Constrained Ramsey Numbers

Po-Shen Loh
Princeton University

Joint work with Benny Sudakov

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- What if there may be *arbitrarily* many colors?
- Then is there an n which guarantees a monochromatic copy of K_t or a rainbow copy of K_t?

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Every edge can have a different color, so cannot guarantee monochromatic subgraphs.

Erdős-Rado, 1950:

$\forall t, \exists n$ s.t. any edge-coloring of the complete graph on $\{1, \ldots, n\}$, with *arbitrarily* many colors, has a copy of K_t that is one of:

- monochromatic
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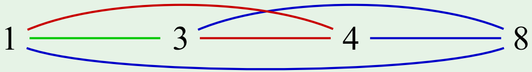
- monochromatic
- rainbow: all edges different colors
- upper lexical: color uniquely determined by larger endpoint
- lower lexical: color uniquely determined by smaller endpoint

Upper lexical coloring:



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Definition:

Constrained Ramsey number $f(S, T)$ = minimum n such that every edge-coloring of K_n, with arbitrarily many colors, has one of:

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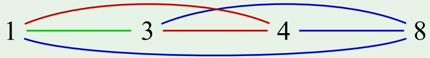

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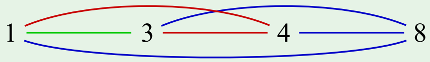

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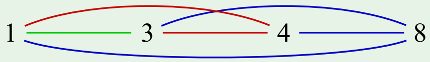

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Thus, $f(S, T)$ exists iff S is a star or T is a forest.

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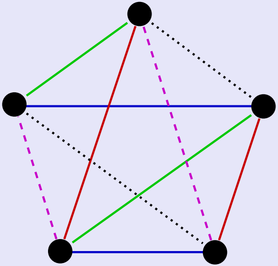

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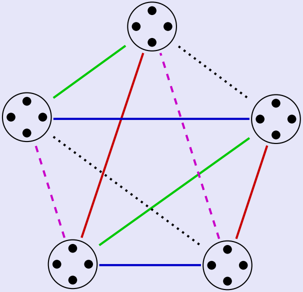

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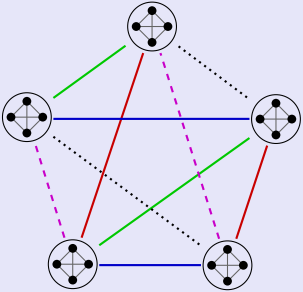

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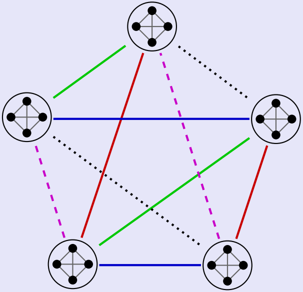

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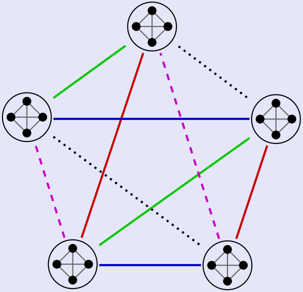

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- Only $t-1$ total colors $\Rightarrow$ no rainbow T.

Upper bounds

Jamison, Jiang, and Ling, 2003: $f(S, T) \leq O(st^2)$

- Proof by induction on diameter of T.
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Major open problem:

Find a sub-cubic upper bound for the diagonal case $f(P_t, P_t)$.

Main result

L. and Sudakov, 2006:

The constrained Ramsey number satisfies $f(S, P_t) \leq O(st \log t)$.

That is, for any tree S with s edges and any integer t, one can always find either a monochromatic copy of S or a rainbow t-path in any edge-coloring of the complete graph on $3600st \log t$ vertices.

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- Proof significantly extends Wagner's idea of orienting edges such that directed paths are automatically rainbow.
- We use the concept of *median order* as an inductive tool, as introduced in Havet and Thomasse (2000).

From colors to directions

The proof proceeds by contradiction; suppose that there is no monochromatic S and no rainbow t-path.

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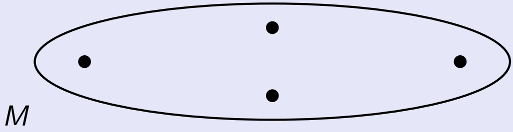
Orientation lemma:

There exists a subset $M \subset V$ such that:



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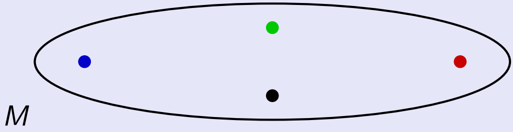

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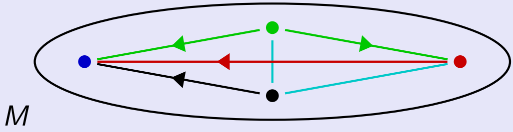

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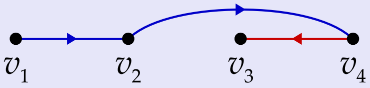
- We may direct most of the edges in M such that if an edge is directed $\overrightarrow{uv}$, then its color was c_u.

Observation: directed paths are automatically rainbow.

Definition:

Given a linear ordering $v_1 < v_2 < \cdots < v_n$ of the vertex set of a directed graph, an edge $\overrightarrow{v_i v_j}$ is called *forward* if $i < j$, and *backward* otherwise.



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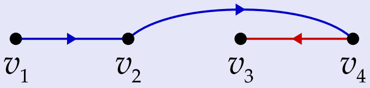

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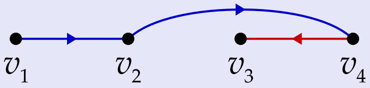

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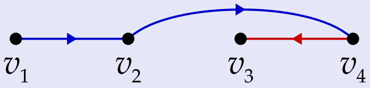

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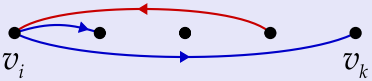
- Always exists, but not necessarily unique.
- Originally arose in theoretical computer science; NP-hard.
- Havet and Thomassé (2000) used median orders to give simple proofs of Dean's Conjecture, and Sumner's Conjecture for arborescences.

Feedback property

Forward bias (helps find directed paths):

Let $v_1 < \cdots < v_n$ be a median order. Then for every $i < k$,



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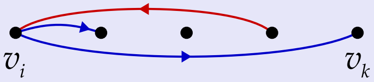

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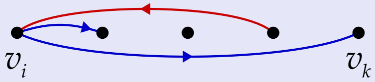

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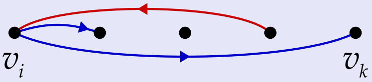

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- But this increases the total number of forward edges in the graph, contradicting the maximality of the median order. □

Summary of results:

- The constrained Ramsey number $f(S, P_t)$ is upper bounded by $O(st \log t)$.
- That is, for any tree S with s edges and any integer t, one can always find either a monochromatic copy of S or a rainbow t-path in any edge-coloring of the complete graph on $3600st \log t$ vertices.
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- Remove the logarithmic term that separates our bound from the simple lower bound $f(S, T) \geq \Omega(st)$. We believe that it is an artifact of the proof.
- It would be very interesting to improve the upper bounds for $f(S, T)$ when T is a general tree (instead of a path).