Collinearity and concurrence

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1 Warm-up

1. Let I be the incenter of $\triangle ABC$. Let A' be the midpoint of the arc BC of the circumcircle of $\triangle ABC$ which does not contain A. Prove that the lines IA', BC, and the angle bisector of $\angle BAC$ are concurrent. **Hint:** you shouldn't need the *Big Point Theorem*[1] for this one!

   Solution: Two of these lines are the angle bisector of $\angle A$, and of course that intersects with side BC.

2 Tools

2.1 Ceva and friends

Ceva. Let ABC be a triangle, with $A' \in BC$, $B' \in CA$, and $C' \in AB$. Then AA', BB', and CC' concur if and only if:

$$\frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} = 1.$$

Trig Ceva. Let ABC be a triangle, with $A' \in BC$, $B' \in CA$, and $C' \in AB$. Then AA', BB', and CC' concur if and only if:

$$\frac{\sin \angle CAA'}{\sin \angle A'AB} \cdot \frac{\sin \angle ABB'}{\sin \angle B'BC} \cdot \frac{\sin \angle BCC'}{\sin \angle C'CA} = 1.$$

Menelaus. Let ABC be a triangle, and let D, E, and F line on the extended lines BC, CA, and AB. Then D, E, and F are collinear if and only if:

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1.$$

Now try these problems.

1. (Gergonne point) Let ABC be a triangle, and let its incircle intersect sides BC, CA, and AB at A', B', C' respectively. Prove that AA', BB', CC' are concurrent.

   Solution: Ceva. Since incircle, we have $BA' = CA'$, etc., so Ceva cancels trivially.

2. (Isogonal conjugate of Gergonne point) Let ABC be a triangle, and let D, E, F be the feet of the altitudes from A, B, C. Construct the incircles of triangles AEF, BDF, and CDE; let the points of tangency with DE, EF, and FD be C'', A'', and B'', respectively. Prove that AA'', BB'', CC'' concur.

   Solution: Trig ceva. Easy to check that triangles AEF and ABC are similar, because, for example, $BFEC$ is cyclic so $\angle ABC = \angle AEF$. Therefore, the line AA'' in this problem is the reflection across

[1] A classic act of desparation in Team Contest presentations.

the angle bisector of the AA' of the previous problem. So, for example, $\angle CAA'' = \angle A'AB$ and $\angle A''AB = \angle CAA'$.

In particular, since we knew that the previous problem's AA', BB', and CC' are concurrent, Trig Ceva gives

$$\frac{\sin \angle CAA'}{\sin \angle A'AB} \cdot \frac{\sin \angle ABB'}{\sin \angle B'BC} \cdot \frac{\sin \angle BCC'}{\sin \angle C'CA} = 1.$$

Now each ratio flips, because, e.g., $\frac{\sin \angle CAA''}{\sin \angle A''AB} = \frac{\sin \angle A'AB}{\sin \angle CAA'}$. So the product is still $1^{-1} = 1$, hence we have concurrence by Trig Ceva again.

2.2 The power of *Power of a Point*

Definition. *Let ω be a circle with center O and radius r, and let P be a point. The* ***power of P with respect to ω*** *is defined to be the difference of squared lengths $OP^2 - r^2$. If ω' is another circle, then the locus of points with equal power with respect to both ω and ω' is called their* ***radical axis****.*

Use the following exercises to familiarize yourself with these concepts.

1. Let ω be a circle with center O, and let P be a point. Let ℓ be a line through P which intersects O at the points A and B. Prove that the power of P with respect to ω is equal to the (signed) product of lengths $PA \cdot PB$.

   Solution: Classical.

2. Show that the radical axis of two circles is always a line.

   Solution: You can even use coordinates! Put both circles on x-axis, with centers $(x_i, 0)$. Let their radii be r_i. Locus is points of the form (x, y) with $(x - x_1)^2 + y^2 - r_1^2 = (x - x_2)^2 + y^2 - r_2^2$. But y^2 cancels, and only x remains, so it is a vertical line at the solution x.

