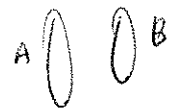
(*) EX. Also, same argument shows that # of copies of $K_{2,2}$ is $\leq (1+\epsilon^*) \frac{1}{4} d^4 n^4$.

<u>EX</u>. Do this for embedding $K_{t,t}$. Find $(1 \pm \epsilon^*) \frac{1}{(t!)^2} n^{2t} d^{t^2}$ copies.

REGULARITY LEMMA AND APPLICATIONS

Recall:

DEF. Pair (A,B) is ϵ-REGULAR if: every:



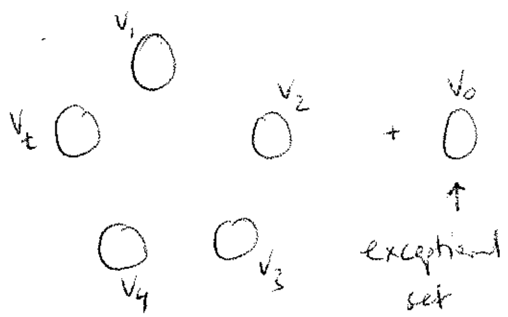
$$\left.\begin{array}{ll} A' \subseteq A & A' \geq \epsilon A \\ B' \subseteq B & B' \geq \epsilon B \end{array}\right\} \Rightarrow |d(A',B') - d(A,B)| < \epsilon.$$

Very surprising fact is that this is not a rare structure in a graph.

Szemerédi first used this in proof that sets of $\mathbb{Z}$ of positive density contain long AP.

TH. (SZEMERÉDI REGULARITY LEMMA)

Given $\epsilon > 0$. There exists M constant s.t. every graph has partition of vtxs:



1. $|V_1| = |V_2| = \cdots = |V_t|$.
2. $|V_0| < \epsilon n$.
3. $\frac{1}{\epsilon} < t < M$.
   meaningless if $t=1$, and we in applications will want t to be large in some sense.
4. All but ϵt^2 of the pairs (V_i, V_j) are ϵ-regular. $i,j > 0$

KEY FEATURE: M is CONSTANT independent of n.

This has given linear time algorithms for some problems (property testing).

BUT: Bound which emerges from proof is: $M < 2^{2^{2^{2^{\cdot^{\cdot}}}}} \Big/ \frac{1}{\epsilon^5}$.

TH. (GOWERS) There exist graphs which require $M > 2^{2^{\cdot^{\cdot^{2}}}} \Big/ \frac{1}{\epsilon}$ for a regularity partition.

We show several features of the regularity lemma first, including applications. Prove next week.

TH. (EQUIV FORM). Can avoid V_0 and instead say that all $|V_i|$ are within ± 1 of each other.

2010-02-01

2

First classical application

TH. (ROTH) Given any $\delta > 0$, constant. Then for suff. large n, EVERY subset of $[n]$ of size δn contains a 3-term AP.

Start by proving the (6,3)-problem, or the:

TH. (TRIANGLE REMOVAL LEMMA)

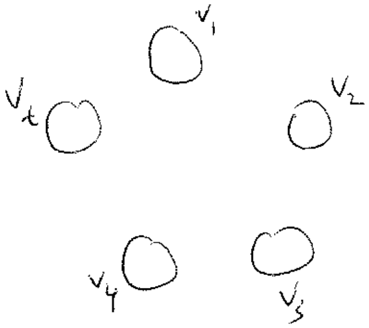
Given $\epsilon^* > 0$. Then there is $\delta > 0$ s.t.:

~~If a graph~~

If one needs to delete $> \epsilon^* n^2$ edges from an n-vtx graph in order to destroy all triangles (not completely delete Δ, but hit ≥ 1 edge), then the # of triangles is $\geq \delta n^3$.

PF. Assume the hypothesis.

Take ~~ε~~ Regularity Partition with parameter ϵ, and no exceptional set.



STANDARD PROCEDURE

1. Delete all edges in each single V_i
2. Delete all edges between pairs (V_i, V_j) that are NOT ϵ-REGULAR.
3. Delete all edges between pairs (V_i, V_j) that are SPARSE: density below δ.

How many edges deleted?

1. At most $t \cdot \binom{n/t}{2} \leq t \cdot \frac{(n/t)^2}{2} = \frac{1}{2}\frac{n^2}{t} \leq \frac{1}{2}\epsilon n^2$. since $t > \frac{1}{\epsilon}$

THIS IS WHY WE WANT LOTS OF PARTS

2. At most $\epsilon t^2 \left(\frac{n}{t}\right)^2 = \epsilon n^2$.

3. At most $\binom{t}{2}\delta\left(\frac{n}{t}\right)^2 \leq \frac{t^2}{2}\delta\frac{n^2}{t^2} = \frac{1}{2}\delta n^2$.

ALL TOGETHER: $\leq$ ~~...~~ $\left(\frac{3}{2}\epsilon + \frac{1}{2}\delta\right)n^2$ ← Will use $\epsilon \ll \delta$, say $\epsilon = \delta^2$, $\delta = \epsilon^*$

~~So we should have used $\frac{\epsilon}{2}$ regular partition to satisfy hypothesis.~~

CONCLUSION: Deleted $\le \epsilon^* n^2$ edges, so still have triangle

DEF ~~REGULARIZE~~ REGULARITY GRAPH: PARAMETERS: ϵ, δ

t vtxs, one representing each cluster.

Edge if the pair of clusters was a dense regular pair.

In particular, edge in regularity graph $\Leftrightarrow$ any edge remains between the (V_i, V_j) after these deletions.

So: REGULARITY GRAPH has triangle

Recall: ~~edge~~ single edge in regularity graph let us find lots of things:

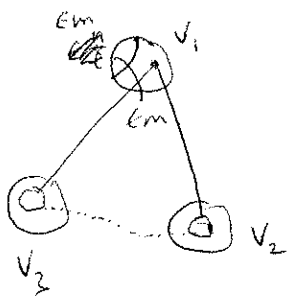
1. As big $K_{t,t}$ as we want, for constant t.
2. Moreover, it had a counting version:

   # of $K_{2,2}$ was WISB: $\Theta(n^4)$.

$\gamma = \gamma(\epsilon^*)$

LEM. If regularity graph has Δ, then graph contains $\ge \gamma n^3$ Δ's, for ~~$\delta \ge \delta(\epsilon)$~~

PF 1.



Let $m = \frac{n}{t}$. KEY FACT: Since $t < M$ bounded, n and m are of the same magnitude!

Delete $\le \epsilon m$ vtxs of V_1 that have degree into V_2 smaller than $(\delta - \epsilon)m$. Also for degree into V_3 smaller than $(\delta-\epsilon)m$.

Now CHOOSE remaining vtx in V_1. $\longleftarrow \ge (1-2\epsilon)m$ choices.

See how many Δ's can be completed thru it.

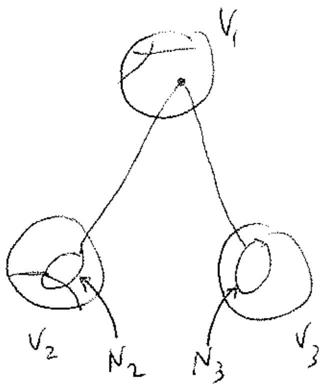
This # is precisely the # of edges between the neighbor sets in V_2, V_3.

But we may assume $(\delta-\epsilon)m > \epsilon m$, since $\epsilon \ll \delta$.

So # edges across is $\ge \underbrace{[(\delta-\epsilon)m]^2}_{\text{sizes of nbh}} \underbrace{(\delta-\epsilon)}_{d(V_2,V_3)} = (\delta-\epsilon)^3 m^2 \ge \underbrace{\frac{(\delta-\epsilon)^3}{M^2}}_{\gamma} n^2$ $\square$

$\Rightarrow$ TRIANGLE REMOVAL LEMMA.

PF 2. (This generalizes more easily to finding copies of graphs)



We eventually need to find lots of nbs of the guy we chose in V_1. So make sure:

1. Delete $\le \epsilon m + \epsilon m$ vtxs, so that V_1 guys have right degree into V_2, V_3: $\geq (\delta - \epsilon) m$
2. Choose one vtx left.
3. Consider its nbs in V_2, V_3, $\geq (\delta-\epsilon) m$ of them. Since this is $\geq \epsilon m$, still have that most vtxs have right degree:
4. Delete $\le \epsilon m$ vtxs of V_2 that have degree $\le (\delta - \epsilon) N_3$ into N_3
5. Still have $\geq (\delta - 2\epsilon) m$ choices. → choose one.
6. Then nbs into N_3 are $\geq (\delta-\epsilon)(\delta-\epsilon) m$, so can choose 3rd vtx.

GETS BOUND: $\geq (\delta - 2\epsilon) m \cdot (\delta - 2\epsilon) m \times (\delta - \epsilon)^2 m$. □

Similar order to before.

IN GENERAL: to get K_r, cut off $(r-1)\epsilon m$ from V_1, then have neighborhoods, then repeat procedure for K_{r-1}.

And to get blow-up of K_r, use the trick as in $K_{2,2}$ to get second vtx in V_1 that sees $\geq (\delta - \epsilon)$-fraction of every neighborhood of the first. Costs another $(r-1)\epsilon m$ vtxs of V_1.

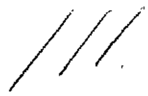
PF. (ROTH, by SOLYMOSI).

Mark points of $[n]^2$ if their coords sum $x+y$ to member of set.

# of pts marked $\geq \frac{1}{2}(\delta n)^2$, extremized if all pack into lower-left corner,

Define graph:

$$V = \begin{cases} n & \text{vtxs, one corresp. to each horiz line.} \\ n & \text{——— vert.} \\ 2n & \text{——— diagonals} \end{cases}$$



$\overline{\quad 4n. \quad}$

Edges between lines if they intersect on a marked point.

Lots of edge-disjoint triangles: For each marked point,

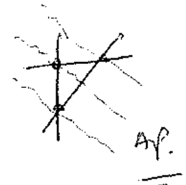
Use $\epsilon^* = \frac{1}{2}\delta^2$. ← Deleting an edge means forgetting one incidence witnessed by that marked vtx. Since too many marked vtxs, can't destroy all.

$\Downarrow$ REMOVAL LEMMA.

Actually have γn^3 triangles.

Take n suff. large s.t. $\gamma n^3 > n^2$.

So have a triangle which is not degenerate:



$\square$

TH. (ERDŐS-STONE-SIMONOVITS) Given H.

For any $c > 0$, if n is suff. large, then any graph with $\geq \left(1 - \frac{1}{\chi - 1} + c\right)\frac{n^2}{2}$ edges contains copy of H.

PF.

HOMEWORK #5

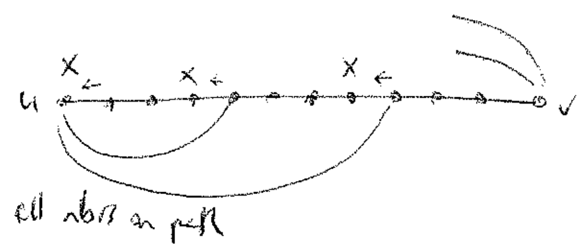
<u>TH</u>. (ERDÖS-GALLAI). Every graph with $E > \frac{n}{2}(t-2)$ contains t-vtx path.

<u>LEM</u>. If all degrees of <u>CONNECTED</u> graph are $\geq \delta$, then there is path with $\min\{2\delta+1, n\}$ vtxs.

<u>PF</u>. Same trick as Dirac:

(*) If longest path contains $L^{\leq 2\delta}$ vtxs, then there is cycle of L vtxs.

<u>PF</u>. Longest path:



Marked δ "x"'s in first $L-1$ vtxs. But $L-1 \leq 2\delta-1$, so not enough space to put nbs of <u>v</u> without making cycle.

So if graph is connected, then cycle extends  to longer path, <u>unless</u> exhausted all vtxs.

So either: $L = n$ (no extension due to space), <u>OR</u>

$L \geq 2\delta+1$. $\square$

<u>PF</u> (TH). Induction on <u>n</u>. Note that $n = t-1$ is impossible, since that is $\binom{t-1}{2}$.

So <u>BASE CASE</u>: $n \leq t-1$. (holds vacuously)

$E \leq 3V$.

Now exploit that bound is proportional to <u>n</u>. (useful trick for planar graphs)

Break into conn cpts. One satisfies $E' > \frac{n'}{2}(t-2)$.

In here, what if $\exists$ vtx of degree $\leq \frac{t-2}{2}$?

Pull it out, leaves $E'' > \frac{n'}{2}(t-2) - \frac{t-2}{2} = \frac{n''}{2}(t-2)$. Can use induction

So assume all degrees $\geq \frac{t-1}{2}$.

<u>LEM</u> $\Rightarrow$ either get path of $2(\frac{t-1}{2})+1 = t$ vtxs, or of n' vtxs.
But from initial observation, $n' > t-1$. $\Big\}$ done $\square$

2010-02-07

(1)

HW #3.

Equivalence classes?

min } degree
avg }
degeneracy

$>$ χ

$\uparrow$

Example: bipartite graph:

$k=2$.

For every t, $\exists$ graph with,

say, min degree $\geq t$, but all subgraphs have $\chi \leq 2$.

So large min degree cannot force large χ.

REGULARITY LEMMA: APPLICATIONS (CONTINUED)

Last time:

LEM. 

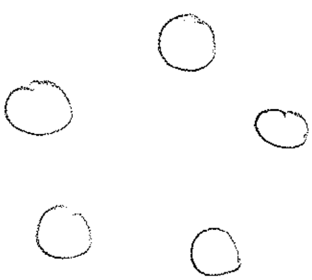
All pairs ϵ-regular and density $\geq \delta$, where $\delta \geq 3\epsilon$. (actually $\delta \gg \epsilon$)
Then # of $\Delta \geq (1 - 30\epsilon)\delta^3 m^3$.

Today we generalize this. But first, recap last argument.
Seeing second time helps you internalize better.

TH. (TRIANGLE REMOVAL LEMMA). For any $\eta > 0$, there is $\gamma > 0$ s.t. following holds for large n: if deleting any ηn^2 edges still leaves a Δ, then there were originally $\geq \gamma n^3$ Δ's

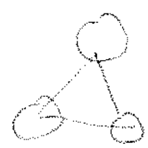
PF. Let $\epsilon = \frac{1}{3}\eta^2$ $\quad \delta = \eta$

Regularity partition: Get t parts of $\approx$ equal size, $\frac{1}{\epsilon} < t < M$.



All but ϵt^2 of the pairs are ϵ-regular.
1. Delete all edges inside each part
2. Delete edges between not-ϵ-regular parts
3. Delete edges between parts with density $< \delta$.

Total Loss: (1): $t \cdot \binom{n/t}{2} \leq t \cdot \frac{1}{2}\left(\frac{n}{t}\right)^2 = \frac{1}{2}\frac{1}{t}n^2 < \frac{\epsilon}{2}n^2$.
(2): $\epsilon t^2 \cdot \left(\frac{n}{t}\right)^2 \leq \epsilon n^2$.
(3): $\binom{t}{2} \cdot \delta\left(\frac{n}{t}\right)^2 \leq \frac{1}{2}\delta n^2$.

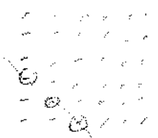
All together, under ηn^2, so still have Δ. Must be  $\Rightarrow$ actually have dense, ϵ-regular Δ.

LEM gives result with $\gamma \approx \frac{\eta^3}{M^3} \cdot$ since $m \geq \frac{n}{M}$.

This game:

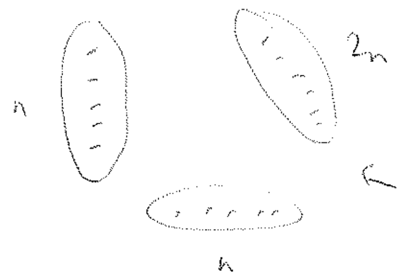
TH. (ROTH) For any $\delta > 0$, for suff. large n, every subset of δn integers in $[n]$ contains 3AP.

PF.



Mark pts in $[n]^2$ that have $x+y \in A$.
marked pts $\geq \frac{1}{2}(\delta n)^2$.

GRAPH $V = \begin{cases} n & \text{horiz lines} \\ n & \text{vert lines} \\ 2n & \text{diagonal lines} \end{cases}$



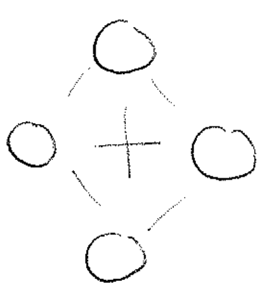
← Graph. 3-partite

E: if the lines intersect at a marked point.

Now suppose we delete only $\eta = \frac{1}{4}\delta^2$, ηn^2 edges.
Each edge deletion corresponds to a marked point, which will be a "witness". Mark it again.
So there is a marked point (actually many) that don't get touched at all.
In particular, untouched point $\Rightarrow$  still have triangle
REMOVAL LEMMA $\Rightarrow$ originally had γn^3 triangles
If any of form  then done, since AP.
Other ones are "degenerate", but only $\leq n^2$ of them,
$\gamma n^3 > n^2$ for large n, so done. □

Now we extend the ~~removal~~ counting lemma.

PROP. Suppose ϵ-regular pairs forming K_r, each cluster of size m, density $> \delta$.

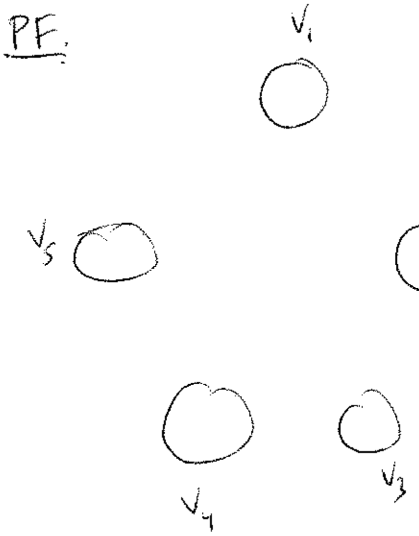


Suppose $\boxed{(\delta-\epsilon)^{rt} > rt\epsilon.}$
Then there is $K_r(t)$, the complete r-partite graph with parts of size t.
$[(\delta-\epsilon)^{rt} - rt\epsilon]m > t$.
actually $(r-1)t$. I was being sloppy. But $(r-1)t = \Delta$, so that is better.

2010-02-07

3

PF.



(1) Cut off $\le \epsilon m \times (r-1)$ vtxs from V_1 so that all vtxs in V_1 have degree into each other V_i of $\geq (\delta-\epsilon)m$.

(2) CHOOSE vtx left in V_1

Shrink $V_2 .. V_5$ to just its neighbors

Still have $V_2 .. V_5 \geq (\delta-\epsilon)m$.

