Chasing a fast robber

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Joint work with Alan Frieze and Michael Krivelevich

Setting (Nowakowski, Winkler)

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1. Cops choose starting positions first.
2. Then Robber chooses starting position.
3. Each cop either moves by 1 edge, or stays put.
4. Robber either moves by 1 edge, or stays put.
5. Repeat steps 3 and 4.

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Remark: It is possible for Robber to win, e.g., if G is a cycle on 4 or more vertices and there is only 1 cop.

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If P is a shortest path between some $a, b \in G$, then one cop is sufficient to keep Robber off P.

Proof idea. Cop maintains invariant: for every vertex $v \in P$, he is closer to v than Robber is. (Possible since P is a shortest path.) □

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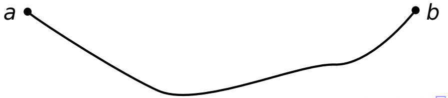

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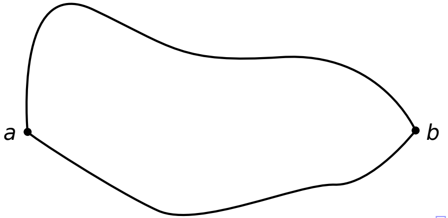


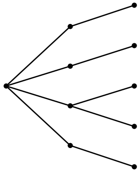
General graphs: lower bound

Projective plane graph

There are C_4-free bipartite graphs with all degrees $\Theta(\sqrt{n})$.

Corollary

The cop number of a general graph can be as large as $\Omega(\sqrt{n})$.



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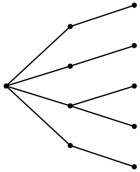

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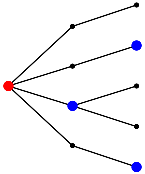
- Robber stays put, unless a cop moves to an adjacent vertex.



- Since no C_3 or C_4, total number of robber's neighbors dominated/occupied by cops is $< \delta$, so robber can escape. □

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Meyniel's Conjecture

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Theorem (FGKNS, 2010)

If the robber can traverse up to 2 edges per move, then $\Omega(\sqrt{\log n})$ cops are required to catch the robber on the $n \times n$ grid.

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- Projective plane graphs were used for the old lower bound, but these are hard to analyze for fast robber strategies.
- Previous upper bound arguments used diameter lemma, which does not apply for fast robber.

Results

Theorem 1 (Frieze, Krivelevich, L.)

Let R be the robber's speed. There exist n-vertex graphs which:

- require $n^{1-\frac{1}{R-2}}$ cops to catch the robber, if $3 \leq R \leq \infty$;
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Theorem 2 (Frieze, Krivelevich, L.)

For any $R \geq 1$ and any connected graph G on n vertices, $n/\alpha^{\sqrt{\log_\alpha n}}$ cops are sufficient to catch any speed-R robber, where $\alpha = 1 + \frac{1}{R}$.

This smoothly extends the best upper bound to fast robbers.

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Earlier: if cannot guard G efficiently, then deduce structure:

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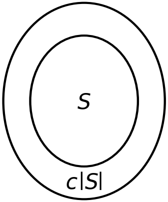

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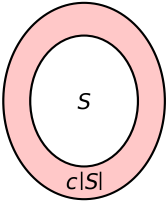

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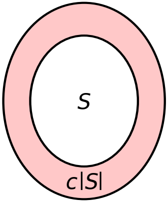

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- The robber can never pass this barrier, so the problem reduces to either S or $G \setminus (N(S) \cup S)$.
- Cost in cops is only c-fraction of vertices removed.

