Matching and Planarity

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1 Warm-up

1. (Bondy 1.5.9.) There are n points in the plane such that every pair of points has distance ≥ 1. Show that there are at most $3n$ (unordered) pairs of points that span distance exactly 1 each.

   Solution: The unit-distance graph is planar. If there are 2 crossing unit-distance edges, then the triangle inequality implies that some pair of points is at distance strictly less than 1.

2 Matching

Consider a bipartite graph $G = (V, E)$ with partition $V = A \cup B$. A *matching* is a collection of edges which have no endpoints in common. We say that A *has a perfect matching to* B if there is a matching which hits every vertex in A.

Theorem. *(Hall's Marriage Theorem) For any set $S \subset A$, let $N(S)$ denote the set of vertices (necessarily in B) which are adjacent to at least one vertex in S. Then, A has a perfect matching to B if and only if $|N(S)| \geq |S|$ for **every** $S \subset A$.*

This has traditionally been called the "marriage" theorem because of the possible interpretation of edges as "acceptable" pairings, with the objective of maximizing the number of pairings. In real life, however, perhaps there may be varying degrees of "acceptability." This may be formalized by giving each vertex (in both parts) an ordering of its incident edges. Then, a matching M is called *unstable* if there is an edge $e = ab \notin M$ for which both a and b both prefer the edge e to their current partner (according to M).

Theorem. *(Stable Marriage Theorem) A stable matching always exists, for every bipartite graph and every collection of preference orderings.*

2.1 Problems

1. (Classical.) Show that every k-regular bipartite graph can have its edges partitioned into k edge-disjoint perfect matchings.

   Solution: Suffices to find one perfect matching. Every set S expands because it has k edges out, and each vertex on the other side can only absorb up to k of them in.

2. (Petersen, 1891.) A *2-factor* of a graph is a 2-regular spanning subgraph (i.e., containing all vertices, and having all degrees equal to 2). For every positive integer k, show that every $2k$-regular graph can be partitioned into k edge-disjoint 2-factors.

   Solution: Suffices to find one 2-factor. Take an Eulerian orientation. Split each vertex v into v^+, v^-. This gives a bipartite graph with twice as many vertices. If there was an edge $\overrightarrow{vw}$, now put it from v^- to w^+. It is a k-regular bipartite graph, so it has a perfect matching by above. Collapsing back the v^+, v^-, we get a 2-factor.

3. (Canada 2006/3.) In a rectangular array of nonnegative reals with m rows and n columns, each row and each column contains at least one positive element. Moreover, if a row and a column intersect in a positive element, then the sums of their elements are the same. Prove that $m = n$.

   Solution: Create bipartite graph, LHS is rows, RHS is columns. Edge if row and column intersect at positive entry. Suppose some set S of rows only has a total set $|T| < |S|$ of columns in which positive entries appear. Let the sums of the $|S|$ rows be $s_1, \ldots, s_k$. By the property, each of the $|T|$ columns has sum equal to one of the s_i. So the total sum of the elements in the S rows, when calculated from the column point of view (since entries outside are all nonnegative) is at most a sum of a subset of the s_i. Yet from the row point of view, it is the full sum. As all $s_i > 0$, this is a contradiction.

   Hall's theorem therefore says that there is a matching from the rows to the columns, implying that the number of columns is at least the number of rows. By symmetry, the reverse inequality is true, and we have equality.

4. (Iran 1998/2.) Suppose an $n \times n$ table is filled with the numbers $0, 1, -1$ in such a way that every row and column contains exactly one 1 and one -1. Prove that the rows and columns can be reordered so that in the resulting table each number has been replaced by its negative.

   Solution: This is the adjacency matrix of a directed bipartite graph. Two sides of the graph, A and B, each have n vertices. We write $+1$ in position (i, j) if there is an edge $\overrightarrow{a_i b_j}$, and -1 if there is an edge $\overleftarrow{a_i b_j}$. Since every row has exactly one of each ± 1, every a_i has out-degree and in-degree exactly 1. Similarly, the column condition implies that every b_j has this property.

   Therefore, the directed graph decomposes into disjoint directed cycles, which we may consider independently since they use disjoint sets of rows/columns. Consider just one directed cycle, WLOG $(a_1 b_1 a_2 b_2 \ldots a_k b_k)$. Note that permuting the rows/columns is equivalent to relabeling the vertices, and redrawing the incidence matrix. Getting the negative matrix means that all edges now need to be in opposite direction. But this is easy on a cycle; go backwards: $a'_1 = a_1$, $b'_1 = b_k$, $a'_2 = a_k$, $b'_2 = b_{k-1}$, ... Now the unsigned incidence matrix is still the same because we have edges between consecutive a's and b's, but all directions are reversed, as desired.