(3) Apply ϵ-regularity again.

Want vtxs left in V_1 to have right degrees into shrunken V_i.

Delete again $\le \epsilon m \times (r-1)$ vtxs from V_1 for this

NEED $(\delta-\epsilon) \geq \epsilon$

$(\delta-\epsilon)^t \geq \epsilon$

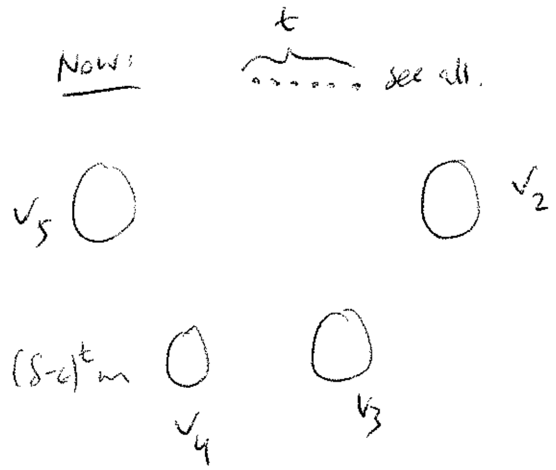
(4) CHOOSE vtx left in V_1

Shrink $V_2 .. V_5$, still have $V_i \geq (\delta-\epsilon)^2 m$.

(5) Repeat until we chose t vtxs in V_1.

$\#\text{ways} \geq [(1-rt\epsilon)m]^t$.

If we find $K_{r-1}(t)$ inside remainder, done since they all connect to first t.



(6) Delete $\le \epsilon m \times (r-2)$ vtxs from V_2 ...

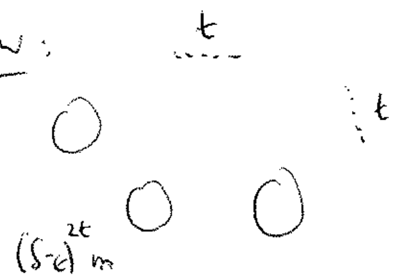
CHOOSE one, shrink $V_3 .. V_5$.

(7) Do this t times.

Total ways $\geq \{[(\delta-\epsilon)^t - rt\epsilon]m\}^t$.

(8) Do same process to pick t from V_3.

Total ways $\geq \{[(\delta-\epsilon)^{2t} - rt\epsilon]m\}^t$.



⋮

(9) Do same process to pick t from V_{r-1}.

Total ways $\geq \{[(\delta-\epsilon)^{(r-2)t} - rt\epsilon]m\}^t$

(10) Pick any t from V_r.

Total ways $\geq [(\delta-\epsilon)^{(r-1)t} m]^t$.

GRAND TOTAL $\gtrsim m^t \cdot \delta^{t^2} m^t \cdot \delta^{2t^2} m^t \cdots \delta^{(r-1)t^2} m^t$

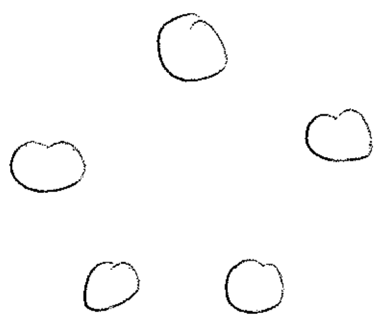
$= m^{rt} \delta^{t^2 \frac{r(r-1)}{2}} = m^V \delta^E$ in $K_r(t)$.

□.

TH. (ERDŐS–STONE–SIMONOVITS),
For any $H, c > 0$, following holds for suff. large n.
Every graph with edges $\geq \left(1 - \frac{1}{\chi(H)-1} + c\right)\frac{n^2}{2}$ contains copy of H.

PF. let $\epsilon = \frac{\delta^{v(H)^3}}{\ }$, $\delta = \frac{c}{3}$.

Apply Regularity Lemma



(1) Delete all edges in clusters.
(2) Delete all edges between not-ϵ-regular pairs
(3). Delete all edges in pairs of density $\leq \delta$.

TOTAL LOSS:
(1). $t \cdot \binom{n/t}{2} \leq t \cdot \frac{1}{2}\left(\frac{n}{t}\right)^2 = \frac{1}{2}\frac{n^2}{t} \leq \frac{\epsilon}{2} n^2$.
(2). $\epsilon t^2 \cdot \left(\frac{n}{t}\right)^2 = \epsilon n^2$.
(3). $\binom{t}{2} \cdot \delta \left(\frac{n}{t}\right)^2 \leq \frac{1}{2}\delta n^2$.

So after all deletions, still have edges $\geq \left(1 - \frac{1}{\chi(H)-1} + \frac{c}{2}\right)\frac{n^2}{2}$.
TURÁN $\Rightarrow$ graph contains K_χ.
By deletions, it must be in the regularity graph
COUNTING LEM $\Rightarrow$ have copy of $K_\chi(v(H))$.
But H lives in $K_\chi(H)$ since it has χ-coloring. (color class decomposition)
And $t \leq v(H)$, so $v(H)$ suffices. $\square$

COR. $ex(n, H) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right)\frac{n^2}{2}$.

TH. Given $\delta > 0$, $\Delta \geq 1$, $\exists \epsilon_0 > 0$ s.t.:
If G's REGULARITY GRAPH with parameters δ, ϵ_0 contains any subgraph H with $\Delta(H) \leq \Delta$, then G contains blow-up of H by factor $\frac{1}{2}\delta^\Delta m$, where m is the size of the cluster in the vtx partition

DEF. H is graph. $R(H) = \min n$ s.t. every 2-coloring of K_n contains mono H.

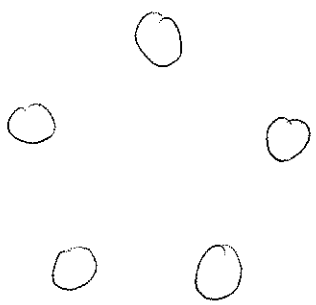
Of course, $R(H)$ exists since ~~2~~ $4^{v(H)}$ vtxs give mono $K_{v(H)}$.

But can we do better than exponential in vtxs?

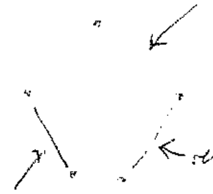
TH. (CHVÁTAL, RÖDL, SZEMERÉDI, TROTTER) For any Δ, $\exists$ $\underline{c}$ constant s.t. every H with max degree Δ has $R(H) \leq c|H|$.

PF. Let $\epsilon =$ ______.

Regularity partition into $\frac{1}{\epsilon} \leq t \leq M$ parts.



$\longrightarrow$ AUX graph:



No edge if not ϵ-regular.

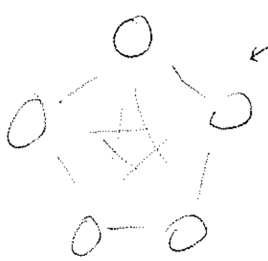
one vtx per cluster.

red if ϵ-regular & density $< \frac{1}{2}$.

blue if ϵ-regular and density $\geq \frac{1}{2}$

What if have red+blue K_N where $N = R(\Delta+1)$? Just need $N = 4^{\Delta+1}$.

Then have mono, say blue, $K_{\Delta+1}$.



$\Delta+1$ parts of size $\frac{n}{t}$.

$\leftarrow$ all ϵ-regular, density $\geq \frac{1}{2}$.

So TH $\Rightarrow$ G contains blow-up of this by factor $\frac{1}{2}\left(\frac{1}{2}\right)^{\Delta} m$.

What if ~~[illegible]~~ $\frac{1}{2}\left(\frac{1}{2}\right)^{\Delta} m > |H|$?

Then it contains H.

To ensure this, need $\frac{1}{2}\left(\frac{1}{2}\right)^{\Delta} \cdot \frac{n}{M} > |H|$.

so need $n > 2^{\Delta+1} M\ |H|$.

Restriction on M? How small $\underline{\epsilon}$ do we need?

TURÁN $\Rightarrow$ for $\epsilon = \frac{1}{2N}$, we already have that if ϵt^2 edges in K_t are missing, then have K_N. $\square$

AJTAI-SZEMERÉDI CORNERS THM

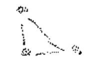
2010-02-16

1

This will give a warm-up to proof of Szemerédi Regularity Lemma.

Density increment argument.

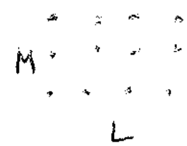
<u>TH</u>. (AJTAI-SZEMERÉDI) For any $\delta > 0$, following holds for suff. large $\underline{n}$.

Any subset of $[n]^2$ of density $\geq \delta$ contains right triangle: .

<u>PF</u>. <u>AIM TO SHOW</u>: Given δ, output $\delta + \eta$. Then given [illegible], determines $n_0 = n_0(\delta, \text{[illegible]})$ s.t.

(*). If no Δ in set of density δ in $[n]^2$,

then there is an $L \times L$ subgrid s.t. density there is $\geq \delta + \eta$.

<u>DEF</u>. $L \times M$ grid in $[n]^2$: collection of positions 

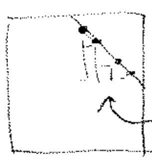
Equal spacing in both vert/horiz direc.

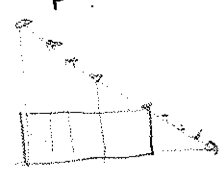
<u>CLAIM 1</u> No $\Delta \Rightarrow \exists$ diagonal with $\geq \frac{\delta}{2} n$ points of set.

<u>PF</u>.  $\leftarrow$ $2n$ diagonals, so if all have $< \frac{\delta}{2} n$, total # of pts $< \delta n^2$.

<u>CLAIM 2</u> $\exists$ Cartesian product $U \times V \subseteq [n]^2$ with density $\geq \rho = \delta + \frac{\delta^3}{16}$.

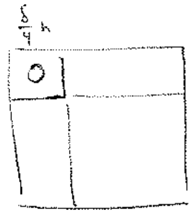
and $|U|, |V| \geq \frac{\delta}{4} n$.

<u>PF</u>.  No pts of A allowed in intersections, else Δ.

Take rectangle in lower subsquare: 

It can only have 1 pt in A

Say its density is essentially 0.

So: 

$\Rightarrow$ one of other 3 rectangles has increased density.

The total density δ is distributed over region of area $1 - \left(\frac{\delta}{4}\right)^2$ fraction of original.

$\Rightarrow$ one of 3 rectangles has density $\geq \rho := \delta\left(1 + \frac{\delta^2}{16}\right)$

$= \delta + \frac{\delta^3}{16}$.

$$\frac{1}{1 - \left(\frac{\delta}{4}\right)^2} > 1 + \left(\frac{\delta}{4}\right)^2$$

CLAIM 3. $U \times V$ can be perfectly partitioned into:

disjoint union of $L \times L$ grids, rubbish, $\le \underline{4\epsilon}\, n^2$ points.

PF. Let $\epsilon = \underline{\delta^6}$.

Define: M = integer s.t. Szemerédi's Thm holds for ϵ and $T = \frac{L}{\epsilon}$.

TH. (SZEMERÉDI) For any $\epsilon > 0$ and integer T, $\exists n_0$ s.t. every subset of $[n]$ with density $\ge \epsilon$ contains T-AP.

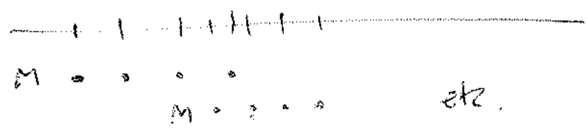
PF. (Hard. But many proofs. Apparently new one on arxiv this week by Terry Tao/Ben Green)

Assume: ~~$\mathcal{M}$~~ n large enough s.t. Szemerédi's Thm holds for ϵ and $T = \frac{M}{\epsilon}$.

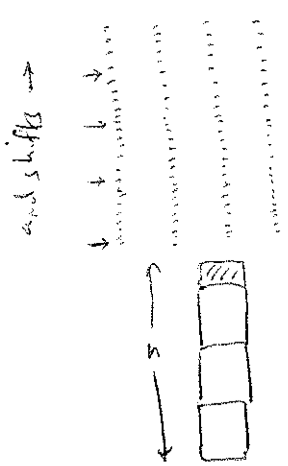
GREEDY ALGORITHM. $|U| \ge \frac{\delta}{4} n$.

Find AP in U of length $\frac{M}{\epsilon}$, then break further into $\frac{1}{\epsilon}$ AP's of length M.
Remove it from set.
Continue.
Until only $\le \epsilon n$ ~~pts~~ values left in U. WASTE: $\le \epsilon n \cdot n$ pts in $U \times V$

Now: $U =$



So we can almost partition $U \times [n]$ into GRIDS.
Consider the fiber above one M-AP:



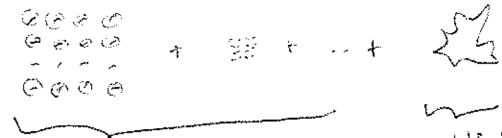
Decompose into $M \times M$ grids that are translates up.
But how much waste (uncovered)?
Block size is ~~...~~ $\le \epsilon n$ coordinates tall since we took $\frac{M}{\epsilon}$ AP and cut it into $\frac{1}{\epsilon}$ pieces.
So we lose at most ϵn coordinates of space on top.

(Actually, block size is very slightly bigger, but doesn't matter)

Now we have:

marked y-coords in V.

$U \times V =$



rubbish $\leq 2\epsilon n^2$.

$M \times M$ partially filled grids. by $U \times V$, not A.

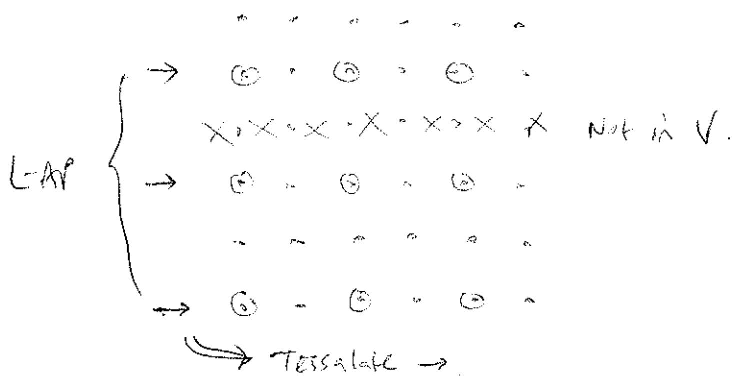
On each $M \times M$ grid, do same Szemerédi trick in y-direction (V).

GREEDY ALGORITHM While still have ϵM values left in the part of V corresponding to an $M \times M$ grid, pull out $\frac{L}{\epsilon}$-AP.

Then chop each AP into $\frac{1}{\epsilon}$ many L-AP's.

WASTE = $\leq$ ~~ϵ~~ ϵ-fraction of each $M \times M$ grid, so $\leq \epsilon n^2$ aggregate

One $M \times M$ grid:

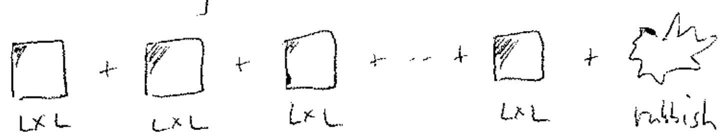


Again, since we have each L-AP taking up "block" space which is ϵ-fraction of total, the wasted space in tesselation is $\leq \epsilon$-fraction of total on right side.

After doing this to every $M \times M$ partially-filled grid, done, total waste $\leq 4\epsilon n^2$.

CLAIM 4. Have density increment.

PF.



Know that on this total, we had A-density $\geq \rho = \delta + \frac{\delta^3}{16}$.

$S = |U \times V|$. $\geq \frac{\delta^2}{16} n^2$.

Want to claim that on one of the L×L's, density is big.

Worst case is if rubbish is full.

Then ~~only~~ still $\geq (\delta + \frac{\delta^3}{16}) S - 2\epsilon n^2$ pts of A in the L×L's.

So avg density on L×L's is $> \delta + \frac{\delta^3}{16} - \frac{2\epsilon n^2}{S}$.

should divide by total area of L×L's. Dividing by more (S).

$> \delta + \frac{\delta^3}{32}$ Using $\epsilon = \delta^6$. $\square$.

$\boxed{\text{FLOW}}$. Given initial δ, calculate how many (finitely) steps before density hits 1. Then define in REVERSE the sequence of L's. At the end, find initial bound on n.

<u>Regularity Lemma</u>

(1)

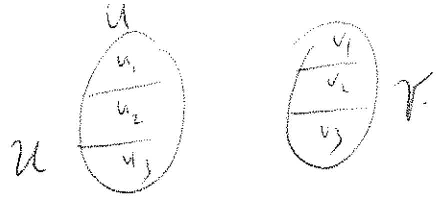
$$q(U,V) = \frac{|U|\cdot|V|}{n^2}\, d^2(U,V)$$

$$q(\mathcal{U},\mathcal{V}) = \sum_{\substack{U\in\mathcal{U}\\ V\in\mathcal{V}}} q(U,V)$$

~~Energy~~ $q(P) = \sum_{\substack{\text{unord.}\\ U,V}} q(U,V)$. Sum of $\frac{|U|\cdot|V|}{n^2}$ over all unordered $U\neq V$:

$$\le \frac{1}{2}\sum_{U,V} \frac{|U||V|}{n^2} = \frac{1}{2}.$$

Lem (i)



$q(\mathcal{U},\mathcal{V}) \geq q(U,V)$.

Pick random $u\in U$, $v\in V$.

$Z = d(\text{part of } \mathcal{U} \text{ on } u,\ \text{part of } \mathcal{V} \text{ on } v)$

~~pf~~) $E[Z] = \sum_{i,j} \frac{|U_i|}{|U|}\frac{|V_j|}{|V|}\, d(U_i,V_j) = \sum_{i,j} \frac{\cancel{|U_i|\,|V_j|}}{|U|\cdot|V|}\,\frac{e(U_i,V_j)}{\cancel{|U_i|\,|V_j|}}$

$= \frac{1}{|U|\cdot|V|}\cdot e(U,V) = d(U,V)$.

$E[Z^2] =$

$= \sum_{i,j} \frac{|U_i|\,|V_j|}{|U|\cdot|V|}\, d^2(U_i,V_j)$

$= \frac{n^2}{|U|\cdot|V|}\sum_{i,j} q(U_i,V_j) = \frac{n^2}{|U|\cdot|V|}\, q(\mathcal{U},\mathcal{V})$

<u>Jensen</u>: $E[Z^2] \geq (E[Z])^2$.

$\frac{n^2}{|U||V|}\, q(\mathcal{U},\mathcal{V}) \geq d^2(U,V)$.