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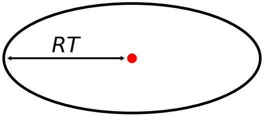

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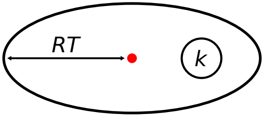

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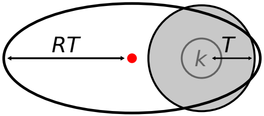

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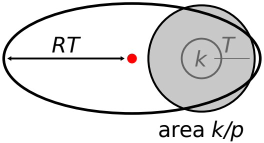

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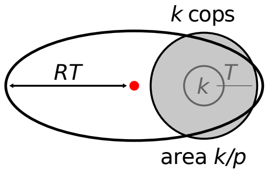

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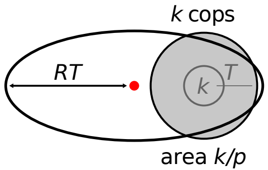

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- Works as long as $\left(\frac{1}{p}\right)^{RT} \ll n$. □

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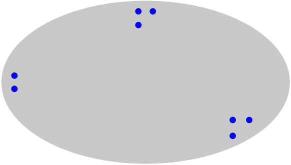
Robber strategy on $G_{n,p}$

- Let C be vertices occupied by cops.



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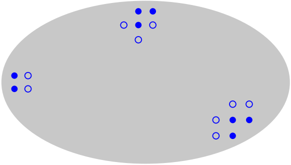

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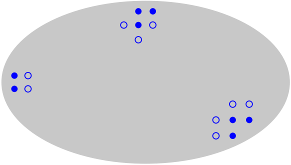

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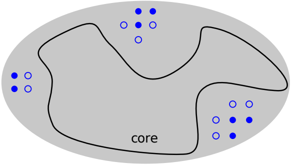

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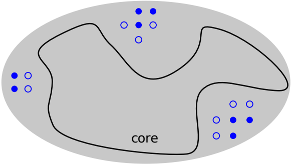

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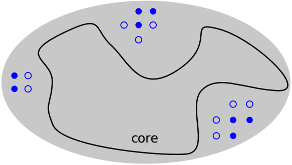

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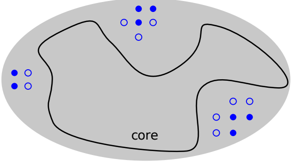
- Say robber is in core, and cops move.



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- Let C^+ be C, together with neighboring vertices.
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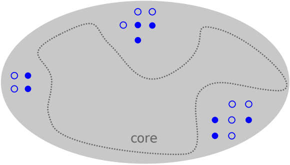
Proof. For $np = n^c$, show speed-$\frac{1}{c}$ robber can elude n^{1-c} cops.

- In $G_{n,p}$, any $H = G \setminus C^+$ has $\frac{np}{3}$-core of size $0.99n$.
- Say robber is in core, and cops move.
- Let C' be new cop positions.



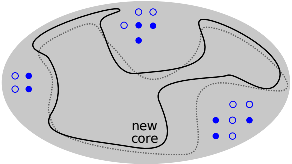
Lower bound

Robber strategy on $G_{n,p}$

- Let C be vertices occupied by cops.
- Let C^+ be C, together with neighboring vertices.
- Robber stays in $\frac{np}{3}$-core of $G \setminus C^+$.

Proof. For $np = n^c$, show speed-$\frac{1}{c}$ robber can elude n^{1-c} cops.

- In $G_{n,p}$, any $H = G \setminus C^+$ has $\frac{np}{3}$-core of size $0.99n$.
- Say robber is in core, and cops move.
- Let C' be new cop positions.
- Since $C' \subset C^+$, robber can still move within H.



- New core also has size $0.99n$, so it overlaps old core.
- By properties of $G_{n,p}$, robber can reach new core fast. □

Remarks.

- Our lower bound robber strategy is (necessarily) more complex, so we use $G_{n,p}$ instead of the projective plane.
- Our upper bound matches the first-order constants of Lu-Peng and Scott-Sudakov, using expansion instead of diameter.

Final comments

Remarks.

- Our lower bound robber strategy is (necessarily) more complex, so we use $G_{n,p}$ instead of the projective plane.
- Our upper bound matches the first-order constants of Lu-Peng and Scott-Sudakov, using expansion instead of diameter.

Open problems.

- Are $\omega(\sqrt{n})$ cops required to catch a speed-2 robber? Our bound only exceeds $\sqrt{n}$ for $R \geq 5$.
- What if cops and robber move at the *same* speed $R \geq 2$?