3. Let ω_1 and ω_2 be two circles intersecting at the points A and B. Show that their radical axis is precisely the line AB.

   Solution: Clearly, points A and B have equal power (both zero) with respect to the circles. From previous problem, we know that locus is a line, and two points determine that line.

The above exercises make the following theorem useful.

Theorem. *(Radical Axis) Let ω_1, ω_2, and ω_3 be three circles. Then their (3) pairwise radical axes are concurrent (or are parallel).*

Proof. Obvious from transitivity and the above definition of radical axis. □

Now try these problems.

1. (Russia 1997/15) The circles S_1 and S_2 intersect at M and N. Show that if vertices A and C of a rectangle $ABCD$ lie on S_1 while vertices B and D lie on S_2, then the intersection of the diagonals of the rectangle lies on the line MN.

   Solution: The lines are the radical axes of S_1, S_2, and the circumcircle of $ABCD$.

2. (USAMO 1997/2) Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, AB as their respective bases. Prove that the lines through A, B, C perpendicular to the lines EF, FD, DE, respectively, are concurrent.

   Solution: Use the three circles: (1) centered at D with radius DB, (2) centered at E with radius EC, and (3) centered at F with radius FA.

2.3 Pascal and company

Pappus. Let ℓ_1 and ℓ_2 be lines, let $A, C, E \in \ell_1$, and let $B, D, F \in \ell_2$. Then $AB \cap DE$, $BC \cap EF$, and $CD \cap FA$ are collinear.

Pascal. Let ω be a conic section, and let $A, B, C, D, E, F \in \omega$. Then $AB \cap DE$, $BC \cap EF$, and $CD \cap FA$ are collinear.

Brianchon. Let the conic ω be inscribed in hexagon $ABCDEF$. Then the diagonals AD, BE, and CF are concurrent.

Remark. Typically, the only "conics" we need to consider are circles. Also, we can apply to degenerate cases where some of the points coalesce. For example, if we use $A = B$, then the line AB should be interpreted as the tangent at A.

Now try these problems.

1. (Half of Bulgaria 1997/10) Let $ABCD$ be a convex quadrilateral such that $\angle DAB = \angle ABC = \angle BCD$. Let G and O denote the centroid and circumcenter of the triangle ABC. Prove that G, O, D are collinear. **Hint:** Construct the following points:

   - M = midpoint of AB
   - N = midpoint of BC
   - $E = AB \cap CD$
   - $F = DA \cap BC$.

   Solution: Direct application of Pappus to the hexagon $MCENAF$. Recognize the intersection points as G, O, and D.

2. (From Kiran Kedlaya's *Geometry Unbound*) Let $ABCD$ be a quadrilateral whose sides AB, BC, CD, and DA are tangent to a single circle at the points M, N, P, Q, respectively. Prove that the lines AC, BD, MP, and NQ are concurrent.

   Solution: Brianchon on $BNCDQA$ gives concurrence of BD, NQ, CA, and do again on $AMBCPD$ to get the rest (use transitivity).

3. (Part of MOP 1995/?, also from Kiran) With the same notation as above, let BQ and BP intersect the circle at E and F, respectively. Show that B, $MP \cap NQ$, and $ME \cap NF$ are collinear.

   Solution: Pascal on $EMPFNQ$.

2.4 Shifting targets

Sometimes it is useful to turn a collinearity problem into a concurrence problem, or even to show that different collections of lines/points are concurrent/collinear.

Identification. Three lines AB, CD, and EF are concurrent if and only if the points A, B, and $CD \cap EF$ are collinear.

Desargues. Two triangles are perspective from a point if and only if they are perspective from a line. Two triangles ABC and DEF are **perspective from a point** when AD, BE, and CF are concurrent. Two triangles ABC and DEF are **perspective from a line** when $AB \cap DE$, $BC \cap EF$, and $CA \cap FD$ are collinear.

False transitivity. If three points are pairwise collinear, that is not enough to ensure that they are collectively collinear, and similarly for lines/concurrence.

True transitivity. If distinct points A, B, C and B, C, D are collinear, then all four points are collinear, and similarly for lines/concurrence.