5. (IMO Shortlist 2006/C6.) A *holey triangle* is an upward equilateral triangle of side length n with n upward unit triangular holes cut (in the standard lattice positions). A *diamond* is a $60°$–$120°$ unit rhombus. Prove that a holey triangle T can be tiled with diamonds if and only if the following condition holds: Every upward equilateral triangle of side length k in T contains at most k holes, for $1 \le k \le n$.

   Solution: First show that if it can be tiled by diamonds, then there are at most k holes in every k-triangle. Each downward triangle requires exactly one upward triangle, and there is a bijection upward and downward triangles by taking an upward triangle and looking at the downward triangle sharing its bottom edge. (This gives a bijection except for the k upward triangles across the bottom.) So there are exactly k more upward than downward triangles, implying that we cannot have more than k holes.

   The converse uses Hall's theorem to match downward triangles with adjacent upward triangles. Suppose for contradiction that the Hall condition is false, i.e., there is a set S of downward triangles such that there are strictly fewer non-hole neighbors $N(S)$. Highlight those S downward triangles, along with all upward triangles directly adjacent to them. We will make the highlighted region into a disjoint union of large upward triangles.

   Indeed, note that if we have a large upward triangle of side length 3, and exactly 2 of the downward triangles are in S, then we can add the third downward triangle to S (increasing $|S|$ by 1), and only increase $|N(S)|$ by at most 1, thus maintaining $|N(S)| < |S|$. Repeating this procedure eventually grows the set S such that $N(S)$ is a disjoint union of large upward triangles. So, one of these large upward triangles, say of side length k, has its set S' of all downward triangles satisfying $|N(S')| < |S'|$. Yet $N(S')$ is the whole set of upward triangles in that part, so this means that there are more than k holes, contradiction.

3 Planarity

When we represent graphs by drawing them in the plane, we draw edges as curves, permitting intersections. If a graph has the property that it can be drawn in the plane without any intersecting edges, then it is called *planar*. Here is the tip of the iceberg. One of the most famous results on planar graphs is the Four-Color Theorem, which says that every planar graph can be properly colored using only four colors. But perhaps the most useful planarity theorem in Olympiad problems is the Euler Formula:

Theorem. *Every connected planar graph satisfies $V - E + F = 2$, where V is the number of vertices, E is the number of edges, and F is the number of faces.*

Solution: Actually prove that $V - E + F = 1 + C$, where C is the number of connected components. Each connecting curve is piecewise-linear, and if we add vertices at the corners, this will keep $V - E$ invariant. Now we have a planar graph where all connecting curves are straight line segments.

Then induction on $E + V$. True when $E = 0$, because $F = 1$ and $V = C$. If there is a leaf (vertex of degree 1), delete both the vertex and its single incident edge, and $V - E$ remains invariant. If there are no leaves, then every edge is part of a cycle. Delete an arbitrary edge, and that will drop E by 1, but also drop F by 1 because the edge was part of a cycle boundary, and now that has merged two previously distinct faces.

3.1 Problems

1. Every planar graph satisfies $E \leq 3V$.

   Solution: Break into connected components. Special case is the components with only 1 or 2 vertices, but still true in those cases. Otherwise, for each face, calculate its perimeter, and add all of these up. This double-counts each edge. Each face has perimeter ≥ 3, so we get $2E \geq 3F$. Plugging in, we have

   $$2 = V - E + F \leq V - E + \frac{2}{3}E = V - \frac{1}{3}E.$$

2. Prove that every planar graph is 6-colorable.

   Solution: Every planar graph contains a vertex of degree at most 5. To see this, use the bound $E \leq 3V - 6$ for $V \geq 3$, and if $V \leq 2$ it is trivial. Keep pulling out the vertex of this degree, and then greedily color the graph in the reverse order.

3. Prove that every planar graph is 5-colorable.

4. **Open:** Prove that every planar graph is 4-colorable, without using a computer.

5. Show that K_5 is not planar.

   Solution: $V = 5$, $E = 10$, so we must have $F = 2 - V + E = 7$. But as in the previous solution, we need to have $2E \geq 3F$, which is not the case.

