Rainbow Hamilton cycles

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Carnegie Mellon University

Joint work with Alan Frieze

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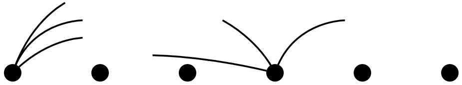

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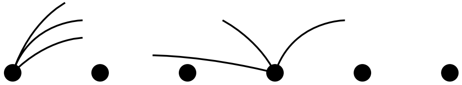
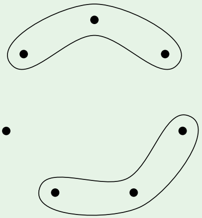


- (Cooper, Frieze) $D_{2\text{-in},2\text{-out}}$ is Hamiltonian **whp**.

Definition (3-uniform hypergraph)

$H_{n,p;3}$: each triple appears independently with probability p.



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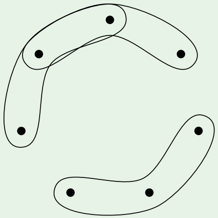

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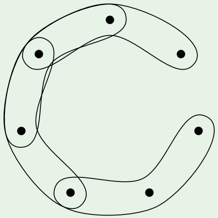

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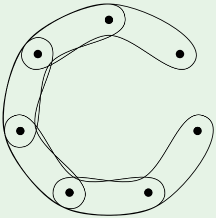

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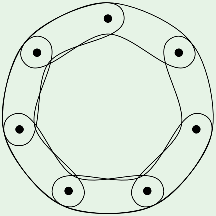

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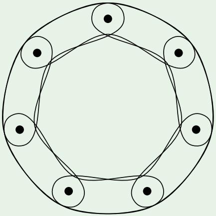

Tight H-cycle

Loose H-cycle

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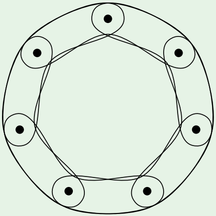

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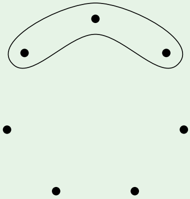

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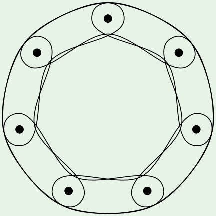

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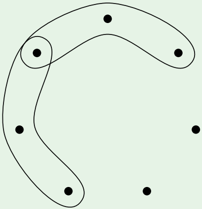

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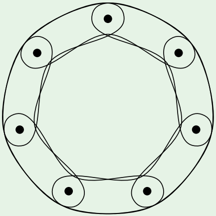

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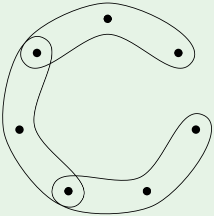

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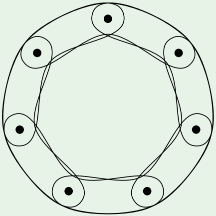

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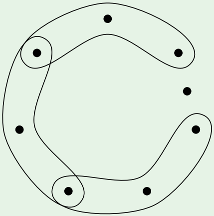

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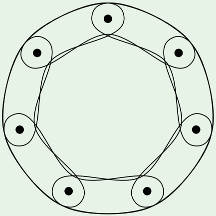

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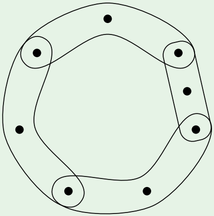

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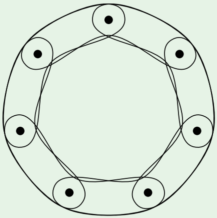

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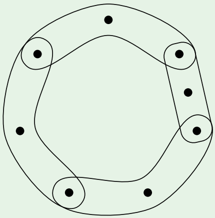

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- (Frieze) $H_{n,p;3}$ has loose H-cycle **whp** if $p > \frac{K \log n}{n^2}$, $4 \mid n$.

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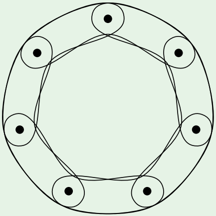

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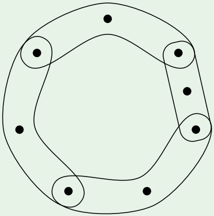

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- (Frieze) $H_{n,p;3}$ has loose H-cycle **whp** if $p > \frac{K \log n}{n^2}$, $4 \mid n$.
- (Dudek, Frieze) Asymptotically answered for all uniformities, and all degrees of loose-ness.

