VI. Collinearity and Concurrence

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June 26, 2003

1 Your Weapons

Ceva Let ABC be a triangle, and let $D \in BC$, $E \in CA$, and $F \in AB$. Then AD, BE, and CF concur if and only if:

$$\frac{AF}{FB}\frac{BD}{DC}\frac{CE}{EA} = 1.$$

Trig Ceva Let ABC be a triangle, and let $D \in BC$, $E \in CA$, and $F \in AB$. Then AD, BE, and CF concur if and only if:

$$\frac{\sin CAD}{\sin DAB}\frac{\sin ABE}{\sin EBC}\frac{\sin BCF}{\sin FCA} = 1.$$

Radical Axis Let $\{\omega_k\}_1^3$ be a family of circles, and let ℓ_k be the radical axis of ω_k and ω_{k+1}, where we identify ω_4 with ω_1. Then $\{\ell_k\}_1^3$ are concurrent. The **radical axis** of ω_1 and ω_2 is the locus of points with equal power with respect to the two circles. This locus turns out to be a straight line. (You can prove it with coordinates!)

Brianchon Let circle ω be inscribed in hexagon $ABCDEF$. Then the diagonals AD, BE, and CF are concurrent.

Identification Three lines AB, CD, and EF are concurrent if and only if the points A, B, and $CD \cap EF$ are collinear.

Desargues Two triangles are perspective from a point if and only if they are perspective from a line. Two triangles ABC and DEF are **perspective from a point** when AD, BE, and CF are concurrent. Two triangles ABC and DEF are **perspective from a line** when $AB \cap DE$, $BC \cap EF$, and $CA \cap FD$ are collinear.

Menelaus Let ABC be a triangle, and let D, E, and F line on the extended lines BC, CA, and AB. Then D, E, and F are collinear if and only if:

$$\frac{AF}{FB}\frac{BD}{DC}\frac{CE}{EA} = -1.$$

Pappus Let ℓ_1 and ℓ_2 be lines, let $A, C, E \in \ell_1$, and let $B, D, F \in \ell_2$. Then $AB \cap DE$, $BC \cap EF$, and $CD \cap FA$ are collinear.

Pascal Let ω be a conic, and let $A, B, C, D, E, F \in \omega$. Then $AB \cap DE$, $BC \cap EF$, and $CD \cap FA$ are collinear.

2 Problems For PWNage (Warm-Ups)

1. (Gergonne Point) Let ABC be a triangle, and let its incircle intersect sides BC, CA, and AB at A', B', C' respectively. Prove that AA', BB', CC' are concurrent.

   Solution: Ceva

2. Let ABC be a triangle, and let D, E, F be the feet of the altitudes from A, B, C. Construct the incircles of triangles AEF, BDF, and CDE; let the points of tangency with DE, EF, and FD be C', A', and B', respectively. Prove that AA', BB', CC' concur.

   Solution: Isogonal conjugate of Gergonne point; trig ceva

3. (Russia97) The circles S_1 and S_2 intersect at M and N. Show that if vertices A and C of a rectangle $ABCD$ lie on S_1 while vertices B and D lie on S_2, then the intersection of the diagonals of the rectangle lies on the line MN.

   Solution: Radical Axis

4. (Simson Line) If P is on the circumcircle of ABC, then the feet of the perpendiculars from P to the (possibly extended) sides of ABC are collinear.

   Solution: Angle chasing shows it with vertical angles

3 Problems

1. (Zeitz96) Let $ABCDEF$ be a convex cyclic hexagon. Prove that AD, BE, CF are concurrent if and only if $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$.

   Solution: Trig Ceva

2. (StP96) The points A' and C' are chosen on the diagonal BD of a parallelogram $ABCD$ so that $AA' \parallel CC'$. The point K lies on the segment $A'C$, and the line AK meets CC' at L. A line parallel to BC is drawn through K, and a line parallel to BD is drawn through C; these meet at M. Prove that D, M, L are collinear.

   Solution: StP96/17

3. (Bulgaria97) Let $ABCD$ be a convex quadrilateral such that $\angle DAB = \angle ABC = \angle BCD$. Let H and O denote the orthocenter and circumcenter of ABC. Prove that D, O, H are collinear.

   Solution: Bulgaria97/10

4. (Korea97) In an acute triangle ABC with $AB \neq AC$, let V be the intersection of the angle bisector of A with BC, and let D be the foot of the perpendicular from A to BC. If E and F are the intersections of the circumcircle of AVD with CA and AB, respectively, show that the lines AD, BE, CF concur.

   Solution: Korea97/8

4 Harder Problems

1. (MOP98) Let ABC be a triangle, and let A', B', C' be the midpoints of the arcs BC, CA, AB, respectively, of the circumcircle of ABC. The line $A'B'$ meets BC and AC at S and T. $B'C'$ meets AC and AB at F and P, and $C'A'$ meets AB and BC at Q and R. Prove that the segments PS, QT, FR concur.

   Solution: They pass through the incenter of ABC, prove with Pascal on $AA'C'B'BC$. See MOP98/2/3a.

2. (MOP98) The bisectors of angles A, B, C of triangle ABC meet its circumcircle again at the points K, L, M, respectively. Let R be an internal point on side AB. The points P and Q are defined by the conditions: RP is parallel to AK and BP is perpendicular to BL; RQ is parallel to BL and AQ is perpendicular to AK. Show that the lines KP, LQ, MR concur.

   Solution: MOP98/5/4

3. (MOP98) Let ω_1 and ω_2 be two circles of the same radius, intersecting at A and B. Let O be the midpoint of AB. Let CD be a chord of ω_1 passing through O, and let the segment CD meet ω_2 at P. Let EF be a chord of ω_2 passing through O, and let the segment EF meet ω_1 at Q. Prove that AB, CQ, EP are concurrent.

   Solution: MOP98/12/3

4. (MOP97) Let $ABCD$ be a cyclic quadrilateral, inscribed in a circle ω, whose diagonals meet at E. Suppose the point P has the following property: if we extend the line AP to meet ω again at F, and we extend the line BP to meet ω again at G, then CF, DG, EP are all parallel. Similarly, suppose the point Q is such that if we extend the line CQ to meet ω again at H, and we extend the line DQ to meet ω again at I, then AH, BI, EQ are all parallel. Prove that E, P, Q are collinear.

   Solution: MOP97/11/5

5 Problems to PWN You

1. (MOP98) Let $A_1A_2A_3$ be a nonisosceles triangle with incenter I. For $i = 1, 2, 3$, let C_i be the smaller circle through I tangent to A_iA_{i+1} and A_iA_{i+2} (indices being taken mod 3) and let B_i be the second intersection of C_{i+1} and C_{i+2}. Prove that the circumcenters of the triangles A_1B_1I, A_2B_2I, and A_3B_3I are collinear.

   Solution: MOP98/4/5

2. (MOP97) Let ABC be a triangle and D, E, F the points where its incircle touches sides BC, CA, AB, respectively. The parallel through E to AB intersects DF in Q, and the parallel through D to AB intersects EF in T. Prove that CF, DE, QT are concurrent.

   Solution: MOP97/2/5

3. (MOP97) Let P be a point in the plane of a triangle ABC. A circle Γ passing through P intersects the circumcircles of triangles PBC, PCA, PAB at A_1, B_1, C_1, respectively, and lines PA, PB, PC intersect Γ at A_3, B_3, C_3. Prove that:

   (a) the points A_2, B_2, C_2 are collinear

   (b) the lines A_1A_3, B_1B_3, C_1C_3 are concurrent