$q(U,V)$ <u>ok</u>

(ii). If $\mathcal{P}'$ is refinement of $\mathcal{P}$ then $q(\mathcal{P}') \geq q(\mathcal{P})$. (2)

Since: $\frac{1}{U} \sum$

$q(\mathcal{P}') =$ sum of $q(U', U'')$ with U', U'' within each bulb of $\mathcal{P}$.

$+$ sum of $q(U', V')$ between different bulbs

these grow.

(iii). If U, V not ϵ-regular, then $\exists (U_1, U_2), (V_1, V_2)$

s.t. $q(\mathcal{U}, \mathcal{V}') \geq q(U, V) + \epsilon^4 \cdot \frac{u \cdot v}{n^2}$.

U: U_1, U_2; V: V_1, V_2

From above, ① $(E[Z])^2 = d(U,V)^2 = \frac{n^2}{u \cdot v} q(U, V)$.

② $E[Z^2] = \frac{n^2}{u \cdot v} q(\mathcal{U}, \mathcal{V})$.

Need to show $Var(Z)$ is big.

$\|$

② $-$ ①

U_1, V_1 = wrong density. Say deviates from $d(U,V)$ by ϵ.

Then w.p. $\geq \epsilon^2$ we see squared deviation of Z from mean by $\geq \epsilon^2$.

$Var(Z) = E[\text{sq. dev}] \geq \epsilon^2 \cdot \epsilon^2 \Rightarrow \frac{n^2}{u \cdot v}\left[q(\mathcal{U}, \mathcal{V}) - q(U, V)\right] \geq \epsilon^4$.

sum of ≥ 0 — small prob part

Now if not ϵ-regular, can boost index by $\geq \frac{\epsilon^5}{2}$. $q(\mathcal{P})$ has V_0 as singletons

treat as singletons in the partition

V_0 $\leq \epsilon n$. $\epsilon \epsilon^4$. V_1 m. V_2 m. V_t. m m m

If (V_i, V_j) already ϵ-regular, do nothing

Else, split both.

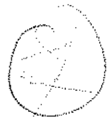
New partition has q grow, not shrink. $q = (\text{initial}) + \sum$ all parts $V_1 \leftrightarrow V_2$

But for each of ϵt^2 bad pairs, we gain by $\geq \epsilon^4 \frac{m^2}{n^2}$

GAIN: $\epsilon^5 t^2 \frac{m^2}{n^2}$.

Say $mt \geq \frac{3}{4} n$. $\geq \frac{\epsilon^5}{2}$.

Had t parts. Need to say # parts now is bdd.

m  at most $t-1$ other cuts ~~So~~

Altogether, $\le 2^{t-1}$ pieces

$\longrightarrow$ $t \cdot 4^t$ parts since each m gets $\le 4^t$ parts

Each piece cut further into blocks of size $\left\lfloor \frac{m}{4^t} \right\rfloor =: B$.

How much new rubbish?

$$\underset{\substack{\uparrow \\ \#\text{parts}}}{t} \cdot \underset{\substack{\uparrow \\ \text{pieces in part}}}{2^{t-1}} \cdot \underset{\substack{\uparrow \\ \text{elements per piece}}}{B} \qquad = t \cdot 2^{t-1} \cdot \frac{m}{4^t} \le \frac{n}{2^t}.$$

So rubbish grows by $\le \frac{n}{2^t}$.

<u>All together:</u>

Each step boost by $\frac{\epsilon^5}{2}$.

So # steps $\le \frac{4}{\epsilon^5}$.

Initially, rubbish at

$$n\Bigg[\underbrace{\frac{1}{2^t} + (\frac{1}{2^t} + (\frac{1}{2^t} \cdots}_{\frac{2}{\epsilon^5}\text{ steps.}}\Bigg]$$

So need $\frac{1}{2^t} \cdot \frac{4}{\epsilon^5} < \epsilon$.

$\frac{4}{2^t} < \epsilon^6$

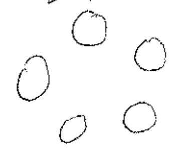
So start with t big, so that

Now each step, t grows. $t \to t 4^t$.

Also, rubbish grows

REGULARITY AND STABILITY

TH. (TURÁN) The unique graph on $\underline{n}$ vtxs with max # edges is:

$T_r(n) =$  $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$.

~~TH? (SIMONOVITS) let F be graph with $\chi = r+1$, s.t. $\exists$ edge whose deletion~~

~~DEF? If F is called COLOR CRITICAL~~

DEF. Graph F has a CRITICAL EDGE if $\chi(F) = r+1$ and $\exists$ edge whose deletion makes $\chi \le r$.

TH. (SIMONOVITS). Let F have $\chi = r+1$ and a CRITICAL EDGE. Then for all suff. large $\underline{n}$, the <u>UNIQUE</u> extremal graph for F is $T_r(n)$.

This was the first application of the Simonovits stability method.

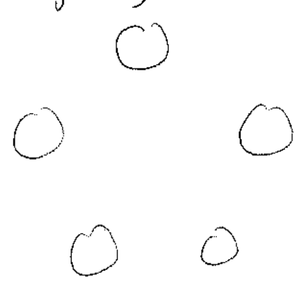
The proof will use the following natural result.

TH (ERDŐS-SIMONOVITS) STABILITY THEOREM

Let F have $\chi = r+1$. For any $\gamma > 0$, $\exists \eta > 0$ s.t. for all suff. large $\underline{n}$, if (F-free) $e(G) \ge (1 - \frac{1}{r} - \eta)\frac{n^2}{2}$, then G can be made r-partite by <u>deleting $\le \gamma n^2$ edges.</u> (carefully chosen)

PF. Let $\epsilon = \underline{\epsilon_0}$, $\delta = \underline{\frac{\gamma}{3}}$.

Apply Regularity Lemma.



t parts

1. Delete edges inside V_i's
2. Delete edges between NOT-ϵ-regular pairs
3. Delete edges between pairs with density $< \delta$.

TOTAL LOSS: $\le (\frac{\delta}{2} + \frac{3}{2}\epsilon) n^2$.

$\le \frac{\gamma}{3} n^2$ $\boxed{\text{CALL THIS } G'}$

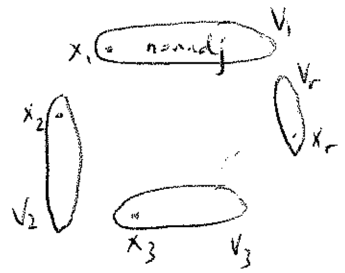
TH ~~KEY~~. Given $\delta > 0$, Δ, $\exists \epsilon_0 > 0$ s.t.:

IF G's REGULARITY GRAPH with parameters ϵ_0, δ contains any subgraph H with $\Delta(H) \le \Delta$, then G contains blow-up of H by factor $\frac{1}{2}\delta^\Delta m$, where $\underline{m}$ is regularity partition cluster size.

In particular, there is no K_{r+1} remaining, or else would find blowup of K_{r+1} by $|F|$, in which there is a copy of F.

LEM. Given any graph without K_{r+1}, with #edges $\geq e(T_r(n)) - k$, one can delete k more edges and make the graph r-partite.

PF.



(1) let x_1 be max degree vtx.
(2). V_1 = set of all non-nbrs of x_1. Delete V_1.
(3) x_2 = max degree vtx in remainder.
considered inside remaining induced subgraph
(4) V_2 = set of all non-nbrs of x_2. Delete
(5). x_3 = max degree vtx left...

Process only runs for $\leq r$ rounds, or else find K_{r+1} since all x_i adjacent.

SHOW: deleting all internal edges in V_i makes r-partite.
(show this costs $\leq k$ edges).

DEF. let $d^+(v)$ = degree of v in induced subgraph on $V_i \cup V_{i+1} \cup \cdots \cup V_r$, where V_i is part containing v.

By def, $\sum_v d^+(v)$ is almost the #edges, except that double-count all edges internal to V_i's.

$$e(G) = \sum_v d^+(v) + \sum_i e(V_i)$$

$$\leq \sum_i d^+(x_i)\cdot|V_i| + \sum_i e(V_i)$$

$$= \left(\text{\# edges in complete } r\text{-partite graph with sizes } |V_1|, \dots, |V_r|\right) + \sum_i e(V_i)$$

$$\leq e(T_r(n)) + \sum_i e(V_i)$$

$$\sum_i e(V_i) \leq e(T_r(n)) - e(G) \leq k. \qquad \square$$

Back to proof of Erdős-Simonovits

~~LEM~~ $\Rightarrow$

$$e(G') \geq \left(1-\frac{1}{r}-\eta\right)\frac{n^2}{2} - \frac{\gamma}{3}n^2.$$

$$= e(T_r(n)) - \left[\frac{\eta}{2}+\frac{\gamma}{3}\right]n^2.$$

LEM $\Rightarrow$ can delete $\left[\frac{\eta}{2}+\frac{\gamma}{3}\right]n^2$ more edges, and now r-partite.

Total loss of edges: $\left[\frac{\gamma}{3}n^2\right] + \left[\frac{\eta}{2}+\frac{\gamma}{3}\right]n^2$, so can choose $\frac{\eta}{2} = \frac{\gamma}{3}$

$\boxed{\eta = \frac{2}{3}\gamma}$, and done. □

OUTLINE of EXACT RESULT for $\chi(F)=3$.

(1). Start with an arbitrarily extremal F-free graph, G.

(2). Since $T_r(n)$ is F-free, $e(G) \geq e(T_r(n)) \approx \left(1-\frac{1}{r}\right)\frac{n^2}{2}$,

so for ~~any~~ $\eta > 0$ we choose, we will satisfy conditions of ERDŐS-SIMONOVITS.

(3). ES $\Rightarrow$ G is bipartite graph (L,R) + ϵn^2 edges, for some suitably small ϵ.

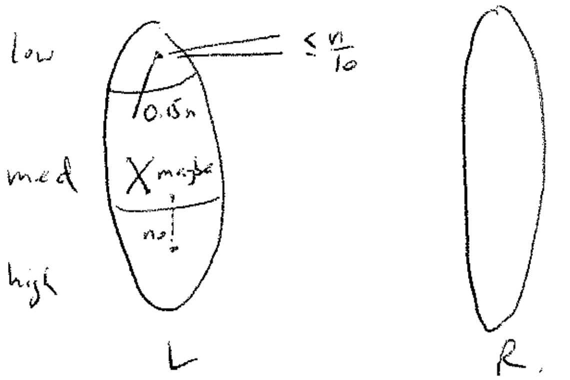
(4). Use this strong structural information ~~to~~ in "local" arguments to get exact structure

PF. Let $\epsilon = \frac{1}{900 f^2}$. PROVE for $\chi(F)=3$. Larger values similar.

By (3), $e(G) \leq L\cdot R + \epsilon n^2$, so must have ~~XXX~~ $0.49n < L, R < 0.51n$,

or else already worse than $K_{\frac{n}{2},\frac{n}{2}}$.

Assume we have partition with closest edit distance to the complete bipartite graph. But #edges deleted from complete $\leq \epsilon n^2$, or else worse than Turán. So edit dist $\leq 2\epsilon n^2$.



(1). Split L by degree across. $f = |F|$.

low = cross degree $\leq \frac{n}{10}$.

med = cross degree up to $R - \frac{n}{20f}$.

high = all others.

(2). Internal degree of low vtxs is $\leq 0.15n$.

If not, then moving the vtx to R would give lower edit distance from complete bipartite.

EDIT $> 0.15n + (0.49n - 0.1n)$
$=$ ~~XXX~~ $0.54n$.

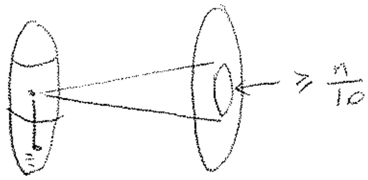
EDIT $< 0.1n + (0.51n - 0.15n)$
$= 0.46n$

(3). Maybe there are edges within <u>med</u> vtxs. But $\# \text{med} \le \dfrac{2\epsilon n^2}{n/20f} = 40\epsilon f n$.

(4). <u>No</u> edges between <u>med</u> and <u>hi</u> vtxs.

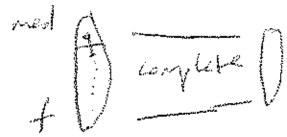
(assume $\ge f$ high vtxs. will see at end this is OK.)

If had edge:



Can choose $f-2$ other high vtxs, and will get:

And this structure contains F by EDGE-CRITICAL.



$\ge \frac{n}{10} - (f-1)\frac{n}{20f}$

$> \frac{n}{20}$

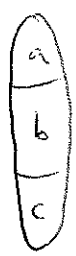
$> f$.

(5) <u>No</u> edges between <u>high</u> vtxs.

(same proof).

(6). $\left[2\times(\text{internal edges } L) + 1\times(\text{cross edges})\right] + \left[\text{same for } R\right] = 2\,e(G)$.

Calculate for L:



$$\le a\left[2\times 0.15n + 0.1n\right] \qquad a\,[0.4n]$$
$$+\,b\left[R - \tfrac{n}{20f}\right] + b^2 \quad \le \; +\,b\left[R\boxed{-\tfrac{n}{20f}+b}\right]$$
$$+\,cR \qquad\qquad +\,cR.$$

Make $\frac{n}{20f} > 40\epsilon f n$, i.e., choose $\boxed{\epsilon < \frac{1}{800f^2}}$

Then this is $\le LR$, with equality IFF $c=L$. <u>and</u> all vtxs have full degree R.

Same true for R.

So, $[\ \ ] + [\ \ ] \le 2LR \Rightarrow e(G) \le LR. \le e(T_2(n))$ as claimed.

EQUALITY only at complete bipartite (L,R). $\square$.

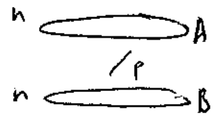
(*) Note: since $b \le 40\epsilon f n$ is tiny, only way we could have had $|\text{high}| < f$ is if almost all vtxs were in <u>low</u>. But those have worst bound $(0.4n)$, so that would easily give strict equality.

(**) For $\chi(F) \ne 2$, adapt def. of <u>low</u> to be that $\exists$ other part with degree into it $< 0.1n$.
Similar exchanging argument $\Rightarrow$ internal degrees $< 0.11n$ for those.
Med $= \exists$ other part with degree into it $<$ ~~[illegible]~~ partsize $- \frac{n}{20f}$. Still bound $|\text{med}|$.
Then all <u>high</u> basically see everyone else, as needed in the argument.

QUASIRANDOMNESS

2010-03-14

(1)

TH.  Bipartite graph with $n^2 p$ edges, p can be function of $\underline{n}$.
If ~~every~~ ϵ-regular (every pair of $\geq \epsilon n$ vtxs has $(1 \pm \epsilon) xyp$ edges),
then $\# C_4$ (labeled) $= (1 \pm \epsilon^{\frac{1}{2}}) n^4 p^4$.

OBVIOUSLY need conditions: if p too small, maybe no C_4 at all.
So need $p \gg n^{-\frac{1}{2}}$. Write it as condition: $\boxed{1 \leq \epsilon n p^2}$
Also, since we are letting p be tiny, will have $\underline{\epsilon}$ function of $\underline{n}$ too. Need to make sure $\underline{\epsilon}$ is even smaller. $\boxed{\epsilon \leq p^9}$.

Then the result will hold no matter what.

PF. Lower bound on C_4:

$$\# \overset{1\ \ 3}{\underset{2}{V}} = \sum_{v \in B} d_v (d_v - 1) \underset{\text{cvx}}{\geq} n \cdot (np)(np-1).$$

$\#$ of ordered pairs of distinct vtxs in A is $\leq n^2$,
so avg is $\geq p(np-1) = np^2 - p$.

$$\# \bowtie = \sum_{\text{distinct } A^2} (\text{codeg})(\text{codeg}-1) \underset{\text{cvx}}{\geq} n(n-1) \times p(np-1) \times (np^2 - p - 1).$$

$$\# \overset{1\ \ 3}{\underset{2}{V}} = \sum_{B} d_v^2 \underset{\text{cvx}}{\geq} n \cdot (np)^2.$$

So avg codegree over ordered pairs ~~of distinct~~ in A is $\geq np^2$.

$$\# \overset{1\ \ 3}{\underset{2\ \ 4}{\bowtie}} = \sum_{A^2} (\text{codeg})^2 \underset{\text{cvx}}{\geq} n^2 \cdot (np^2)^2 = n^4 p^4.$$

Now use $1 \leq \epsilon n p^2 \leq \epsilon n p \leq \epsilon n$.

i.e. $n - 1 \geq n - \epsilon n = (1-\epsilon) n$.

$np - 1 \geq np - \epsilon np = (1-\epsilon) np$.

$np^2 - p - 1 \geq np^2 - 2\epsilon n p^2 = (1 - 2\epsilon) np^2$

So $\# \bowtie \geq n^4 p^4 \cdot (1-\epsilon) \cdot (1-\epsilon) \cdot (1-2\epsilon)$

$\geq (1 - 4\epsilon) n^4 p^4$.

$> (1 - \epsilon^{\frac{1}{2}}) n^4 p^4$.

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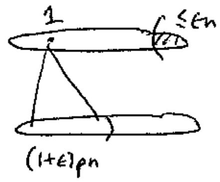
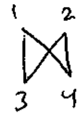
(2)

Upper bound on C_4.



$\le \epsilon n$ vtxs have over $(1+\epsilon)np$ degree to B. else violate ϵ-regularity.

~~So of the n choices~~



Embed vtx 1. If lands in the ϵn, then even in worst case, $\#(2,3,4)$ making C_4 is a further $\le n^3$.

$$\text{So } \#C_4 \le \underbrace{(\epsilon n \times n^3)}_{1 \text{ is wrong}} + \underbrace{n}_{\substack{\text{right}\\\text{choices}\\\text{for } 1}} \times \Big[\underbrace{(\epsilon n \times n^2)}_{\substack{2 \text{ is wrong}\\\text{degree to } N(1)}} + n \times \big(\underbrace{(1+\epsilon)^2 p^2 n}_{\substack{\text{Upper bound}\\\text{on codeg}(1,2)}}\big)^2 \Big]$$

$$= 2\epsilon n^4 + (1+\epsilon)^4 n^4 p^4.$$

$$< n^4 p^4 \left[1 + 5\epsilon + \frac{2\epsilon}{p^4} \right].$$

But $\epsilon \le p^9$, so $\frac{1}{p^4} \le \frac{1}{\epsilon^{4/9}}$.

$$< n^4 p^4 \left[1 + \epsilon^{\frac{1}{2}} \right]. \qquad \square$$

How would you verify this quasirandomness? Need to check all pairs of subsets

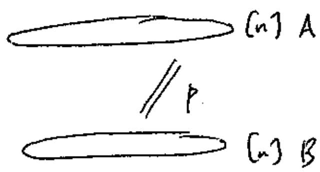
How many checks is that (complexity)? $\approx 2^n \times 2^n$ (all pairs of subsets).