6. Show that $K_{3,3}$ is not planar.

   Solution: $V = 6$, $E = 9$, so we must have $F = 2 - V + E = 5$. But as in the previous solution, we need to have $2E \geq 3F$. Actually, we need $2E \geq 4F$, because $K_{3,3}$ has no triangles. But this stronger inequality is false.

7. Show that $K_{4,4}$ is not planar.

   Solution: $K_{4,4}$ contains $K_{3,3}$.

8. (AoPS.) Prove that every convex polyhedron which has no quadrilateral or pentagonal faces must have at least 4 triangular faces.

   Solution: First note that at every vertex, at least 3 faces meet, so the degree of each vertex (in the edge graph) is at least 3. Therefore, $3V \le \sum d_v = 2E$, i.e., $V \le \frac{2}{3}E$. Also, by summing over faces, if f_k is the number of faces with k sides, then

$$\begin{aligned} 2E &= 3f_3 + 6f_6 + 7f_7 + \cdots \\ \frac{E}{3} &= \frac{1}{2}f_3 + f_6 + \frac{7}{6}f_7 + \cdots . \end{aligned}$$

   From Euler's formula,

$$2 = V - E + F \le F - \frac{E}{3} = \frac{1}{2}f_3 - \frac{1}{6}f_7 - \frac{2}{6}f_8 - \cdots \le \frac{1}{2}f_3,$$

   giving the result.

9. (IMO Shortlist 2006/C7.) Consider a convex polyhedron without parallel edges and without an edge parallel to any face other than the two faces adjacent to it. Call a pair of points of the polyhedron *antipodal* if there exist two parallel planes passing through these points and such that the polyhedron is contained between these planes.

   Let A be the number of antipodal pairs of vertices, and let B be the number of antipodal pairs of midpoints of edges. Determine the difference $A - B$ in terms of the numbers of vertices, edges, and faces.

   Solution: Use the following map to S^2, together with Euler's $V - E + F = 2$. Send each face of the polyhedron to its normal vector. Send each edge of the polyhedron to the set of normal vectors of its supporting planes, which is a (shorter) great circle arc between the images of the faces under this map. Send each vertex of the polyhedron to the set of normal vectors of supporting planes, which is a region between the great circle arcs drawn in the previous step. This produces a "spherical polyhedron," to which we translate the conditions of the problem and apply Euler's Formula.

10. (St. Petersburg 1997/13.) The sides of a convex polyhedron are all triangles. At least 5 edges meet at each vertex, and no two vertices of degree 5 are connected by an edge. Prove that this polyhedron has a face whose vertices have degree 5, 6, 6, respectively.

   Solution: By Euler, $E \le 3V - 6$, so in particular the sum of degrees is less than $6V$. We will use this for a contradiction. Suppose there are no 5,6,6 faces. We will count the number of edges which connect vertices of degree 5 to vertices of degree ≥ 7.

   Let x_i be the number of vertices of degree i for each i. No 5,6,6 implies that each 5-vertex has at most 2 neighbors of degree 6, thus it contributes 3 edges which cross from degree 5 to degree ≥ 7. On the other hand, any vertex of degree d has at most $\lfloor d/2 \rfloor$ neighbors of degree 5 because no two degree-5 guys are adjacent. Thus, double-counting gives:

$$\begin{aligned} 3x_5 &\le \sum_{d=7} x_d \cdot \left\lfloor \frac{d}{2} \right\rfloor \\ x_5 &\le \sum_{d=7} x_d \cdot \frac{1}{3} \left\lfloor \frac{d}{2} \right\rfloor . \end{aligned}$$

   Note that for $d \ge 7$, the cumbersome expression satisfies $\lfloor d/2 \rfloor / 3 \ge d - 6$. Adding to the LHS so that it becomes 6 times the number of vertices:

$$\begin{aligned} x_5 &\le \sum_{d=7} x_d \cdot (d-6) \\ 6x_5 + 6x_6 + \sum_{d=7} 6x_d &\le 5x_5 + 6x_6 + \sum_{d=7} x_d \cdot d. \end{aligned}$$

Recognize the LHS as $6V$ and the RHS as sum of degrees, and this contradicts our opening observation.

11. (Conway.) A *thrackle* is a drawing of a graph in the plane, in such a way that every pair of edges crosses exactly once.

    Open: Prove that every thrackle has $E \leq V$.

    Note: It has been shown that every minimal counterexample is a pair of even cycles intersecting at a single vertex. Hence it suffices to prove that such graphs cannot be drawn as thrackles.