Rainbow Hamilton cycles

Question

Does $G_{n,p}$ have a rainbow Hamilton cycle if edges are randomly colored from κ colors?

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Observations

- Must have $p > \frac{\log n + \log\log n + \omega(n)}{n}$ with $\omega(n) \to \infty$.
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- Must have $p > \frac{\log n + \log\log n + \omega(n)}{n}$ with $\omega(n) \to \infty$.
- Must have $\kappa \geq n$.

- (Cooper, Frieze) True if $p = \frac{20 \log n}{n}$ and $\kappa = 20n$.
- (Janson, Wormald) True if $G_{2r\text{-reg}}$ is randomly colored with each of $\kappa = n$ colors appearing exactly $r \geq 4$ times.

Loose vs. rainbow H-cycles

- Connect 3-uniform hypergraphs to colored graphs

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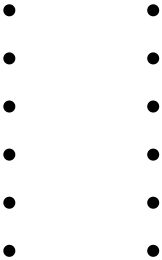

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(bisected vertex set)

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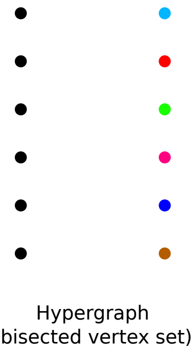

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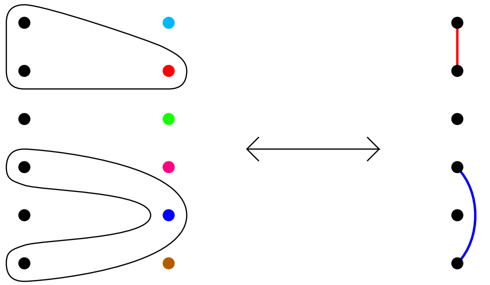

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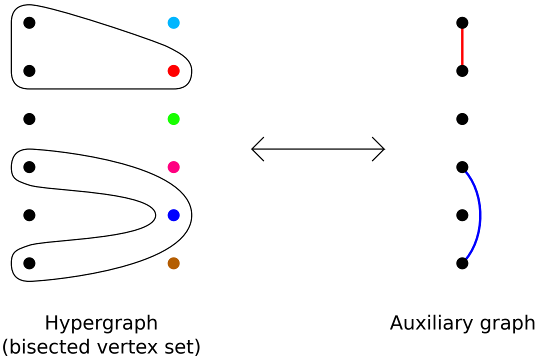
- Connect 3-uniform hypergraphs (loose Hamiltonicity) to colored graphs (rainbow Hamilton cycles).



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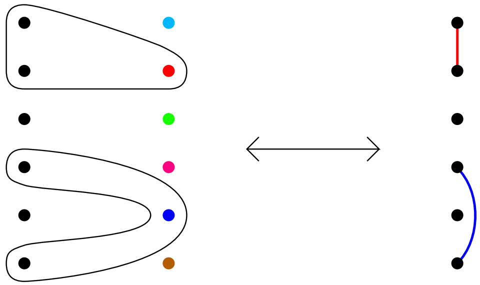

Hypergraph (bisected vertex set)

Auxiliary graph

- Frieze applied Johansson-Kahn-Vu to find perfect matchings.
- Apply Janson-Wormald to find rainbow H-cycle in randomly colored random regular graph.

Theorem (Frieze, L.)

For any fixed $\epsilon > 0$, if $p = \frac{(1+\epsilon)\log n}{n}$, then $G_{n,p}$ contains a rainbow Hamilton cycle **whp** when its edges are randomly colored from $\kappa = (1+\epsilon)n$ colors.

Remarks:

- Asymptotically best possible, both in terms of p and κ.
- Still holds when ϵ tends (slowly) to zero.

Proof ideas

Observation

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- By Chernoff, $\mathbb{P}\left[\deg(v) < \frac{1}{10}\mathbb{E}\right] < e^{-\frac{2}{3}\mathbb{E}} = n^{-\frac{2}{3}}$.
- Typically, all but $< \sqrt[3]{n}$ vertices have degree $\geq \frac{1}{10}\log n$. $\square$

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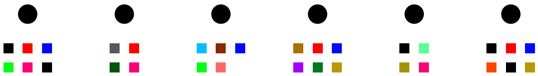
First attempt to find rainbow H-cycle:

- Suppose all degrees $\geq \frac{1}{10}\log n$.
- At each vertex, expose list of colors that appear.