Not just a CS question either: if incorporating in a proof, say given a random graph would take a union bound over exponentially many events $\longrightarrow$ need exponential concentration on failure probability.

Sometimes we don't have this, e.g., recent work with Alan Frieze

AMAZING CONVERSE

TH.



$$\left.\begin{array}{l} \#\text{edges} = n^2 p \\ \# C_4 \le (1+\epsilon^{12}) n^4 p^4 \\ \epsilon^{12} n p^2 \ge 1 \end{array}\right\} \Rightarrow \begin{array}{l} \text{every pair of sets } X, Y \ge \epsilon n \\ \text{have } e(X,Y) = (1 \pm \epsilon) xy\, p. \end{array}$$

Prove via reparameterization $\epsilon^{12} \leftrightarrow \epsilon$.

THINK of p const, say $\frac{1}{2}$.

Actually we prove this for the case $\left[1 + \left(\frac{\epsilon}{2}\right)^{12}\right] n^4 p^4$, and $\left(\frac{\epsilon}{2}\right)^{12} n p^2 \ge 1$.

<u>CLAIM 1</u>. $\#\Lambda \le (1+\epsilon) n^3 p^2$.

<u>PF</u>. $\displaystyle \#\bowtie = \sum_{\text{diff } B^2} (\text{codeg})(\text{codeg}-1) \underset{\text{cvx}}{\ge} n(n-1) \times (\text{avg codeg})(\text{avg codeg} - 1)$

IF CLAIM WERE FALSE, then avg codeg $\ge (1+\epsilon)np^2 \ge np^2 \ge \frac{1}{\epsilon}$

$$\ge n(n-1) \times \left[\frac{\#\Lambda}{n(n-1)}\right]^2 (1-\epsilon)$$

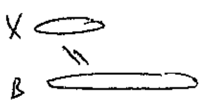
$$\ge (1-\epsilon)\frac{[\#\Lambda]^2}{n^2}$$

Yet $\#\bowtie \le (1+\epsilon) n^4 p^4$, so:

$$[\#\Lambda]^2 \le (1+2\epsilon) n^6 p^4$$

$$[\#\Lambda] \le (1+\epsilon) n^3 p^2. \qquad \square.$$

<u>CLAIM 2</u> For any $X \subseteq A$ of size $\ge \delta n$, where $\delta = 2\epsilon^{\frac{1}{3}}$.



$\#\text{edges} = (1 \pm \delta) x n p.$

<u>PF</u>. From <u>CLAIM 1</u>, $\#\Lambda \le (1+\epsilon) n^3 p^2$.

This also is $\displaystyle \sum_X d(d-1) + \sum_{\text{not } X} d(d-1) = \sum_X d^2 + \sum_{\text{not } X} d^2 - (\#\text{edges})$.

$$\text{So} \quad \sum_X d^2 + \sum_{\text{not } X} d^2 \le (1+\epsilon) n^3 p^2 + n^2 p \le \boxed{(1+2\epsilon)\, n^3 p^2}$$

since $n^2 p \le \epsilon n^3 p^2$.
$\Uparrow$
$1 \le \epsilon n p$.

close

<u>REMARK</u> $\displaystyle \sum_X d^2 + \sum_{\text{not } X} d^2 \ge n \cdot (\text{avg deg})^2 = \boxed{n \cdot (np)^2}$

SAME INTUITION AS PROOF OF REGULARITY LEMMA. Then d's must be pretty even.

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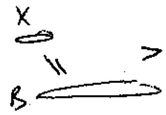
(4)

let Z = degree of uniformly random vtx in A.

$E[Z] = np$

$E[Z^2] \le (1+2\epsilon) n^2 p^2$.

$\mathrm{Var}[Z] \le n^2 p^2 [2\epsilon]$.

Now suppose for contradiction that  $> (1+\delta) x n p$.

~~Then in X, avg degree~~

Then $\underset{X}{\mathrm{avg}}(\text{degree}) > (1+\delta) np$.

$\mathrm{Var}[Z] = \mathbb{E}[(Z-\bar{Z})^2]$.

Already the $\mathbb{P} \ge \delta$ section of X is going to contribute:

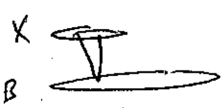
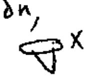
$$\mathbb{E}[(Z-\bar{Z})^2 \mid \text{vtx} \in X] \ge \left(\mathbb{E}[(Z-\bar{Z}) \mid \text{vtx in } X]\right)^2$$

$$\ge (\delta np)^2$$

So already get $> \delta \times (\delta np)^2$ contribution to Var.

Contradiction since $\delta = 2\epsilon^{\frac{1}{3}}$.

Similar argument if ~~e(X,B)~~  $< (1-\delta) x n p$. $\square$.

CLAIM 3  If $X > \delta n$, then #  is $\le (1+\delta) x^2 n p^2$, where $\delta = 2\epsilon^{\frac{1}{4}}$

PF. SIMILAR INTUITION: Need some random variable to have low variance.

let Z = codegree of uniformly random ordered pair from A.

$$\mathbb{E}[Z] = \frac{\# V}{n(n-1)} = \frac{1}{n(n-1)} \sum_B d(d-1) \quad \text{~~[illegible]~~}$$

$$\underset{\text{cvx}}{\ge} \frac{1}{n^2} n \times (np)(np-1)$$

$$\ge (1-\epsilon) np^2.$$

~~$\mathbb{E}[Z^2] \ge \frac{\ldots}{n(n-1)}$~~

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(5)

For $\sum (\text{codeg})^2$, look to C_4.

$$\#\bowtie = \sum_{\text{diff } A^2} (\text{codeg})(\text{codeg}-1)$$

$$= \sum_{\text{diff } A^2} (\text{codeg})^2 - \#V$$

CLAIM 1

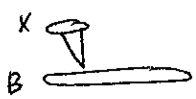
So $\mathbb{E}[Z^2] \le \frac{1}{n(n-1)}\left[\#\bowtie + \boxed{\#V}\right]$

$\le \frac{1}{n(n-1)}\left[(1+\epsilon)n^4p^4 + \boxed{(1+\epsilon)n^3p^2}\right]$

Now $n^3p^2 \le \epsilon n^4 p^4$
$\Uparrow$
$1 \le \epsilon n p^2$

$\le (1+3\epsilon) n^2 p^4$.

$\mathrm{Var}[Z] \le n^2p^4[5\epsilon]$

Assume for contradiction that  $> (1+\delta)x^2np^2$.

Then again, on $>\delta^2$-fraction of $\mathbb{P}$-space, we have avg value of Z $> (1+\delta)np^2$.

Global avg value of Z is $\frac{\#V}{n(n-1)} \le (1+2\epsilon)np^2$ by CLAIM 1.

So: Contribute $> \delta^2 \times (\delta - 2\epsilon)^2 n^2 p^4$ to Var.

$*$ for $\delta = 2\epsilon^{\frac{1}{4}}$.

PF (TH) Know:

$\#\text{edges} = (1 \pm 2\epsilon^{\frac{1}{3}})xnp.$ AND, $\#\overset{X}{\vee} \le (1+2\epsilon^{\frac{1}{4}})x^2np^2.$

INTUITION: concentrate d_X.

But $\#\overset{X}{\vee} = \sum_B d_X(d_X-1) = \sum_B d_X^2 - (\text{edges})$

So $\sum_B d_X^2 \le (1+2\epsilon^{\frac{1}{4}})x^2np^2 + (1+2\epsilon^{\frac{1}{3}})xnp.$

NOTE $xnp \le \frac{\epsilon^{\frac{11}{12}}}{2}x^2np^2$

$\Updownarrow$

$1 \le \frac{\epsilon^{\frac{11}{12}}}{2}xp$

$\Uparrow$

$1 \le \frac{\epsilon^{\frac{11}{12}}}{2}2\epsilon^{\frac{1}{12}}np$

$\le (1+3\epsilon^{\frac{1}{4}})x^2np^2.$

Let $Z = d_X$ of uniformly random vtx in B.

$\mathbb{E}[Z^2] \le (1+3\epsilon^{\frac{1}{4}})x^2p^2.$

$\mathbb{E}[Z] = (1\pm 2\epsilon^{\frac{1}{3}})xp.$

$\operatorname{Var} Z \le x^2p^2\left[4\epsilon^{\frac{1}{4}}\right].$

Yet if $\underset{Y}{\text{avg}}\, d_X < (1-2\epsilon^{\frac{1}{12}})xp$, then we contribute to Var: $2\epsilon^{\frac{1}{12}} \times \left[2\epsilon^{\frac{1}{12}} - 2\epsilon^{\frac{1}{3}}\right]^2 x^2p^2$

overwhelms Var. $\divideontimes$.

Similar for $\underset{Y}{\text{avg}}\, d_X > (1+2\epsilon^{\frac{1}{12}})xp.$

$\square$.

2010-03-16

1

EXTREMAL SET THEORY

~~PROBLEM~~ (TH) (LITTLEWOOD-OFFORD).

Given n (not nec. distinct) complex #s $z_1, \ldots, z_n$ of abs val ≥ 1, and a complex number z, prove that $\mathbb{P}\left[|\varepsilon_1 z_1 + \cdots + \varepsilon_n z_n - z| < 1\right] = o(1)$.

(Here, ε_i are i.i.d. ± 1)

TH (KATONA-KLEITMAN). Actually, $\mathbb{P} \leq O\left(\frac{1}{\sqrt{n}}\right)$

Specifically, $\mathbb{P} \leq \dfrac{\binom{n}{\lfloor n/2 \rfloor}}{2^n}$.

Proof via Extremal Set Theory

~~TH~~. ~~KSRE~~

DEF. Collection of ~~subsets~~ sets is ANTICHAIN if no ~~pair of sets satisfies~~ set contains another.

TH. (SPERNER) Largest antichain in collection of all subsets of $[n]$ is of size $\binom{n}{\lfloor n/2 \rfloor}$.

PROP. Given antichain $\{A_1, A_2, \ldots, A_t\}$, must have $\displaystyle\sum_i^t \frac{1}{\binom{n}{|A_i|}} \leq 1$.

PF. For each A_i, let E_i = event that in random permutation $n!$, the first $|A_i|$ elements are the set A_i.

Since antichain, E_i disjoint.

$\mathbb{P}[E_i] = \dfrac{1}{\binom{n}{|A_i|}}$.

$\Rightarrow \mathbb{P}[\text{some } E_i] = \sum \dfrac{1}{\binom{n}{|A_i|}}$, but ≤ 1. □

PF (SPERNER)

$$1 \geq \sum^t \frac{1}{\binom{n}{|A_i|}} \geq \sum^t \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} = \frac{t}{\binom{n}{\lfloor n/2 \rfloor}}$$

$$\Rightarrow t \leq \binom{n}{\lfloor n/2 \rfloor}.$$ □

PF (L-O FOR $\mathbb{R}$)

WLOG, all $z_i \geq 0$.

Now for each satisfying (ε_i), define set A to be indicator of $+1$'s.

Antichain: if $A \subseteq B$, then the additional $+z_i$ would push you out of the <2-width window.

□

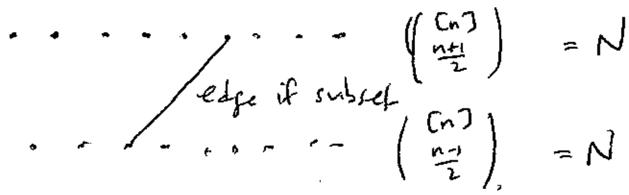
<u>TH</u>. (SPERNER) The only families that attain equality are $\binom{[n]}{k}$ for $k \in \{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil\}$

<u>PF</u>. From probabilistic proof, only tight if all A_i satisfy $\binom{n}{|A_i|} = \binom{n}{\lfloor n/2 \rfloor}$, i.e. $|A_i| \in \{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil\}$.

So if n-even, done.

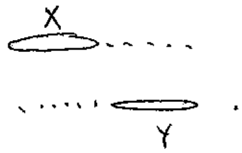
Else, all sets of size $\frac{n \pm 1}{2}$.

Construct bipartite graph:



Looking for largest independent set in this bipartite graph.

Suppose max indep. set is:



Then no edges between $X \leftrightarrow Y$.

However, bipartite graph is <u>regular</u> with degree $n - \frac{n-1}{2} = \frac{n+1}{2}$.

So, $|N(X)| \geq |X|$, with equality IFF $N(N(X)) = X$.

Suppose $|X| + |Y| = N$, ie, it is a contender for max antichain.

Then not enough space for $|N(X)| > |X|$, ie, forces $N(N(X)) = X$.

Similarly, $N(N(Y)) = Y$.

So:

And bipartite graph disconnected.

<u>BUT</u> obviously it is connected: to move from set $A \in \binom{[n]}{\frac{n-1}{2}}$ to $B \in \binom{[n]}{\frac{n-1}{2}}$, just repeatedly add one missing element to A, then delete one extraneous element, etc.

So must have $X = \phi$ or $Y = \phi$, precisely as claimed. $\square$

2010-03-16

③.

OBS. MAX size of intersecting family is $\frac{1}{2}2^n$ by A vs A^c.

TH (ERDŐS–KO–RADO). Maximum size of intersecting family of k-sets of $[n]$ is: $\binom{n-1}{k-1}$ OR $\binom{n}{k}$ if $n \le 2k-1$.

EXTREMAL EXAMPLES: all sets containing fixed elt, or all sets

PF. Obvious if $n \le 2k-1$ since pigeonhole

For $n=2k$, $\binom{2k-1}{k-1} = \frac{k}{2k}\cdot\binom{2k}{k} = \frac{1}{2}\binom{n}{k}$, true since we can go A vs A^c.

For $n \geq 2k$, Katona cycle trick:

Generate random k-set by:

① Random permutation π.

② Random starting index $i \in [n]$.

$\pi_1 \quad \pi_2 \quad (\pi_3 \quad \cdots) \quad \pi_n$.

$\uparrow$ $i=3$ $\quad$ k guys.

$\mathbb{P}[k\text{-set in } \mathcal{F}] = \mathbb{P}[k\text{-set in } \mathcal{F} \mid \pi]$.

But once π is fixed, clearly $\le$ ~~2k-1~~ $\underline{k}$ values of $\underline{i}$ can work.

So $\quad \le \frac{k}{n}$

$$\Rightarrow |\mathcal{F}| \le \frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}.$$

<u>Now for tightness</u> (extremal constructions).

Observe above: every permutation MUST have exactly k-window of i's that work

So start with one π: $(1,2,\dots,n)$.

WLOG, the window is $(1,2,\dots,k)$, $(2,\dots,k+1)$, $\dots$ $(k,\dots,2k-1)$.

<u>CLAIM!</u> For every set S of $k-1$ elements of $[n]$ which not include $\{2k\}$, or $\{k\}$,

$S \cup \{k\} \in \mathcal{F}$.

PF. Suppose $T = S \cap \overbrace{\{k+1, \dots, 2k-1\}}^{k-1}$.

Consider $\sigma =$

$1 \quad 2 \quad \cdots \quad k \quad \overbrace{k+1 \quad \cdots \quad 2k-1}^{k-1} \quad 2k \quad \cdots$ positions

$\cdots \quad \boxed{S-T} \; k \; \boxed{\boxed{T} \quad} \quad 2k \quad \cdots$

permute $k+1 \dots 2k-1$

The moment of switch from NOT $\mathcal{F}$ to YES $\mathcal{F}$ must start a k-window of IN $\mathcal{F}$.

⊢ NOT IN $\mathcal{F}$ ⊣

⊢ IN $\mathcal{F}$ ⊣

⊢ SO THIS IS IN $\mathcal{F}$ ⊣

□.

Now consider case $\underline{n>2k}$.

So far, know that every k-set containing $\underline{k}$ is in $\mathcal{F}$, <u>except possibly</u> those that contain $\underline{2k}$.

Say $\underline{S}$ is a k-set containing $\underline{k}$ and $\underline{2k}$.

Choose arbitrary element $z \in [n]$ NOT IN $\underline{S}$, and build permutation:

DIFFERENT ?

position   1   2   3 ...   k   k+1 ...   (2k) ...   (n)

(swap $\underline{n}$, $\underline{z}$)   1   2   (n)   ..   k   k+1 ——   2k ——   (z)

NOT IN $\mathcal{F}$

IN $\mathcal{F}$

IN $\mathcal{F}$

$\Rightarrow \{z, 1, 2, n, \ldots, k-1\} \notin \mathcal{F}$.

So it's the same pattern, but backwards!

Now let T be those guys in $\{1, 2, n, \ldots, k-1\} \cap S$, and again permute:

including 2k

1 2 n $\boxed{T}$ k $\boxed{\text{rest of } S}$ ————— z

Still have same signature of:

NOT IN $\mathcal{F}$

IN $\mathcal{F}$

$\Rightarrow$ IN $\mathcal{F}$

So all sets containing $\underline{k}$ are in $\mathcal{F}$. $\square$.

<u>Case $n=2k$</u>.

Reason for split is that there is another example.

(1). All sets containing one element.

(2). $\binom{n-1}{k-1} = \binom{2k-1}{k-1} = \binom{2k-1}{k}$, so could take all k-subsets of some ~~2k-1~~ n-1 set (ignore one element)

Case 1. EVERY ~~subset~~ k-set with both $\{k, 2k\}$ is NOT in $\mathcal{F}$.
Show every set S containing $2k$ is NOT in $\mathcal{F}$ (only left to consider S that have $2k$ but NOT k).
Consider permutation:

position 1 2 3 ... k k+1 $2k = n$.

k * * * * 2k. $\boxed{S - \{2k\}}$ *

CLAIM 1 ⇒ — IN $\mathcal{F}$ —| k excluded |—

Since tight via $A \Leftrightarrow A^c$ ⇒ |— NOT IN $\mathcal{F}$ —|

□.

Case 2. SOME k-set with both $\{k, 2k\}$ is IN $\mathcal{F}$.
Show every set S containing k is in $\mathcal{F}$. (only left to consider S that have both $k, 2k$)
Consider permutation: S has $k-2$ elements not $\{k, 2k\}$.

1 2 3 ... k k+1 ... 2k-1 $2k = n$.

k .... 2k (z) ← Element NOT in (S), NOT k, 2k, NOT in A } k elts; k-2 more; 2k-2 excluded ⇒ 2 left.

|— IN $\mathcal{F}$ by Assumpt —|

|— IN $\mathcal{F}$ by CLAIM 1 —|

|— IN $\mathcal{F}$ by CLAIM 1 —|

⇒ |— NOT IN $\mathcal{F}$ —|

Now permute: $T = S \cap (A - \{k, 2k\})$

... $\boxed{S - T - k, 2k}$ k 2k $\boxed{\boxed{T} \text{ rest of } A}$ (z)

|— NOT IN $\mathcal{F}$ (Permut) —|

|— IN $\mathcal{F}$ (Permut) —|

⇐

|— IN $\mathcal{F}$ by window —|

$S \in \mathcal{F}$.