Now try these problems.

1. (Full Bulgaria 1997/10) Let $ABCD$ be a convex quadrilateral such that $\angle DAB = \angle ABC = \angle BCD$. Let H and O denote the orthocenter and circumcenter of the triangle ABC. Prove that D, O, H are collinear.

   Solution: In the previous section, we showed that G, O, D were collinear, where G was the centroid of ABC. But G, H, O are collinear because they are on the Euler Line of ABC, so we are done by transitivity.

2. (Full MOP 1995/?) Let $ABCD$ be a quadrilateral whose sides AB, BC, CD, and DA are tangent to a single circle at the points M, N, P, Q, respectively. Let BQ and BP intersect the circle at E and F, respectively. Prove that ME, NF, and BD are concurrent.

   Solution: Combine previous section's problems. We know from one of them that B, $MP \cap NQ$, and D are collinear. From the other, we know that B, $MP \cap NQ$, and $ME \cap NF$ are collinear. Identification/transitivity solves the problem.

3 Problems

1. (Zeitz 1996) Let $ABCDEF$ be a convex cyclic hexagon. Prove that AD, BE, CF are concurrent if and only if $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$.

   Solution: Trig Ceva

2. (China 1996/1) Let H be the orthocenter of acute triangle ABC. The tangents from A to the circle with diameter BC touch the circle at P and Q. Prove that P, Q, H are collinear.

   Solution: Let A' be the foot of the altitude from A, and let C' be the foot of the altitude from C. Then $H = AA' \cap CC'$. Let ω be the circle with diameter BC. Construct the circle ω' with diameter AO. The intersection of these two circles is precisely P, Q, since $\angle APO = 90° = \angle AQO$. So we need to show that H is on the radical axis, i.e., that H has equal power wrt the two circles. Power of H wrt ω is $CH \cdot HC'$, and power wrt ω' is $AH \cdot HA'$ since $\angle AA'O = 90° \Rightarrow A' \in \omega'$. But it is a well-known fact that $AH \cdot HA' = CH \cdot HC'$ for any triangle, which can be verified by observing that $ACA'C'$ is cyclic.

3. (Turkey 1996/2) In a parallelogram $ABCD$ with $\angle A < 90°$, the circle with diameter AC meets the lines CB and CD again at E and F, respectively, and the tangent to this circle at A meets BD at P. Show that P, F, E are collinear.

   Solution: Use Menelaus. Need to show:

   $$\frac{CE}{EB} \cdot \frac{BP}{PD} \cdot \frac{DF}{FC} = -1.$$

   Actually, the configuration is already OK, so suffices to consider only unsigned lengths. Construct the point $X = PA \cap CE$. By similar triangles, $BP/DP = BX/DA$. Also by similar triangles, $DF/BE = DA/BA$. So it suffices to show that $CE/FC \cdot BX/BA = 1$, i.e., that $\triangle ABX \sim \triangle ECF$. We already have $\angle B = \angle C$, and we can see that $\angle X = \angle F$ by observing that $\angle X = \frac{1}{2}(AC - AE)$ and $\angle F = \frac{1}{2}EC$, where AC, AE, EC stand for the measures of those arcs in radians. But this is immediate because AC is a diameter.

4. (St. Petersburg 1996/17) The points A' and C' are chosen on the diagonal BD of a parallelogram $ABCD$ so that $AA' \parallel CC'$. The point K lies on the segment $A'C$, and the line AK meets CC' at L. A line parallel to BC is drawn through K, and a line parallel to BD is drawn through C; these meet at M. Prove that D, M, L are collinear.

   Solution: Can be done with bare hands.

5. (Korea 1997/8) In an acute triangle ABC with $AB \neq AC$, let V be the intersection of the angle bisector of A with BC, and let D be the foot of the perpendicular from A to BC. If E and F are the intersections of the circumcircle of AVD with CA and AB, respectively, show that the lines AD, BE, CF concur.

   Solution: Can be done with Ceva.

