Edges and Triangles

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Joint work with Jacob Fox

Edges in triangles

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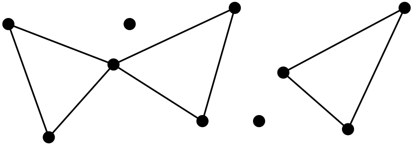

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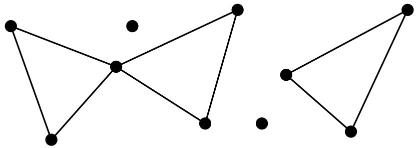

Question (Erdős-Rothschild)

What if the total number of edges must be at least $0.001n^2$? Must some edge be in many triangles?

Regularity Lemma

Szemerédi Regularity Lemma

For every ϵ, there is M such that every graph can be ϵ-approximated by an object of complexity bounded by M.

Triangle Removal Lemma

For any ϵ, there is a δ such that every graph with $\leq \delta n^3$ triangles can be made triangle-free by deleting only ϵn^2 edges.

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Dependency between parameters:

- (From Regularity Lemma.) $\frac{1}{\delta}$ is tower of height power of $\frac{1}{\epsilon}$.
- (Fox.) $\frac{1}{\delta}$ is tower of height logarithmic in $\frac{1}{\epsilon}$.

Lower bound for Erdős-Rothschild

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- Either case gives an edge in at least $\min\{\frac{3\delta n}{c}, \frac{c}{3\epsilon}\}$ triangles.
- Take $\frac{1}{\delta} = \sqrt{n}$, and $\frac{1}{\epsilon} =$ power of $\log^* n$. $\square$

Previous work

Theorem (Alon-Trotter)

For any constant $c < \frac{1}{4}$, there is a cn^2-edge graph with every edge in a triangle, but the most popular edge only in $\sqrt{n}$ triangles.

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Question (Erdős, 1987)

Given cn^2 edges, each of which is in a triangle, is there always some edge which is in at least n^ϵ triangles, for a constant ϵ?

New result

Theorem (Fox, L.)

There are n-vertex graphs with

$$\frac{n^2}{4}\left(1 - e^{-(\log n)^{1/6}}\right)$$

edges, each of which is in a triangle, but with no edge in more than $n^{14/\log\log n}$ triangles.

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Remarks:

- Every edge is in under $n^{o(1)}$ triangles.
- The edge density approaches $\frac{1}{4}$ from below.
- Sharp transition: after edge density $\frac{1}{4}$, some edge is in a linear number of triangles.

Theorem (Hoeffding-Azuma)

For any L-Lipschitz random variable X determined by n independent samples,

$$\mathbb{P}\left[|X - \mathbb{E}[X]| > t\right] \leq 2e^{-\frac{t^2}{2L^2 n}}.$$

Construction materials

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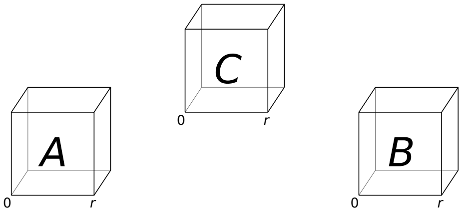
Classical result

In even dimensions d, the Euclidean ball of radius r has

$$\mathrm{Vol}\left(B_r^{(d)}\right) = \frac{\pi^{d/2} r^d}{(d/2)!}.$$

Core tripartite graph:

- Take 3 copies of the lattice cube of side r in dimension $d = r^5$.