□.

TH. (BOLLOBÁS). Max size of a (k,l)-system is $\binom{k+l}{k}$.

DEF. (k,l) system is family of pairs of sets (A_i, B_i) s.t.:

1. $A_i \cap B_i = \phi$.
2. $i \neq j \Rightarrow A_i \cap B_j \neq \phi$. (intersect)
3. $|A_i| = k$, $|B_i| = l$.

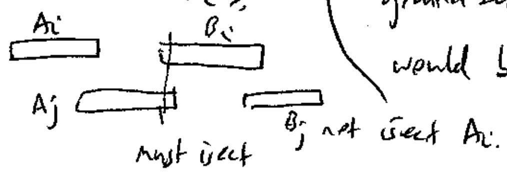
PF. WLOG, ground set is some $[n]$.

(Can take $n < \infty$ since will prove same bound for all arbitrary large finite $\underline{n}$. If answer for $n = \infty$ was $> T$, then since T pairs (A_i, B_i) take up $\leq T(k+l)$ ground set space, then result for that $\underline{n}$ would be $> T$.)

Take random permutation of $[n]$.

$E_i = \mathbb{P}[\text{all of } A_i \text{ before all of } B_i]$

Events disjoint:



So $\sum \mathbb{P}[E_i] \leq 1$.

Each $\mathbb{P}[E_i] = \dfrac{k!\, l!}{(k+l)!} = \dfrac{1}{\binom{k+l}{k}}$. $\square$

EQUALITY is achieved if $[n] = [k+l]$, and $A_i = k\text{-set of } [n]$, $B_i = A_i^c$.

70s

2010-03-21

TH (LOVÁSZ). Pair of sets (A_i, B_i) s.t.:

1. $|A_i| = k$, $|B_i| = \ell$.
2. A_i ~~intersects~~ disjoint from B_i.
3. For $i<j$, A_i intersects B_j.

$\Rightarrow$ # of set pairs $\le \binom{k+\ell}{k}$

HAS 3 PROOFS, ALL ALGEBRAIC.

APPLICATION

DEF. Graph is F-saturated if adding any edge will create copy of F.

$sat(n, F)$ = min # of edges in any F-SATURATED n-vtx graph.

NOTE. $ex(n, F)$ = max # of edges in any F-SATURATED n-vtx graph

"DUAL" of Turán number.

EX. $sat(n, K_3) = n-1$. (like Mantel's Thm K_3)

PF. star is clearly K_3-saturated

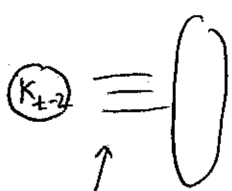
And any graph with $\le n-2$ edges is NOT CONNECTED.

So there is an edge to add between two distinct connected cpts

$\Rightarrow$ this edge doesn't make K_3.

TH (TURÁN) $ex(n, K_t)$ known exactly, of order $\frac{n^2}{2}\left(1-\frac{1}{t-1}\right)$.

How to make K_t-saturated graph?



complete

indep set $n-(t-2)$

$$\#\text{edges} = \binom{t-2}{2} + (t-2)\big(n-(t-2)\big).$$

TH (ERDŐS–HAJNAL–MOON 64). $sat(n, K_t) = \binom{t-2}{2} + (t-2)\big(n-(t-2)\big)$.

To prove this, we actually will prove something stronger.

<u>DEF</u>. Graph G is WEAKLY-F-SATURATED if ~~there is a seq~~ the missing edges (from K_n) can be added in sequence s.t. each addition creates a new copy of F.

<u>DEF</u> w-sat(n,F) = min # of edges in weakly-F-saturated n-vtx graph

<u>NOTE</u>. Every F-saturated graph is also weakly-F-saturated, so w-sat$(n,F) \le$ sat(n,F).

<u>TH</u>. w-sat(n,K_t) = sat$(n,K_t) = \binom{t-2}{2} + (t-2)(n-(t-2))$.

<u>PF</u>. Need to show: any weakly-K_t-saturated graph has $\geq \binom{t-2}{2} + (t-2)(n-(t-2))$ edges.

i.e., # missing edges $\le \binom{n}{2} - \binom{t-2}{2} - (t-2)(n-(t-2))$.

$= \binom{n-(t-2)}{2}$ since the only missing edges were in the indep set of size $n-(t-2)$

Define (A_i, B_i), one set pair per missing edge.

Ground set $= [n]$ (vtxs).

Suppose missing edges added in order $e_1, e_2, \ldots, e_m$ make new K_t at each step.

Let B_i = (vtxs) endpts of e_i.

A_i = vtxs <u>NOT IN</u> the K_t created in i^{th} step.

(if several new K_t, arbitrarily pick one).

<u>Check</u>: $|A_i| = n-t$, $|B_i| = 2$, obviously A_i and B_i disjoint.

But if $i<j$, <u>only WAY</u> that A_i disjoint B_j is if:

j^{th} edge.

t vtxs for K_t in i^{th} step.

Yet $j > i$, so at i^{th} step, that edge not there, ✗

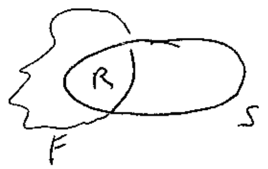
So, LOVÁSZ $\Rightarrow$ # missing edges $\le \binom{n-t+2}{2}$. $\square$

<u>NOTE</u> FOR ordinary <u>sat</u>, same proof plugs into BOLLOBÁS.

2010-03-21

(3)

DEF. A family $\mathcal{F} \subseteq 2^{[n]}$ SHATTERS a set S if every subset $R \subseteq S$ can be obtaind as $R = F \cap S$ for some $F \in \mathcal{F}$



(For every R)

TH. (SAUER-SHELAH) If $|\mathcal{F}| > \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{k}$, then there is set S of size $k+1$ s.t. $\mathcal{F}$ shatters S.

NOTE. TIGHT: if $|\mathcal{F}| = \binom{n}{0} + \dots + \binom{n}{k}$, then can ~~fix some k-set, and take~~ take all subsets of size $\le k$.

Will be missing full $(k+1)$-set

PROOF BY COMPRESSIONS.

DEF. Given $i \in [n]$, and family $\mathcal{F}$, let COMPRESSION $c_i(\mathcal{F})$ be:

~~FOR each $F \in \mathcal{F}$ st. that doesn't contain i, leave it alone.~~

FOR each F that has i, but $(F - i)$ not in $\mathcal{F}$, replace F with $F - i$.

ELSE leave F alone.

"compresses" the ~~fam~~ sets in the family.

PROPERTIES.

(1). # of distinct sets remains constant under ~~shifting~~. compressions.

(2). After finitely many compressions, have DOWN-CLOSED FAMILY.

(3). If $c_i(\mathcal{F})$ shatters a set S, then so does $\mathcal{F}$.

THEN. Start with $\mathcal{F}$ that shatters no sets of size $k+1$.

Keep compressing.

At end, have DOWN-CLOSED FAMILY which shatters no $k+1$ set.

$\Rightarrow$ final family has no sets of size $\ge k+1$ (since down-closed $\Rightarrow$ shatter)

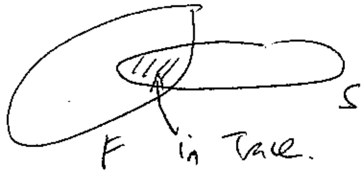
$\Rightarrow$ DONE.

2010-03-21

(4)

REMAINS to establish property (3) of compressions.

<u>DEF</u>. Given family $\mathcal{F}$, set S, TRACE $Tr_{\mathcal{F}}(S)$ = all sets $F \cap S$ from $F \in \mathcal{F}$.



<u>NOTE</u>. SHATTER $\leftrightarrow$ TRACE is all 2^S.

<u>PROP</u>. Given $\mathcal{F}$, S.

Trace of $c_i(\mathcal{F})$ on S $\subseteq$ $c_i\Big($ Trace of $\mathcal{F}$ on $S\Big)$,

<u>PF</u> Consider set $T \in \Big($Trace of $c_i(\mathcal{F})$ on $S\Big)$.

<u>CASE 1</u> $i \in T$,

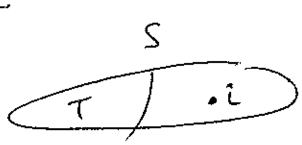
Given: $T = F' \cap S$, $F' \in c_i(\mathcal{F})$.

Since $i \in F'$, and F' still in i-compression, must have <u>BOTH</u> F' and $(F'-i)$ in $\mathcal{F}$.

So Trace of $\mathcal{F}$ on S includes both T and $(T-i)$,

$\Rightarrow c_i$ of trace includes T.

<u>CASE 2</u> $i \notin T$.



Given: $T = F' \cap S$, $F' \in c_i(\mathcal{F})$.

So suppose c_i sent $F' \leftarrow F$.

<u>CASE A</u>. $F' = F$.

Then Trace of $\mathcal{F}$ on S has $F \cap S = T$

and c_i doesn't touch T since no i.

<u>CASE B</u> $F' = F - i$.

Then Trace of $\mathcal{F}$ on S has $F \cap S = T \cup \{i\}$

But c_i then forces $T \in c_i(\text{Trace})$, since if

T was not there, it will be compressed onto.

$\square$.

2010-03-29

5

DEF RANGE SPACE $S=(X,R)$, where:

X = ground set (finite or infinite)

R = collection of subsets of X (finite or infinite), called "ranges".

EX. X = set of n pts in $\mathbb{R}^2$.

R = all subsets that can be realized as intersection of X and a triangle:

DEF VAPNIK-CHERVONENKIS DIMENSION (VC-DIMENSION) of range space is the max cardinality of a shattered set. (could be ∞)

TH If (X,R) is range space with $|X|=n$ and VC-dimension d, then $|R| \le \binom{n}{0}+\dots+\binom{n}{d}$.

PF (was Sauer-Shelah). Proved independently by Sauer / Perles-Shelah / V-C.

TH If X is ~~set of n pts in~~ $\mathbb{R}^2$,

R is all ~~subsets that can be realized as intersection of X and a~~ interiors of d-sided cvx polygon.

Then VC-dim of (X,R) is finite fcn of d.

(roughly $d \log(d)$).

TH. (X,R) range space of VC-dim d, and let $\epsilon>0$ be given. Let $M=\frac{8d}{\epsilon}\log\frac{8d}{\epsilon}$.

Then ANY finite subset $A \subseteq X$, no matter what ~~cardinality~~ size, satisfies:

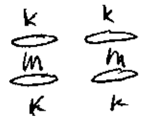
(*) $\exists$ subset $A_0 \subseteq A$ of size $\le M$ s.t. every range that contains $\ge \epsilon$-fraction of A must also contain a point of A_0. "ϵ-NET".

COR. Any set of pts can have bounded size subset s.t. every Δ containing $\ge \epsilon$-fraction of the set also contains a pt in ϵ-net.

$\Rightarrow$ Applications in computational geometry.

APPLICATION OF KRUSKAL-KATONA

Q. (STATISTICAL PHYSICS). What is max # of independent sets that n-vtx k-regular graph can have?

CONJ. (ALON). Maximized by  ·· etc.

~~TH. (ZHAO)~~

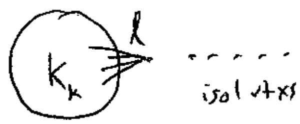
TH. (KAHN) True if graph is assumed bipartite.

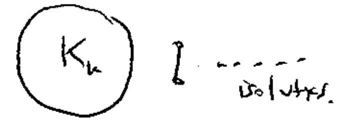
TH. (ZHAO) True for all graphs

Q. What if graph not forced to be regular? EQUIV: how to maximize # cliques

PROP Given n, m, the graphs which have $\underline{n}$ vtxs, $\underline{m}$ edges, and max # of cliques

are: $m = \binom{k}{2} + l$ where $k > l \geq 0$.

$G^* =$  isol vtxs.

UNLESS $l = 1$, can also have  isol vtxs.

PF. CLAIM. For every $\underline{t}$, any G with $\underline{n}$ vtxs, $\underline{m}$ edges has #cliques of size $\underline{t}$ $\leq$ #cliques of size $\underline{t}$ in G^*.

PF Only need to check $2 \leq t \leq k$.

CASE 1. $l+2 \leq t \leq k$.

Then $\#K_t$ in G^* is exactly $\binom{k}{t}$.

Suppose G has more t-cliques.

Form t-uniform hypergraph with $\geq \binom{k}{t} + \binom{t-1}{t-1}$ hyperedges from K_t's in G.

Kruskal-Katona ⇒ # 2-sets that are in some hyperedge in G is $\geq \binom{k}{2} + \binom{t-1}{1}$

$\geq \binom{k}{2} + l + 1$

exceeds # edges of G. ✱.

CASE 2. $2 \leq t \leq l+1$.

Then G^* has $\#K_t = \binom{k}{t} + \binom{l}{t-1}$

Similar ⇒ if G has $\#K_t \geq \binom{k}{t} + \binom{l}{t-1} + \binom{t-2}{t-2}$, then # edges $\geq \binom{k}{2} + \binom{l}{1} + \binom{t-2}{0}$

too many. □.

HENCE. G^* is a maximizer of total # cliques.

EQUALITY ⇒ G has same # of K_t as G^*, for every t.

$t = k$ ⇒ G has K_k too.

$l+1$ ⇒ use $t = l+1$ ⇒ G must be  since that is only way to add another K_{l+1}.

$l = 1$ ⇒ add arbitrary edge works.

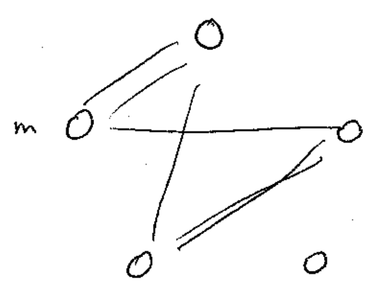
REGULARITY LEMMA HOMEWORK

EX 1. For any $\gamma > 0$, following holds for suff. large n.

Every graph with $\geq \gamma n^2$ edges CANNOT be decomposed into union of n induced matchings. (here, each matching is an induced subgraph of original graph)

PF. Suppose not. Assume we have graph with $\geq \gamma n^2$ edges, and it IS decomposed.

Regularity Lemma with $\epsilon = \frac{\gamma}{4}$, $\delta = \frac{\gamma}{2}$, into $\frac{1}{\epsilon} \leq t \leq M$ parts.



DELETE:

① Edges inside each V_i.

② Edges between not-ϵ-regular pairs.

③ Edges between pairs of density $< \delta$

t parts of size m

TOTAL LOSS:

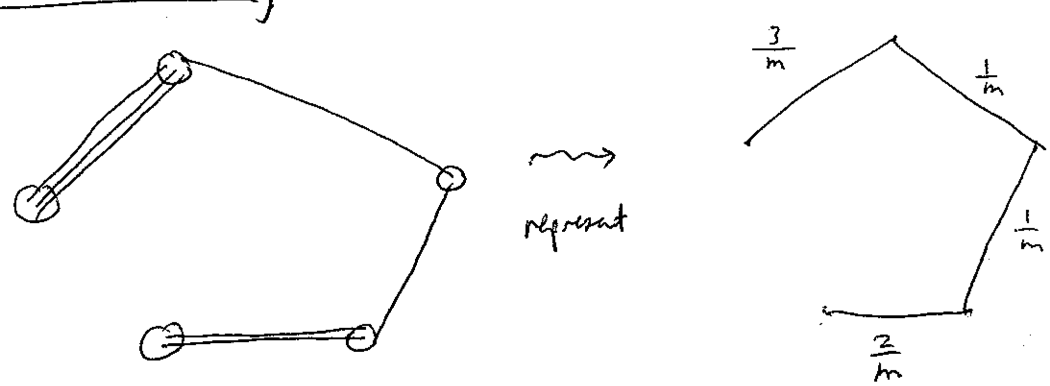
$$\left.\begin{array}{ll} ① & \leq t \cdot \binom{n/t}{2} \leq \frac{1}{2}\frac{n^2}{t} \leq \frac{1}{2}\epsilon n^2. \\ ② & \leq \epsilon t^2 \cdot \left(\frac{n}{t}\right)^2 = \epsilon n^2. \\ ③ & \leq \binom{t}{2} \delta \cdot \left(\frac{n}{t}\right)^2 \leq \frac{1}{2}\delta n^2. \end{array}\right] \to \leq \left(\frac{\delta}{2} + \frac{3}{2}\epsilon\right) n^2 \text{ edges}$$

$$\leq \left(\frac{1}{4}\gamma + \frac{3}{8}\gamma\right) n^2 = \frac{5}{8}\gamma n^2.$$

So we are left with graph G' with $\geq \frac{3}{8}\gamma n^2$ edges,

AND it still is decomposed into n induced matchings, since edge deletion preserves.

CONSIDER ONE induced matching



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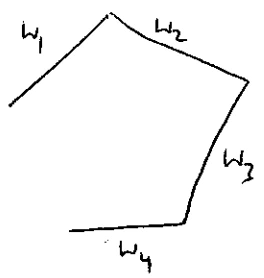
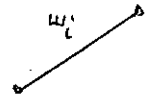
(2)

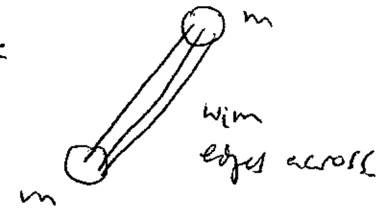
So we have weighted graph on t vtxs:



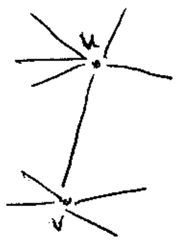
and



means that we have



<u>KEY PROPERTY</u>. IF:



u, v adjacent in weighted graph and weighted degrees of <u>u</u> <u>AND</u> <u>v</u> are both $\geq \epsilon$, then <u>NOT</u> induced matching.

<u>PF</u>

$\geq \epsilon m$ vtxs used in the induced matching that lie in "u". They are all distinct since they came from matching.

$\geq \epsilon m$ in "v".

Since $\overline{uv}$ was edge we must have had the pair ϵ-regular and δ-dense

So, # edges between ~~pa~~ the <u>ϵm</u> bits is

$$\geq (\epsilon m)^2 \times [\delta - \epsilon]$$

$$= \left(\frac{\gamma}{4} m\right)^2 \times \frac{\gamma}{4}$$

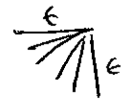
$$\geq \left(\frac{\gamma}{4}\right)^3 \frac{1}{M^2} n^2$$

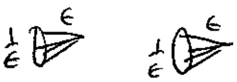
$> n$ for <u>n</u> large enough $\Rightarrow$ induced graph is <u>NOT</u> matching. $\square$

<u>THUS</u>: in schematic, every pair of vtxs with weighted degree $\geq \epsilon$ is NON-adjacent.