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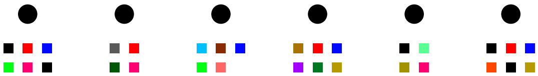

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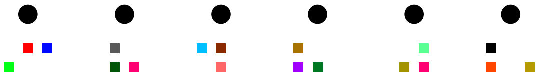

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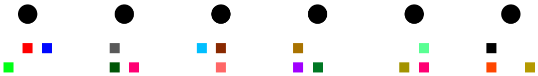

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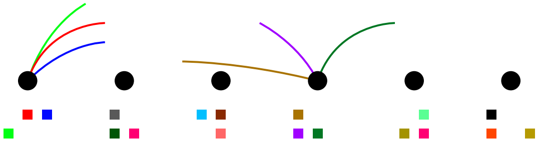

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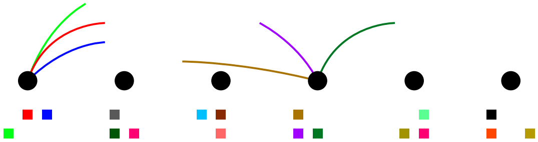

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Already requires 3n colors.

Saving the constant factor

Sprinkling

Reserve $p' = \frac{\epsilon}{2} \cdot \frac{\log n}{n}$ and $\kappa' = \frac{\epsilon n}{2}$ for 2nd phase.

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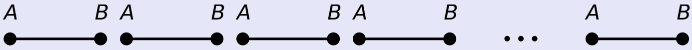
Main lemma

Using only edges and colors from Phase 1, there is a partition into rainbow intervals, such that:



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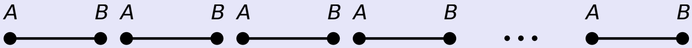

- All intervals have length $L = \frac{14}{\epsilon}$.
- Each A-vertex has $\geq \frac{\epsilon^2}{40L} \log n$ B-neighbors in Phase 2.
- Each B-vertex has $\geq \frac{\epsilon^2}{40L} \log n$ A-neighbors in Phase 2.

Final rainbow linking

- Expose Phase 2 colors between A- and B-vertices.
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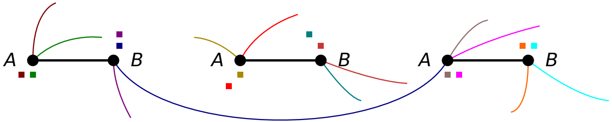

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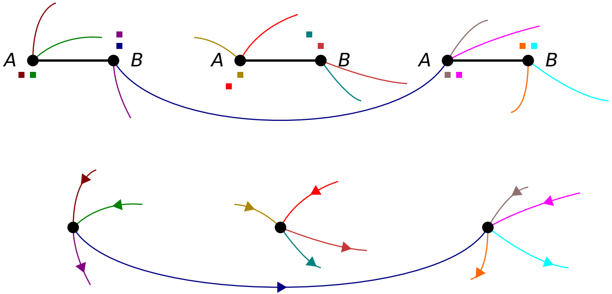

- Aux. digraph: vertices are intervals; edges oriented $B \to A$.
- Directed H-cycle in $D_{\text{2-in,2-out}}$ links all intervals via Phase 2. $\square$

Constructing intervals

Theorem (Ajtai, Komlós, Szemerédi; de la Vega)

Let $p = \frac{\omega}{n}$, where $0 < \omega < \log n - 3 \log\log n$. Then $G_{n,p}$ has a path of length $\left(1 - \frac{1}{\omega}\right) n$ **whp**.

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To obtain intervals:

- Adapting proof of de la Vega, find rainbow path of length $n - o(n)$ in Phase 1.
- Break the long path into intervals of length $L = \frac{14}{\epsilon}$.

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- Absorb all missing vertices into system of intervals, using minimum degree two. $\square$

Conclusion

Theorem (Frieze, L.)

For any fixed $\epsilon > 0$, if $p = \frac{(1+\epsilon)\log n}{n}$, then $G_{n,p}$ contains a rainbow Hamilton cycle **whp** when its edges are randomly colored from $\kappa = (1+\epsilon)n$ colors.

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Edge-colored random graph process:

- Start with n isolated vertices.
- Each round, add a new edge, selected uniformly at random from all missing edges.
- Randomly color the new edge from a set C of size at least n.

Conclusion

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Question

Does a rainbow Hamilton cycle appear as soon as the minimum degree is at least two and at least n colors have arrived?