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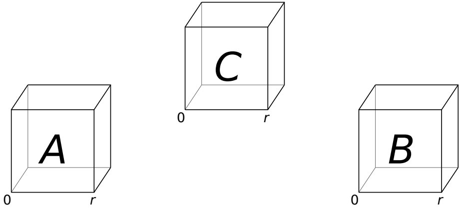

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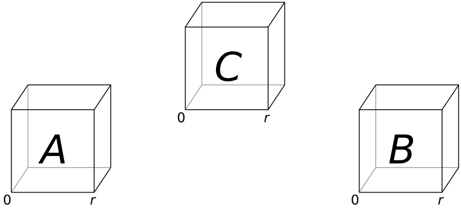

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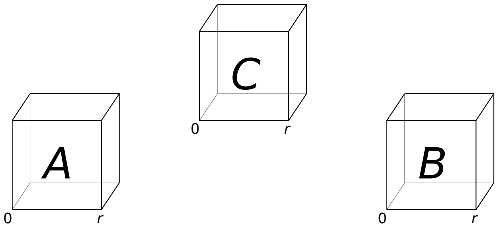
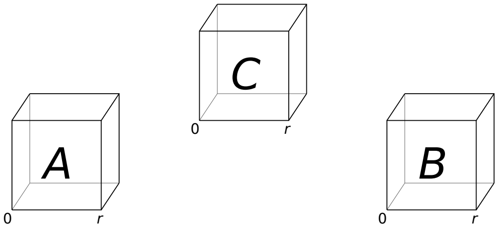


Positive edge density:

- The edge density between A and B is the probability that two random points in the cube have squared-distance within $\mu \pm d$.



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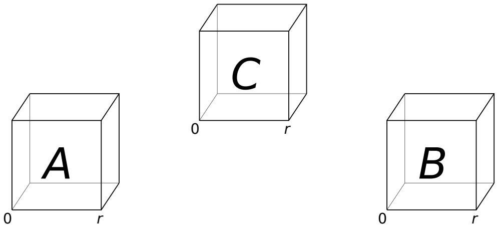

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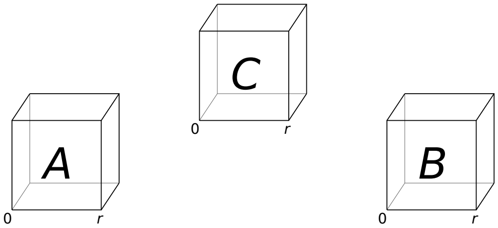
- The typical deviation from μ is $r^2\sqrt{d} = r^{4.5} \ll d$, since $d = r^5$, so the A–B edge density approaches 1!



Every A–B edge is in a triangle:

- A–B endpoints have squared-distance $\mu \pm d$.
- Their integer-rounded midpoint has squared-distance $\frac{\mu}{4} \pm 2d$ from each endpoint.

Properties

Every A–B edge is in few triangles:

- Given $0 = (0, \ldots, 0)$ and $z = (z_1, \ldots, z_d)$ with $\|z\|^2 = \mu \pm d$.
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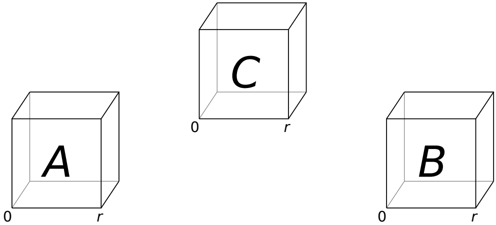

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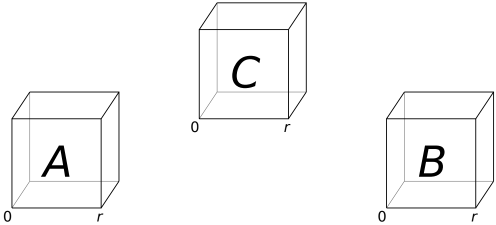
- Now every edge has few triangles; every A–B edge has some.

Final construction



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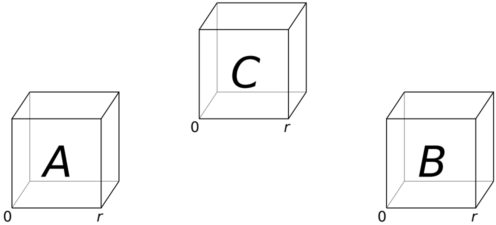

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Blow up (simplification by Alon):

- Replace every vertex in A and B with 2^d copies of itself.
- Now every edge is in at most $30^d \lll (2r)^d$ triangles.