<u>NOTE</u>: total # edges in the induced matching is $\left(\sum_{\text{edges}} w_i\right) m$

FINAL CLAIM. Any weighted graph on t vtxs with property that vtxs with weighted degree $\geq \epsilon$ are nonadjacent, has total weight $\leq \epsilon(t-1)$.

(realized by construction:  .

If we insist that also at each vtx the sum of degrees ≤ 1, then it can be almost realized by ($\frac{1}{\epsilon}$  ... disjoint union).
)

PF. Construction motivates following procedure

Pick vtx of min weighted degree

NOTE: if all weighted degrees $\geq \epsilon$, then no edges by property, ✱.

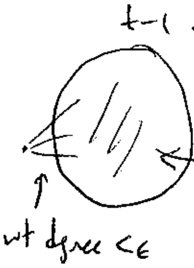
So vtx has weighted degree $\leq \epsilon$.

Pull it off.

Induction on remaining $t-2$ vtxs

Thus total wt degree

$$\leq \epsilon(t-2) + \epsilon.$$
$$= \epsilon(t-1).$$



(BASE CASE is $t=1$, trivially 0).

□.

PUT ALL TOGETHER Total # edges in any induced matching of G' is $< \epsilon t m$.

But $tm = n$.

So this is $< \epsilon n. = \frac{\gamma}{4} n$.

And if we take n induced matchings, use up total of $< \frac{\gamma}{4} n^2$ edges

Yet $e(G') \geq \frac{3}{8}\gamma n^2$, so doesn't completely use it up ✱.

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(4)

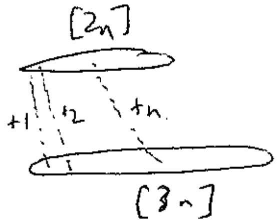
EX 2 Use EX 1 to prove Roth.

Given $A \subseteq [n]$ of size $\geq \epsilon n$.

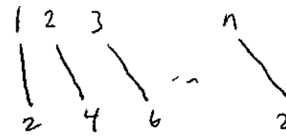
~~Assume~~

Construct graph as union of $\underline{n}$ matchings. We will then be able to claim that one is NOT induced

Match 1.



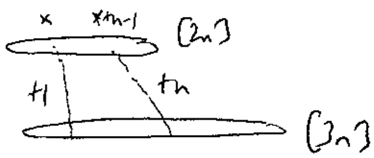
Potential edges are



We add them in if the difference is in A

⋮

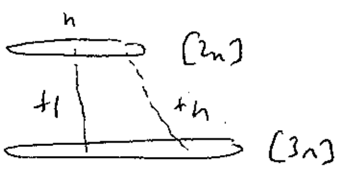
Match $\underline{x}$



All potential edges shifted to start at $\underline{x}$

⋮

Match $\underline{n}$



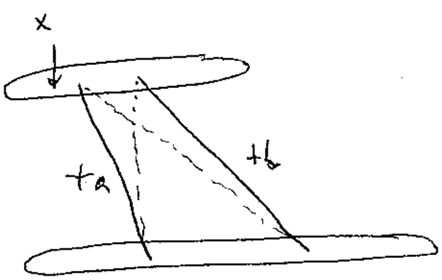
Total # of edges added is $\underline{\text{EXACTLY}}$ $|A| \times n$.

↳ since each matching would send given vtx on top to diff vtx on bottom.

Yet $V = 5n$, so $E \geq \epsilon n^2 \geq \frac{\epsilon}{25} V^2$

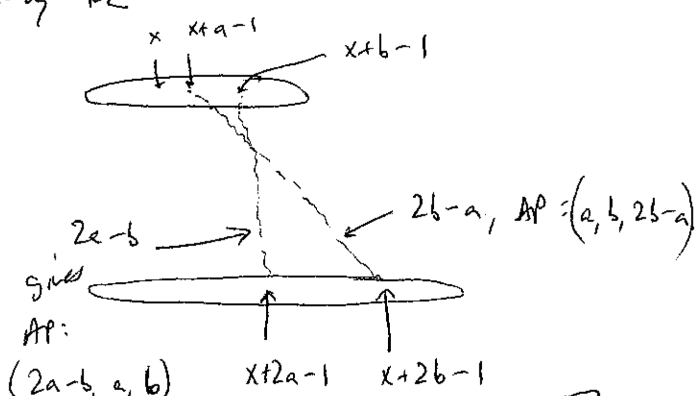
By $\underline{\text{EX 1}}$, E $\underline{\text{cannot}}$ be union of V induced matchings, let alone $n = \frac{V}{5}$.

So some matching, say match $\underline{x}$ is not induced, ie



one of cross edges is present

$\underline{\text{BUT}}$

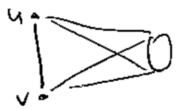


□.

EX 3. Weak Ramsey–Turán.

TH. $\forall \gamma > 0$, $\exists \eta > 0$ s.t. for large n, $E \geq \left(\frac{1}{4}+\gamma\right)n^2 \Rightarrow$ either K_4 or $\alpha \geq \eta n$.

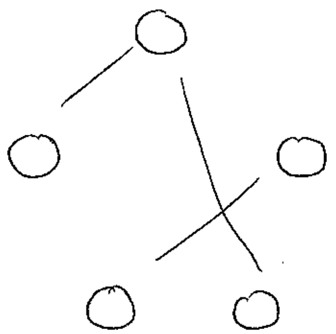
PF. KEY OBSERVATION: Since no K_4,



codegree of any edge is indep set.

So: suff. to show: ~~either~~ graph has an edge with codegree $\geq \eta n$.

Apply Regularity Lemma with $\epsilon = \frac{1}{3}\delta$, $\delta = \frac{1}{2}\gamma$.



t parts of size m.

DELETE

(1) Edges within V_i.

(2) Edges between not ϵ-regular pairs

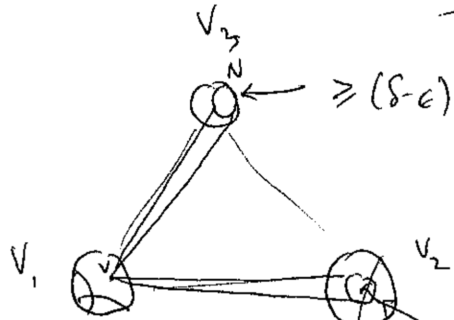
(3) Edges between parts of density $\leq \delta$.

TOTAL LOSS: $\left(\frac{\delta}{2} + \frac{3}{2}\epsilon\right) n^2$.

NEED $\boxed{\left(\frac{\delta}{2} + \frac{3}{2}\epsilon\right) < \gamma.}$ → let $\delta = \frac{\gamma}{2}$, $\epsilon = \frac{1}{3}\delta$.

Then Turán ⇒ there is still a triangle.

So:

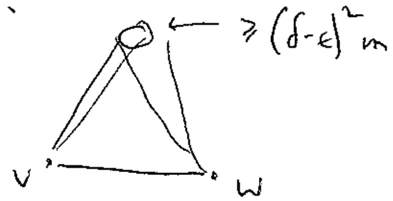


Delete $2\epsilon m$ so all v have degree $\geq (\delta-\epsilon)m$ in V_2, V_3.

NEED: $\boxed{(\delta-\epsilon)m > \epsilon m.}$

Then cut off ϵm s.t. all vtxs $w \sim v$ in V_2 now have degree $\geq (\delta-\epsilon)^2 m$ in N.

Then get:



$$\text{Codegree} \geq \left(\frac{2}{3}\epsilon\right)^2 m \geq \left(\frac{2}{9}\delta\right)^2 \frac{n}{M}$$

So $\eta = \left(\frac{2}{9}\delta\right)^2 \frac{1}{M}$ suffices. □

2010-03-28

6

EX 4 (SZEMERÉDI, IN HUNGARIAN).

TH For any $\gamma > 0$, $\exists \eta > 0$ s.t. $E \geq (\frac{1}{8} + \gamma) n^2$, then either K_4 or $\alpha \geq \eta n$.

PF. ~~For~~ let $\epsilon = \frac{1}{3}\delta$, $\delta = \frac{1}{2}\gamma$.

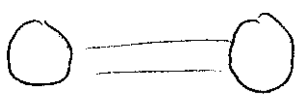
From EX 3, if after regularity lemma, still have $\triangle$ of ϵ-regular, δ-dense pairs, done

$$\# \text{edges left} \geq \left(\tfrac{1}{8} + \gamma\right) n^2 - \Big(\underbrace{\tfrac{1}{2} \cdot \tfrac{1}{2}\gamma}_{\frac{1}{4}\gamma} + \underbrace{\tfrac{3}{2} \cdot \tfrac{1}{3}\delta}_{\frac{1}{4}\gamma}\Big) n^2$$

$$\boxed{\geq \left(\tfrac{1}{8} + \tfrac{1}{2}\gamma\right) n^2.}$$

$\longleftarrow$ just over $\frac{1}{2}$ of Turán $\triangle$-density

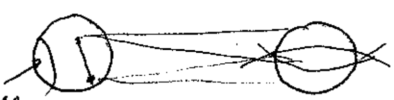
KEY OBSERVATION if:



ϵ-regular,
density $\geq \frac{1}{2} + 2\epsilon$

THEN: in original graph, either have K_4 or $\alpha \geq \eta n$.
(no deletions)

PF.



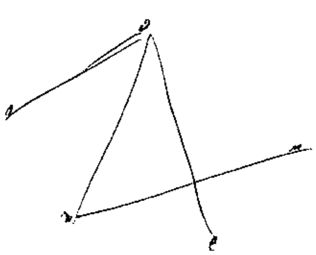
$\longleftarrow$ But 2 vtxs with degree $\geq (\frac{1}{2} + \epsilon) m$ overlap
so codegree $\geq 2\epsilon m$.
Again, if edge there, then K_4.
Else, large indep set $\square$.

cut off ϵm

now all degrees $\geq (\frac{1}{2} + \epsilon) m$.

Cannot be indep set, else done. $\Rightarrow$ edge.

So: In Regularity graph, every edge corresponds to pair with density $\leq \frac{1}{2} + 2\epsilon$



Schematic

No $\triangle$s. $\Rightarrow \# \text{edges} \leq \frac{t^2}{4}$.

$\Rightarrow$ in original graph, after deletions,

$$\# \text{edges} \leq \frac{t^2}{4} \times m^2 \left(\tfrac{1}{2} + 2\epsilon\right)$$

$$= \left(\tfrac{1}{8} + \tfrac{\epsilon}{2}\right) n^2$$

$$\boxed{\leq \left(\tfrac{1}{8} + \tfrac{1}{12}\gamma\right) n^2.}$$

$\divideontimes$.

LINEAR ALGEBRA IN EXTREMAL SET THEORY

RULES & CLUBS

In a town, of n people, too many clubs were being formed.

To restrict the # of clubs, mayor passed the following rule:

1. Every club must have even # of people
2. Every pair of clubs must have even intersection

Seems pretty restrictive.

Q How many clubs can be formed?

TH Max is still $2^{n/2}$ (Say n even)

PF. Can get $2^{n/2}$: split people into pairs, then take collections of pairs

Why can't get more? Will see

Since still too many clubs, mayor ~~passed~~ slightly changed rule.

(1') Every club must have odd # of people

TH. Now max # of clubs is n.

PF. Can get n: one club per person (singleton)

Why not more than n?

For each club, take the CHARACTERISTIC VECTOR over $\mathbb{F}_2$.

Let all these be $v_1, v_2, \dots, v_t$.

RULE (1') $\Rightarrow$ Each $v_i \cdot v_i = 1$

RULE (2) $\Rightarrow$ Each $v_i \cdot v_j = 0$ if $i \neq j$.

LOOKS LIKE ORTHONORMAL SET.

(But this is not a proper dot product, since it doesn't satisfy positivity: there is no notion of positive/negative in $\mathbb{F}_2$)

However, recall same proof for linear independence

DEF. Over arbitrary field $\mathbb{F}$, set of vectors $v_1, v_2, \dots, v_t$ is LIN. INDEP if every $\sum c_i v_i = 0$ with $c_i \in \mathbb{F}$ must have all $c_i = 0$.

TH. Max size of lin. indep set in $\mathbb{F}^n$ is n.

PF. (Uses fact that every basis has same size, and elementary basis of $\mathbb{F}^n$ has size n.)

Why every basis same size

LEM Given 2 bases, can add elt of one to the other

then first is lin dep ~~(since was maximal lin indep)~~ (since could express everyone)

so can pull off at least one elt of first until lin indep again, still spans

So if we start with smaller basis and augment/delete from larger basis, eventually end up with basis which is strict subset of 2^{nd} (larger) basis. $\times$ $\square$.

LEM. If over any $\mathbb{F}$, have $v_i \cdot v_i \neq 0$ and $v_i \cdot v_j = 0$ for $i \neq j$, then $\{v_i\}$ lin. indep.

PF. Take any $\sum c_i v_i = 0$, and dot product both sides by v_i.

Get $c_i \underbrace{(v_i \cdot v_i)}_{\neq 0} + 0 = 0$. so $c_i = 0$. $\square$

THUS in odd town, #clubs $\leq n$.

ALSO: if we fix any prime p, and: ① club sizes $\not\equiv 0 \pmod p$ ② ~~club sizes~~ intersections $\equiv 0 \pmod p$ } $\Rightarrow$ max # is n.

OPEN. Oddtown mod 6. Easy to show #clubs $\leq 2n$ (EXERCISE)

There are some improvements to $(2 - o(1))n$.

But no gain in the constant.

PF (of Eventown $2^{n/2}$ maximum.)

Suppose we have MAX collection of vectors $\{v_i\}$ st. all $v_i \cdot v_j = 0$ (even $i = j$)

OBS. Set is a vector subspace over $\mathbb{F}_2$

Since $\mathbb{F}_2$, just need to check that $(v_i + v_j)$ satisfies the properties.

$(v_i + v_j) \cdot (v_i + v_j) = 0$ by distributive, since all $v_i \cdot v_j = 0$.

$(v_i + v_j) \cdot v_k = 0$ also.

So the question is to find largest dim of vector subspace of $\mathbb{F}_2^n$ which is self-orthogonal

let $A = d\,\overbrace{\boxed{\text{— basis —}}}^{n}$ where each basis vector of the subspace is a row.

rank $A = d$.

Consider $Ax = 0$.

Know (dim of soln space) + (rank A) $= n$.

But all the same vectors are in soln space, so (dim of soln space) $\geq d$.

$\Rightarrow \quad n \geq d + d \Rightarrow d \leq \frac{n}{2}$. $\Rightarrow$ max # of vectors is $2^{n/2}$. $\square$

LINEAR ALGEBRA OVER $\mathbb{R}$.

TH Given $n+1$ subsets $A_1, A_2, \cdots, A_{n+1} \subseteq [n]$.

Then $\exists$ ~~partitions~~ I, J (disjoint) s.t. $\bigcup_I A_i = \bigcup_J A_j$.

PF Characteristic vectors over $\mathbb{R}^n$ are linearly dependent

So $\exists \sum c_i v_i = 0$.

let I be indices with $c_i > 0$
— J ———— $c_i < 0$ $\Rightarrow \sum_I c_i v_i = \sum_J (-c_j) v_j$.

Since all ~~i~~ I have c_i same sign, LHS vector is SUPPORTED precisely on $\bigcup_I v_i$. Similar for RHS. $\square$.

TH. (B. LINDSTRÖM, LOGICIAN) Given $n+2$ subsets of $[n]$,
$\exists$ disjoint $I, J \subseteq [n+2]$ s.t.: $\begin{cases} \bigcup_I A_i = \bigcup_J A_j \\ \bigcap_I A_i = \bigcap_J A_j. \end{cases}$

PF Now need vectors in $\mathbb{R}^{n+1}$. Use $\binom{v_i}{1}$ where v_i is i^{th} characteristic vector.

Then $\sum c_i v_i = 0$, again split into $\oplus$ and $\ominus$: $\sum_\oplus c_i \binom{v_i}{1} = \sum_\ominus |c_i| \binom{v_i}{1}$.

From first n coords, get $\bigcup_\oplus A_i = \bigcup_\ominus A_j$ again.

Last coord $\Rightarrow \sum_\oplus c_i = \sum_\ominus |c_j|$.

Yet $\bigcap_\oplus A_i$ is precisely the ~~indices~~ coords in LHS which get value $\sum_\oplus c_i = \sum_\ominus |c_i|$, so also $\bigcap_\ominus A_j$. $\square$.

FISHER'S INEQUALITY

2010-03-30

(Y)

Q. Collection of subsets of $[n]$, all pairwise intersections have ~~size~~ λ equal size

How many subsets can there be?

(Recall ERDŐS-KO-RADO didn't care about sizes of intersections)

CONSTRUCTIONS

(1) FLOWER:

Since petals disjoint

max # sets $\le \underset{\text{core}}{1} + \underbrace{n-\lambda}_{\text{+1 for each outside vx.}}$

(2) PROJECTIVE PLANE (FINITE GEOMETRY)

In $\mathbb{F}^3$, let POINT = 1-d subspace

LINE = 2-d subspace

So identify POINTS $\leftrightarrow [n]$.

LINES $\leftrightarrow$ sets of POINTS in them.

(*) Every pair of LINES isects at exactly one POINT

since distinct 2-d subspaces have isection still a subspace, and of dim exactly 1 $\Rightarrow$ only one point.

Yet # POINTS $= \frac{F^3-1}{F-1}$ = #LINES (ortho complement)

TH (FISHER). ~~Max~~ # of subsets is $\le n$.

PF. Let λ = isection size. If $\lambda = 0$, disjoint $\Rightarrow$ partition, done

If $\exists$ set of size EXACTLY λ, then must be FLOWER, done

So suppose all sets size $> \lambda$.

INCIDENCE MATRIX over $\mathbb{R}$. $A = \begin{pmatrix} \chi_1 & \chi_2 & \cdots & \chi_t \end{pmatrix}$. Have $A^T A = \overbrace{\begin{pmatrix} >\lambda & & \lambda \\ & >\lambda & \\ & & >\lambda \\ \lambda & & \ddots \; >\lambda \end{pmatrix}}^{t} \Big\} t$ $= \lambda J +$ pos. def. diagonal matrix

Yet J is pos. semidef: $x^T J x = (x_1 x_2 \cdots x_t) \begin{pmatrix} \Sigma x_i \\ \vdots \\ \Sigma x_i \end{pmatrix} = (\Sigma x_i)^2 \ge 0$

So A is pos. def: $x^T A^T A x \ge 0$,

and equality ONLY when $x = 0$

In particular, cannot have nontrivial soln to $A^T A x = 0$, so $A^T A$ is nonsingular

$\Rightarrow$ rank $A^T A = t$.