6. (Bulgaria 1996/2) The circles k_1 and k_2 with respective centers O_1 and O_2 are externally tangent at the point C, while the circle k with center O is externally tangent to k_1 and k_2. Let ℓ be the common tangent of k_1 and k_2 at the point C and let AB be the diameter of k perpendicular to ℓ. Assume that O and A lie on the same side of ℓ. Show that the lines AO_1, BO_2, ℓ have a common point.

   Solution: Can be done with Ceva.

7. (Russia 1997/13) Given triangle ABC, let A_1, B_1, C_1 be the midpoints of the broken lines CAB, ABC, BCA, respectively. Let l_A, l_B, l_C be the respective lines through A_1, B_1, C_1 parallel to the angle bisectors of A, B, C. Show that l_A, l_B, l_C are concurrent.

   Solution: Key observation: l_A passes through the midpoint of AC. Since it is parallel to bisector of $\angle A$, and medial triangle is homothety of ratio $-1/2$ of original triangle, the lines l_A, etc. concur at the incenter of the medial triangle.

   Proof of key observation: construct B' by extending CA beyond A such that $AB' = AB$. Also construct C' by extending BA beyond A such that $AC' = AC$. Then l_A is the line through the midpoints of BC' and $B'C$. This is the midline of quadrilateral $BB'C'C$ parallel to BB', so it hits BC the midpoint of BC.

8. (China 1997/4) Let $ABCD$ be a cyclic quadrilateral. The lines AB and CD meet at P, and the lines AD and BC meet at Q. Let E and F be the points where the tangents from Q meet the circumcircle of $ABCD$. Prove that points P, E, F are collinear.

   Solution: Uses Polar Map

4 Harder problems

1. (MOP 1998/2/3a) Let ABC be a triangle, and let A', B', C' be the midpoints of the arcs BC, CA, AB, respectively, of the circumcircle of ABC. The line $A'B'$ meets BC and AC at S and T. $B'C'$ meets AC and AB at F and P, and $C'A'$ meets AB and BC at Q and R. Prove that the segments PS, QT, FR concur.

   Solution: They pass through the incenter of ABC, prove with Pascal on $AA'C'B'BC$. See MOP98/2/3a.

2. (MOP 1998/4/5) Let $A_1A_2A_3$ be a nonisosceles triangle with incenter I. For $i = 1, 2, 3$, let C_i be the smaller circle through I tangent to A_iA_{i+1} and A_iA_{i+2} (indices being taken mod 3) and let B_i be the second intersection of C_{i+1} and C_{i+2}. Prove that the circumcenters of the triangles A_1B_1I, A_2B_2I, and A_3B_3I are collinear.

   Solution: MOP98/4/5: Desargues

3. (MOP 1998/2/3) Let ABC be a triangle, and let A', B', C' be the midpoints of the arcs BC, CA, AB, respectively, of the circumcircle of ABC. The line $A'B'$ meets BC and AC at S and T. $B'C'$ meets AC and AB at F and P, and $C'A'$ meets AB and BC at Q and R. Prove that the segments PS, QT, FR concur.

   Solution: They pass through the incenter of ABC, prove with Pascal on $AA'C'B'BC$. See MOP98/2/3a.

4. (MOP 1998/12/3) Let ω_1 and ω_2 be two circles of the same radius, intersecting at A and B. Let O be the midpoint of AB. Let CD be a chord of ω_1 passing through O, and let the segment CD meet ω_2 at P. Let EF be a chord of ω_2 passing through O, and let the segment EF meet ω_1 at Q. Prove that AB, CQ, EP are concurrent.

   Solution: MOP98/12/3

5 Impossible problems

- Find (in the plane) a collection of m distinct lines and n distinct points, such that the number of *incidences* between the lines and points is $> 4(m^{2/3}n^{2/3} + m + n)$. Formally, an incidence is defined as an ordered pair (ℓ, P), where ℓ is one of the lines and P is one of the points. (This is known to be impossible by the famous Szemerédi-Trotter theorem.)

  Solution: The constant of 4 can be obtained via the crossing-lemma argument in the Probabilistic Lens after Chapter 15 in *The Probabilistic Method*, by Alon and Spencer.