But rank $A \le n$, so $t \le n$. $\square$.

(rank $PQ \le$ rank Q since rows of PQ are lin. comb of rows of Q, and rank is dim of row space)

GRAHAM – POLLACK

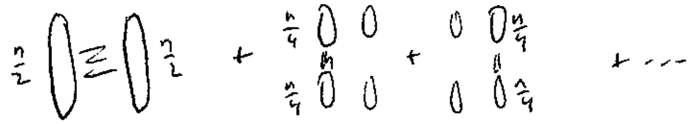
Q The edges of K_n are partitioned into <u>disjoint</u> complete bipartite graphs. How few can there be?

CONSTRUCTIONS

<u>Easy</u>: $\binom{n}{2}$: one per edge

<u>n-1</u> : $n-1$ stars. (Keep pulling out one vtx, connected to all others.)

<u>n-1</u> : If $\underline{n}$ is power of 2,



$n = 2^k$

1 of size $\frac{n}{2}, \frac{n}{2}$.

2 ——— $\frac{n}{4}$

4 ——— $\frac{n}{8}$

$\vdots$

2^{k-1} $\quad \frac{n}{2^k}, \frac{n}{2^k}$

total $1+2+4+\dots+2^{k-1} = 2^k - 1 = n-1.$

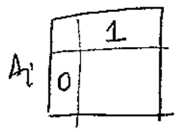
<u>TH</u>. (GRAHAM–POLLAK). Cannot achieve with less than $n-1$ graphs.

<u>NOTE</u>: Since the 2 constructions are completely different, proof cannot be simple induction. In general, if many different optimal structures, proofs are hard (why stability results are nice)

<u>NOTE</u>: PARTITION makes many graphs required.
If only need COVERING, then can do with $\lceil \log_2 n \rceil$ graphs
(Use binary representation of each vtx.
Graph $\underline{i}$ is LHS = those with i^{th} coord 0
RHS = ——— 1.)

<u>PF.</u> Suppose we have $t \le n-2$ complete bipartite graphs in partition

Let A_i be their "adjacency matrices".

That is, A_i 

So since the i^{th} complete bipartite graph is between some ~~[illegible]~~ (X_i, Y_i), we write 1 in r^{th} row, c^{th} column iff $\forall x\ \underline{r} \in X_i$, $\forall x\ \underline{c} \in Y_i$.

Actually, A_i is the adjacency matrix of directed graph $X_i \,()\, \rightrightarrows \,()\, Y_i$.

Now $\operatorname{rank} A_i = 1$.

(Recall in Jin-Yi Cai's talk that bipartite adjacency matrix gave rank 2, even though they are the most trivial.
In his talk, he needed to have special bipartite case to deal with this.
So here, we just use $\frac{1}{2}$ of adj. matrix instead).

Let $B = \sum A_i$.

Then $\operatorname{rank}(B) \le t$. $\leftarrow$ rank of sum $\le$ sum of ranks since rank = dim of row space.
$\le n-2$. So can express each row of sum as sum of basis elements from 2 row spaces $\Rightarrow$ new basis $\subseteq$ union of previous 2

$\Rightarrow \dim \ker(B) \ge 2$.

Always $\dim \ker(J) = n-1$, since J has rank 1

$\Rightarrow \exists$ nonzero x s.t. $Bx = 0$
$Jx = 0$.

Yet partition $\Rightarrow J = B + B^T + I$.

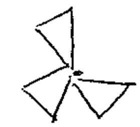
So $x^T J x = x^T B x + x^T B^T x + x^T I x$

$0 = 0 + 0 + x^T x \Rightarrow x = 0$. $\ast$ $\square$.

FRIENDSHIP THM

TH (ERDŐS-RÉNYI-SÓS) Finite graph, every 2 vtxs share EXACTLY ONE neighbor. Then $\exists$ vtx adj. to all other vtxs.

IGNORE ∞ case



REMARKS. If every 2 people have exactly one friend in common, then $\exists$ "politician" who is everybody's friend.

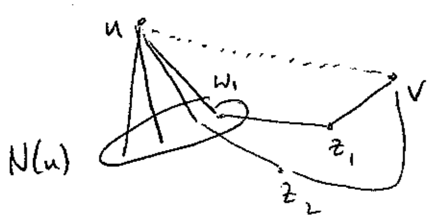
Sounds like ERDŐS-KO-RADO; intersecting families.

FALSE FOR ∞-GRAPHS: let $G_0 = C_5$.
G_1 = add new vtx for every pair of vtxs with codegree 0.
$\vdots$
G = Union of all G_i.

PF. Suppose all codegrees $= 1$, but $\nexists$ politician. So no C_4 subgraph

CLAIM 1. G is regular.

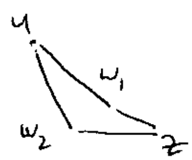
PF Consider NON-ADJ vtxs u, v:



Get z_1 since w_1, v has codegree $= 1$.
— z_2 ——.

And $z_1 \neq z_2$, or else get C_4:

$\operatorname{codg}(w_1, w_2) \neq 1$.



$\Rightarrow d(v) \geq d(u)$.
symm $\Rightarrow d(u) \geq d(v)$.
$\Rightarrow d(u) = d(v)$.

Now all vtxs outside $N(u)$ have degree $= d(u)$.
(must be ≥ 1 v, else u is politician)
Next, $\operatorname{codg}(u,v) = w_1$, say. So all other $w_k \neq v \Rightarrow$ degree $= d(v) = d(u)$.
Finally, w_1 not politician, so it non adj. to somebody $\Rightarrow$ degree $= d(\text{somebody}) = d(u)$.

$\square$

2010-04-04

(4)

Let $\underline{k}$ be regularity of graph G.

<u>Count # of vtxs:</u> Start from arbitrary vtx, $\underline{v}$.

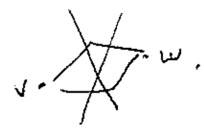
# of walks of length 2 is EXACTLY $\underline{k^2}$ by regularity.

This reaches <u>every</u> vtx since codegrees > 0.

Also, counts:

$\underline{v}$ is counted $\underline{k}$ times.

Everyone else counted <u>EXACTLY</u> once.



So $n = k^2 - k + 1$.

<u>ADJACENCY MATRIX</u> $A^2 = \begin{bmatrix} k & & & 1 \\ & k & & \\ & & \ddots & \\ 1 & & & k \end{bmatrix}$ since codegrees = 1.

$$\left(\text{interpretation: } A_{ij} = \begin{matrix} 1 & \text{if } i\sim j \\ 0 & \text{else.} \end{matrix} \quad (A^2)_{ij} = \sum_k (A_{ik})(A_{kj}) = \sum_k [i\sim k\sim j]. = \#\text{2-walks } i\to j \right).$$

Since $A^2 = J + (k-1)I$ and eig. of $J = n, 0, 0, 0, 0 \ldots 0$, then eig of $A^2 = \begin{matrix} n+k-1 = k^2 \\ k-1 \\ k-1 \\ \vdots \end{matrix}$

So eig of $A = \begin{matrix} k \leftarrow \oplus \text{ since } k\text{-regular} \\ \pm\sqrt{k-1} \end{matrix}$

$$\text{Say } \left.\begin{matrix} \underline{r} \text{ of } +\sqrt{k-1} \\ \underline{s} \text{ of } -\sqrt{k-1} \\ \mathrm{Tr} A = 0 \end{matrix}\right\} \Rightarrow (r-s)\sqrt{k-1} + k = 0.$$

$$(r-s)^2(k-1) = k^2$$

$$\Rightarrow (k-1) \mid k^2.$$

$$\text{But } (k-1) \mid k^2 - 1 \Rightarrow (k-1) \mid 1.$$

$$\Rightarrow k = 2, \quad n = k^2 - k + 1 = 3,$$

has 3 politicians.

<u>ELSE</u>, actually have politician. Graph:

Then cody $\Rightarrow$ must match up the nbrs of the politician. $\square$

FRANKL-WILSON

~~Q.~~ **DEF.** Given a set L of integers, a set system is L-INTERSECTING if every pairwise intersection has size in L.

Q. If $|L| = s$, how big can $\mathcal{F}$ be? (bunch of subsets of $[n]$)

RECALL: FISHER'S INEQ: $|L| = 1$.

CONSTRUCTION. $L = \{0, 1, \ldots, s-1\}$, $\mathcal{F}$ is all sets of size $\leq s$.

$$\# \text{sets} = \binom{n}{0} + \cdots + \binom{n}{s}$$

TH 1. Max size is this.

~~TH 2.~~ Q. If also all sets have same size ("uniform"), how many can there be?

CONSTRUCTION $L = \{0, \ldots, s-1\}$, $\mathcal{F}$ is all sets of size $= s$.

TH 2. Max size is $\binom{n}{s}$.

TH 3 (FRANKL-WILSON) If $\mathcal{F}$ is L-intersecting MOD p (intersection sizes mod p are in L)

AND every set $F \in \mathcal{F}$ has $|F| \bmod_p \notin L$.

THEN. $|\mathcal{F}| \leq \binom{n}{0} + \cdots + \binom{n}{s}$.

COMPARE ODDTOWN IF $s = 1$.

REMARK Uniform version is more complicated to state.

(see that Frankl-Wilson already added new condition $|F| \bmod p \notin L$.)

(Unif. version adds ~~more complicated~~ another conditions, more complicated proof via Möbius inversion.)

PF 3. For each set $F \in \mathcal{F}$, let χ_F be characteristic vector over ~~$\mathbb{R}$~~ $\mathbb{F}_p^n$.

Let $p_F(x_1, x_2, \ldots, x_n) = \prod_{\ell \in L} (\vec{x} \cdot \chi_F - \ell)$, a polynomial over $\mathbb{F}_p$.

NOTE: ①. $p_F(\chi_F) = \prod_{\ell \in L} (|F| - \ell) \neq 0$ (in $\mathbb{F}_p$)

② $p_F(\chi_E) = \prod_{\ell \in L} (|E \cap F| - \ell) = 0$.

2010-04-06

(2)

Same flavor as the orthonormal set from oddtown.

Now $\mathbb{F}_p[x_1,\dots,x_n]$ is vector space over $\mathbb{F}_p$.

So all the p_F are vectors

LIN INDEP. Suppose $\exists c_F$, not all 0, in $\mathbb{F}_p$, s.t. $\sum c_F p_F = 0$, as polynomials.

Then plug in χ_E: $0 = \sum c_F p_F(\chi_E) = \underbrace{c_E p_E(\chi_E)}_{\neq 0}$ so $c_E = 0$.

But what space do they live in?

Note that $p_F(x_1, x_2, \dots, x_n)$ looks like: (if $F = \{2,3\}$)

$$(x_2+x_3-l_1)(x_2+x_3-l_2)\cdots(x_2+x_3-l_s)$$

So ~~we need~~ suff. to have basis elements $x_1^{0..s}\, x_2^{0..s} \cdots x_n^{0..s}$ $\Rightarrow$ all p_F live in vector space of dimension n^{s+1}.

$\Rightarrow |\mathcal{F}| \le n^{s+1}$.

TO DO BETTER. Can say monomial exponents sum to $\le s$. But that's $\ge \binom{n+s-1}{s}$ still, even just from the sum to $= s$. $+\binom{n+s-2}{s-1}+\dots$

OR

In Lin. indep argument, only ever plug in 0 or 1 for each x_i.

If $x_i \in \{0,1\}$, then $x_i^{10} = x_i$.

Define $q_F(x_1,\dots,x_n) =$ MULTILINEAR version of p_F.

i.e. expand out p_F, then shrink all powers, e.g. $2x_1^5 x_2 x_3^7 \to 2x_1x_2x_3$, then recombine to form multilinear polynomial, each monomial either has x_i or not.

STILL LIN INDEP since that follows only from plugging in $0/1$.

BASIS FOR SPACE Only need monomials where either x_i in or not.

And each monomial has $\le s$ vars.

So #monomials $\le \binom{n}{0}+\dots+\binom{n}{s}$. $\square$

REMARKS WHAT IF p not PRIME?

①. mod ~~$\neq$~~ p^r, still get polynomial bound on $|\mathcal{F}|$, but not good answer:

$$|\mathcal{F}| \le ((1+o(1))\binom{n}{2^{s-1}})$$

↑

BABAI, et. al.

OPEN get bound $|\mathcal{F}| \le n^{O(s)}$.

②. mod N if there are 2 distinct prime factors not true

e.g. for mod 6. "GROULMOUS" (?) has construction with $2^{c \cdot \frac{\log^2 n}{\log\log n}}$,

where sizes $\equiv 0 \pmod 6$ and intersections $\not\equiv 0 \pmod 6$.

NOT COUNTEREXAMPLE TO ODDTOWN MOD 6, since the intersections should be $\equiv 0$, and sizes $\not\equiv 0$

PF 1. Order sets in $\mathcal{F}$ by their sizes: $|F_1| \le |F_2| \le \dots \le |F_t|$.

Define $p_i = \prod_{\substack{\ell \in L \\ \ell < |F_i|}} (x \cdot \chi_{F_i} - \ell)$, over $\mathbb{R}$. ONLY THE ℓ's smaller than $|F|$. NECESSARY SINCE MAYBE $|F| \in L$.

Let q_i = MULTILINEAR version of p_i.

NOTE ①. $q_i(\chi_{F_i}) = \prod_{\substack{\ell \in L \\ \ell < |F_i|}} (|F_i| - \ell) \ne 0$.

②. IF $j < i$, then $q_i(\chi_{F_j}) = \prod_{\substack{\ell \in L \\ \ell < |F_i|}} (|F_i \cap F_j| - \ell) = 0$.

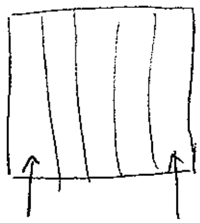
Now $|F_i \cap F_j| \in L$.

And $j < i$, so $|F_j| \le |F_i| \Rightarrow |F_i \cap F_j| \underset{\uparrow}{<} |F_j| \le |F_i|$.

~~so~~ strict since not equal sets

Feels like Lovász vs. Bollobás.

<u>Bookkeeping</u>:

matrix $A =$ 

output of q_1 output of q_{4th}.

that is, 1st column is: $q_1(x_{F_1})$, $q_1(x_{F_2})$, $\vdots$, $q_1(x_{F_t})$.

So, $A = \begin{pmatrix} \neq 0 & 0 & 0 & 0 \\ ? & \neq 0 & & 0 \\ ? & ? & \neq 0 & \\ & ? & & \neq 0 \end{pmatrix}$

<u>SHOW q_i INDEP</u>: if they were dependent then the partial list of evaluations are dependent, i.e., columns of matrix A are dependent

But $\det A \neq 0$ (Lower-Δ, and nonzero diagonal, or can see by obvious),

<u>So</u>: q_i INDEP $\Rightarrow \# \leq \binom{n}{0} + \cdots + \binom{n}{t}$.

<u>PF 2</u>. Assume that EVERY value in L actually appears as an intersection size, else throw it out.

Suppose all sets have size <u>k</u>.

Then no intersection has size <u>k</u>, so $k \notin L$.

Hence can define $p_F(x_1, \ldots, x_n) = \prod_L (x \cdot \chi_F - l)$,

q_F = MULTILINEAR

<u>PROPERTIES</u>. (1). $q_F(\chi_F) = \prod_L (|F| - l) \neq 0$ since $k \notin L$.

(2). $q_F(\chi_E) = \prod_L (|E \cap F| - l) = 0$

But if we use previous ideas, still space has too high dimension.
New ingredient: add more independent vectors.

For each $I \subseteq [n]$ with $|I| \le s-1$, ~~add~~ let $g_I(x_1,\ldots,x_n) = (x_1+\cdots+x_n-k)\prod_I x_i$.
(MULTILINEARIZED)

PROPERTY ① $g_I(\chi_F) = (|F|-k)\prod_I \cdots = 0$ since $|F|=k$.
② $g_I(\chi_I) = (|I|-k)$. And $|I| \le s-1 < k$ since # of intersection sizes $\le |\{q_1,\ldots,k-1\}|$ for unif sets.

INDEP Suppose: $\sum \alpha_F q_F + \sum \beta_I g_I = 0$. $(*)$

① Plug in any χ_E, and get $\alpha_E q_E(\chi_E) + 0 = 0 \Rightarrow \alpha_E = 0$.

②. Left to show all $\beta_I = 0$.

Let I_0 be minimum size s.t. $\beta_{I_0} = 0$.

Plug in χ_{I_0}. Since ① $\Rightarrow$ all $\alpha_E = 0$, our equation $(*)$ is:

$$\sum 0 + \sum \beta_I g_I(\chi_{I_0}) = 0.$$

But for any I with $\beta_I = 0$ that is not I_0, since I_0 is minimum size, must have I containing some $j \notin I_0$.

So, $g_I(\chi_{I_0})$ has x_j appearing in product and $x_j = 0$.

So $g_I(\chi_{I_0}) = 0$.

So: have $\sum 0 + \sum 0\, g_I(\chi_{I_0}) + \beta_{I_0} \underbrace{g_{I_0}(\chi_{I_0})}_{\neq 0.} + \sum \beta_I \cdot 0 = 0$.

$\rightarrow \beta_{I_0} = 0$. ✱.

THUS $\{q_F, g_I\}$ independent, and $\#\{g_I\} = \binom{n}{0} + \cdots + \binom{n}{s-1}$.

Space still dimension $\binom{n}{0} + \cdots + \binom{n}{s}$, so $\#\{q_F\} \le \binom{n}{s}$ $\square$.

APPLICATIONS OF FRANKL-WILSON

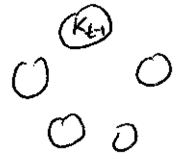
2010-04-12
1

EXPLICIT RAMSEY GRAPHS

TH. $2^{t/2} \le R(t,t) \le 4^t$.

Q. How to construct a large graph with no clique or indep. set of size $\underline{t}$?

EASY: $(t-1)^2$ vtxs:



$\}$ $t-1$ disjoint K_{t-1}.

TH. (NAGY) $V = \binom{[t]}{3}$

$E = \{ A \sim B \text{ if } |A \cap B| = 1 \}$.

Has no ~~edge~~ homogeneous set of size $t+1$

PF. (Exercise)

TH (FRANKL-WILSON) Generalize above construction, so hom $\le t$ but $V \approx t^{c \cdot \frac{\lg t}{\lg \lg t}}$.

PF. Let p = prime.

$V = \binom{[n]}{p^2-1}$.

$E = \left\{ A \sim B \text{ if } |A \cap B| \equiv -1 \pmod p \right\}$

($p=2$ is NAGY)

CLIQUE: sets $A_1, A_2, \ldots, A_t$, all of size p^2-1.

And all $|A_i \cap A_j| \equiv -1 \pmod p$

So let $L = \{ p-1, 2p-1, \ldots, p^2-1 \}$, of p values.

actually can throw away last one

The non-modular pairwise intersection sizes of A_i's are all in L, $p-1$ values.

So ~~FRANKL-WILSON~~ RAY-CHAUDHURI-WILSON $\Rightarrow t \le \binom{n}{p-1}$

INDEP SET: Sets A_i of size $p^2-1 \equiv -1 \pmod p$.

Pairwise isection $\not\equiv -1 \pmod p$.

Let $L = \{0, 1, 2, \ldots, p-2\}$, $p-1$ values.

FRANKL-WILSON $\Rightarrow t \le \binom{n}{p-1}$.

2010-04-12

(2)

<u>So</u>. $V = \binom{n}{p^2-1}$, no clique / indep set of size $\binom{n}{p-1}$.

Optimize parameters: $n = p^3$.

Let $t = \binom{p^3}{p-1} \approx (p^3)^p = e^{3p\log p} \Rightarrow 3p\log p = \log t$

$p = \frac{\log t}{3\log\log t}$

Then $V = \binom{p^3}{p^2-1} \approx (p^3)^{p^2} = e^{3p^2\log p}$

$$= e^{\frac{1}{3}\frac{\log^2 t}{(\log\log t)^2}\log\log t}$$

$$= e^{\frac{1}{3}\frac{\log^2 t}{\log\log t}}$$

$$= t^{\frac{1}{3}\frac{\log t}{\log\log t}}$$

$\Box$.

To THIS DAY this is the best known constructive bound

CHROMATIC NUMBER OF UNIT DISTANCE GRAPH

<u>Q</u>. (\$1000, ORIG. BY NELSON) How few colors req'd to color all points in $\mathbb{R}^2$ s.t. no 2 pts at distance 1 have same color?

<u>RMK</u>. By Tychonov compactness argument, suffices to consider arbitrarily large finite graph

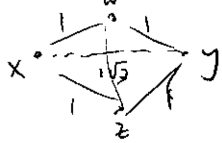
So exists

<u>PROP</u> $4 \le \chi(\mathbb{R}^2) \le 7$.

<u>PF</u>. <u>$\chi \ge 4$</u> (not 4-color thm)

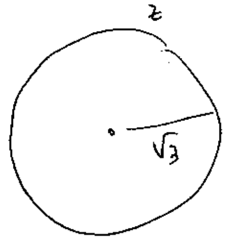
<u>CLAIM</u>: If 3-colored plane then every pair of pts at distance $\sqrt{3}$ has same color

<u>PF</u>:



Since $c(w) \neq c(z)$, so $c(y)$ and $c(x)$ are BOTH the leftover color.

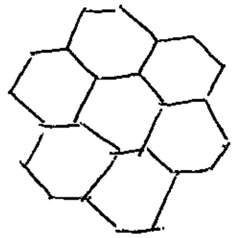
<u>THUS</u>:



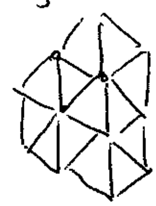
Entire circle is monochromatic $\Rightarrow$ unit distance same color. $\divideontimes$

PF. $\chi \leq 7$ How to color with 7 colors:

Split plane into hexagons of diameter just less than 1:



there is a coloring of this graph:



s.t. 7 colors used
and no 2 colors
within graph distance 0,1,2
of each other

So in geometry, monochromatic patches are far apart $\square$

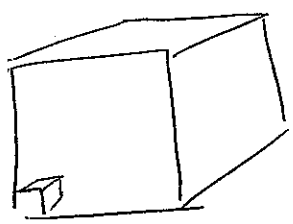
Q. (HADWIGER-NELSON). How fast does $\chi(\mathbb{R}^d)$ grow?

PROP. $\chi(\mathbb{R}^d) \geq d+1$

PF. Simplex with side 1

PROP. $\chi(\mathbb{R}^d) \leq d^{d/2}$

PF. Cut space into unit cubes. Then cut further into side length $\frac{1}{\sqrt{d}}$, so tiny cubes have diam 1.



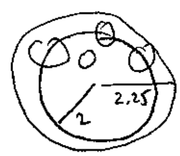
Then give each mini cube its own distinct color.
And tessellate space with the big pattern. $\square$

PROP. $\chi(\mathbb{R}^d) \leq 9^d$ "only" exponential.

PF. Let $\underline{C}$ be maximal subset of $\mathbb{R}^d$ s.t. any 2 pts have distance $\geq \frac{1}{2}$.

Let H be graph on C where we connect any centers at distance ≤ 2.

$\underline{\Delta(H)}$: Open balls of radius $\frac{1}{4}$ are disjoint, so max # inside is:

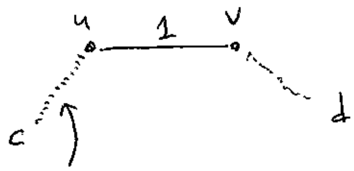


$\leq 9^d$ (-1) Since don't count the starting $B(\frac{1}{4})$ in middle.

__So.__ Can color H with $\leq q^d$ colors by greedy.

Now __RULE__: for each pt in $\mathbb{R}^d$, give it color of closest C.
Break ties arbitrarily

Can we have 2 pts (u,v) same color at distance 1?



Must be pt $\underline{c}$ at distance $\lneq \frac{1}{2}$, or else should add $\underline{u}$ to C.
Same for $\underline{d}$.

Δ-ineq $\Rightarrow$ dist$(c,d) \lneq 2$, so by H, diff. colors. $\divideontimes$ $\square$.

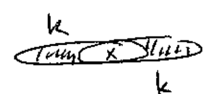
__TH.__ (LARMAN-ROGERS) Got $\chi(\mathbb{R}^d) \geq \Omega(d^2)$.

__TH__ (FRANKL-WILSON). $\chi(\mathbb{R}^d) \geq 1.1^d$.

__PF__ Find large (finite) subset of $\mathbb{R}^d$ which requires many colors

Let V = vtxs in $\{0,1\}^d$ with Hamming weight $\underline{k}$. Let $d = 4p$, $k = 2p-1$ (just below $\frac{1}{2}$).

__CLAIM.__ Largest subset of V with all distances NOT $\sqrt{2p}$ is $\leq \binom{d}{p-1}$.
(INDEP set in the $\sqrt{2p}$-distance subgraph)



__PF__ Identify pts with subsets of $[d]$.
Distances avoid $\sqrt{2p}$ $\Rightarrow$ intersections avoid $\underline{x}$, where $2p = 2(k-x)$,
since all subsets are $\underline{k}$ uniform.

So $x = k - p = p - 1$.

Uniform family, permitted intersections $\{0,1,2,\cdots,p-2\}$ __MOD P__
And uniform size $\equiv -1 \pmod p$.
$\Bigg\} \Rightarrow \# \leq \binom{d}{p-1}$. $\square$

__Now__: dilate by $\frac{1}{\sqrt{2p}}$, get set of $\binom{d}{k}$ pts with all UNIT-indep sets $\leq \binom{d}{p-1}$

$$\Rightarrow \chi \geq \binom{d}{k} \Big/ \binom{d}{p-1} = \binom{4p}{2p-1} \Big/ \binom{4p}{p-1} \approx 2^{4p\,H(\frac{1}{2}) - 4p\,H(\frac{1}{4})}$$

$$= 2^{d\cdot[H(\frac{1}{2}) - H(\frac{1}{4})]}. \qquad H(\tfrac{1}{2}) = 1,\ H(\tfrac{1}{4}) = \tfrac{1}{4}\log_2 4 + \tfrac{3}{4}\log_2 \tfrac{4}{3} = 0.81.$$

$$\longrightarrow 2^{d\cdot 0.19} = (1.14)^d.$$ $\square$.

HOMEWORK 3

(1). Split $\mathcal{F} = \mathcal{F}_2 \cup \mathcal{F}_3$, where:

$\mathcal{F}_2$ = sets with size $\not\equiv 0 \pmod 2$

$\mathcal{F}_3$ = sets with size $\not\equiv 0 \pmod 3$.

(Not a partition, since some sets may fall in both categories)

Covers $\mathcal{F}$ since if A not in $\mathcal{F}_2 / \mathcal{F}_3$, then $|A| \equiv 0 \pmod 2$, $\equiv 0 \pmod 3$ $\Rightarrow |A| \equiv 0 \pmod 6$!

But now $\mathcal{F}_2$ satisfies oddtown since intersections were $0 \pmod 6 \Rightarrow 0 \pmod 2$.

$\mathcal{F}_3$ satisfies mod 3-town

$\Rightarrow$ each $\left.\begin{array}{l}|\mathcal{F}_2| \le n \\ |\mathcal{F}_3| \le n\end{array}\right\} \Rightarrow |\mathcal{F}| \le 2n$. $\square$

(2) TH $\left.\begin{array}{l}\mathcal{F} \text{ set family, all sets } \not\equiv 0 \pmod{p^k}, \\ \text{all intersections } \equiv 0 \pmod{p^k}.\end{array}\right\} \Rightarrow |\mathcal{F}| \le n$.

PF. Take characteristic vectors $v_i \in \mathbb{Q}^n$ for the sets in $\mathcal{F}$.

Know: $v_i \cdot v_i \not\equiv 0 \pmod{p^k}$.

$v_i \cdot v_j \equiv 0 \pmod{p^k}$.

Suppose $\sum c_i v_i = 0$ over $\mathbb{Q}$.

Clear denominators $\rightarrow$ assume all c_i are in $\mathbb{Z}$, and have GCD = 1

(Now entire equation is in $\mathbb{Z}$)

Consider $0 = v_j \cdot (\sum c_i v_i) = \sum c_i (v_i \cdot v_j)$.

Now $p^k \mid$ all $c_i(v_i \cdot v_j)$ except for $c_i(v_i \cdot v_i)$ term.

$\Rightarrow p \mid c_i$.

But holds for all i. $\Rightarrow$ GCD contains p. $*$.

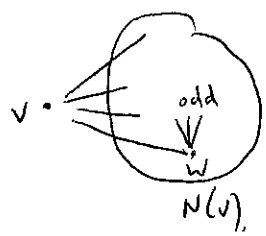
Thus $\{v_i\}$ are indep over $\mathbb{Q}$ $\rightarrow |\mathcal{F}| \le n$. $\square$.

Complete problem: oddtown mod $N = p_1^{k_1} \cdots p_r^{k_r}$.

$\left.\begin{array}{l}\mathcal{F}_1 = \text{sets size } \not\equiv 0 \pmod{p_1^{k_1}} \\ \vdots \\ \mathcal{F}_r\end{array}\right\} \Rightarrow \mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_r$, each $|\mathcal{F}_i| \le n$. $\square$.

(4). <u>CLAIM</u>. All codegrees odd $\Rightarrow$ all degrees even.

<u>PF</u>. Arbitrary vtx $\underline{v}$.



every other $w \in N(v)$ has odd codegree with $\underline{v}$.
$\Rightarrow \underline{w}$ has odd #nbs in $N(v)$.

So subgraph induced by $N(v)$ has all odd degrees $\rightarrow$ even size. $\square$.

<u>TH</u>. All codegrees odd, all degrees even $\Rightarrow$ #vtxs odd.

<u>PF</u>. let $v_1, \cdots, v_n$ be characteristic vectors of nbh's of vertices $1 \dots n$.

<u>Know</u>: $v_i \cdot v_i \equiv 0 \pmod 2$ Work over $\mathbb{F}_2$
$v_i \cdot v_j \equiv 1 \pmod 2$

So $\{\mathbb{1}, v_1, \cdots, v_n\}$ is $(n+1)$ vectors in $\mathbb{F}_2^n$, hence lin. dep!

$\exists$ b, a_i, not all 0: $\quad b\mathbb{1} + \sum a_i v_i = 0$. (over $\mathbb{F}_2$) $\quad (*)$

Take $v_j \cdot$ EQN:

$$
\begin{aligned}
0 &= b(\mathbb{1}\cdot v_j) + \sum a_i (v_i \cdot v_j) \\
&= \underset{\text{even deg}}{b\,(0)} + \sum_{i \neq j} a_i (1) + \underset{v_j \cdot v_j = 0}{a_j\,(0)} \\
&= S - a_j, \text{ where } S = \sum a_i. \qquad (**)
\end{aligned}
$$

$\Rightarrow$ all a_j are equal

<u>CASE 1</u>. all $a_j = 0$. Then $(*)$ $\Rightarrow$ $b = 0$, and ~~a~~ TRIVIAL lin. comb. $\divideontimes$

<u>CASE 2</u>. all $a_j = 1$. But $(**)$ $\Rightarrow$ $0 = S - a_j$
$= n - 1. \pmod 2$

$\Rightarrow \underline{n}$ odd $\square$.

(5). Consider random permutation σ of $[n]$.

Let X = # of initial sequences of σ that are in $\mathcal{F}$.

e.g. if $\sigma = (1,2,5,3,4,6)$

Then X = # of these in $\mathcal{F}$:

$\{1\}, \{1,2\}, \{1,2,5\}, \ldots \{1,2,5,3,4,6\}$.

$$t \geq \mathbb{E}[X] = \sum_{F\in\mathcal{F}} \mathbb{P}[F \text{ is initial } |F|\text{-sequence in } \sigma]$$

$$= \sum_{F\in\mathcal{F}} \frac{1}{\binom{n}{|F|}}.$$

Q. What is max # of F we can take s.t. $\sum \frac{1}{\binom{n}{|F|}}$ is $\leq t$?

A. Looks like Knapsack problem.

But if Knapsack problem has exact soln via GREEDY algorithm, then that is optimal.

Smallest item cost: $\frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$ $\longrightarrow$ can take $\binom{n}{\lfloor \frac{n}{2}\rfloor}$ of them $\rightsquigarrow$ total cost 1

In general, for each layer of the set lattice, we can take the whole layer, and get total cost 1

So we take t full layers as close as possible to the center. $\square$.

(6). Maximum size intersecting family is 2^{n-1} since "A in" $\Rightarrow$ "A^c out".

CLAIM. Every maximal intersecting family $\mathcal{F}$ has size 2^{n-1}.

PF. Suff. show if $E \notin \mathcal{F}$, then $E^c \in \mathcal{F}$, as this implies $|\mathcal{F}| \geq \frac{1}{2} 2^n$.

(*) But note that if $F \in \mathcal{F}$ but some set $F^+ \supsetneq F$ is not in $\mathcal{F}$, then we should have added F^+ to $\mathcal{F}$ for maximality.

~~So if E ∉ F, and E^c~~ Also, maximality $\Rightarrow$ why not add E to $\mathcal{F}$? Must have set $F\in\mathcal{F}$ s.t.

Then (*) $\Rightarrow$ $E^c = F^+ \in \mathcal{F}$ too. $\square$.

2010-04-25

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MEDIAN ORDER

DEF. TOURNAMENT: directed graph, s.t. between every pair of vtxs is EXACTLY ONE edge, in one of the 2 directions.

DEF. MEDIAN ORDER (of digraph, not. nec. tournament): a linear ordering of the vtxs ~~to~~ which maximizes # of FORWARD edges.

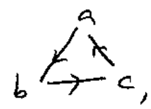
FORWARD EDGE: order $v_1 \quad v_2 \quad v_3 \quad \cdots \quad v_{10} \quad v_n$.

FWD

BACKWD

REMARKS ① Given a digraph median order always exists.

② But not nec. unique: in



all three are median orders: a, b, c; b, c, a; c, a, b.

③ These were originally studied for own sake; known to be NP-hard to find for a general digraph

④ In 2000, Havet and Thomassé discovered that they can be used as powerful inductive tool for digraph problems

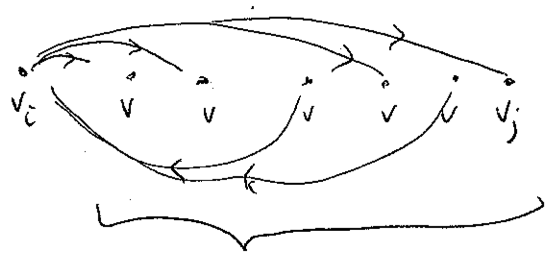
TH (FOLKLORE) Every tournament has a Hamiltonian path

PF. Consider a median order: $v_1 \quad v_2 \quad v_3 \quad \cdots v_t \, v_{t+1} \quad v_n$.

if edge between consecutive vtxs is BACK, then should swap them, increases # of FWD edges in total

So all consec. vtx pairs have FWD edges ⇒ Ham path. $\square$

LEMMA (FEEDBACK PROPERTY) In median order, $\forall\, i<j$:



in region $v_{i+1}, \ldots, v_j$: (# of FWD edges from v_i) $\geq$ (# BACK ~~from~~ to v_i)

PF. move v_i to spot after v_j.

2010-04-25

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DEF. In digraph, vtx $\underline{v}$ is a KING if it can reach everyone in 2 steps:

$$\{v\} \cup N^+(v) \cup N^{++}(v) = \text{all vtxs.}$$

TH. Every tournament has a KING.

PF. Fix median order.

Consider v_1 and any v_j

CASE 1: then done.

CASE 2: $v_1 \ldots\ldots\ldots v_j$.

In here, $\geq \frac{1}{2}(j-1)$ outnbrs of v_1 (since count v_j as in-nbr)

Also $\geq \frac{1}{2}(j-1)$ in-nbrs of v_j.

But only $j-2$ vtxs $\Rightarrow$ they intersect.

So:

$v_1 \to \cdot \to v_j$ $\square$

CONJ (SUMNER). Given any tree T, with $\underline{n}$ vtxs, each edge with orientation

Then every tournament with $2n-2$ vtxs contains copy of T.

Still open, but:

TH. (HAVET–THOMASSÉ) True if T is ARBORESCENCE: all edges out from root

PF. Embedding algorithm.

1. Fix median order $v_1\ v_2 \cdots,\ v_{2n-2}$.
2. Embed root of T at v_1, start WINDOW at $\{v_1\}$.
3. Maintain property: in window, every vtx has $\geq$($\frac{1}{2}$ of its following vtxs in the window) not yet occupied.
4. Select next vtx of T to embed, say is child of v_k.
5. Add next vtx to WINDOW.
6. Property $\Rightarrow v_k$ has outnbr in WINDOW, embed it.
7. Extend window by 1 to maintain PROPERTY.
8. Back to 4

\# exposed:

1 (root)

$+ 2(n-1)$ each next vtx

-1 (can skip 7 at end)

$= 2n-2.$ $\square$