III. Telescoping Sums and Products

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1 Trig (stolen from Titu97)

1. Evaluate:

$$\sum_{k=0}^{\infty} \tan^{-1} \frac{2}{(2k+1)^2}$$

Solution: If a_n is a positive sequence, then:

$$\tan^{-1} a_{n+1} - \tan^{-1} a_n = \tan^{-1} \left(\frac{a_{n+1} - a_n}{1 + a_{n+1} a_n} \right)$$

This telescopes with $a_n = 2n$ from the above notation. The sum is $\pi/2$.

2. Evaluate:

$$\sum_{k=0}^{\infty} \tan^{-1} \frac{2}{k^2}$$

Solution: This telescopes twice; use $a_n = n$ and the fraction is $(a_{n+1} - a_{n-1})/(1 + a_{n-1}a_{n+1})$. So the answer is $\pi/2 + \pi/2 - \pi/4 - 0 = 3\pi/4$.

3. Prove that the numbers $n \sin n°$, $n = 2, 4, \ldots, 180$ average to $\cot 1°$.

Solution: Multiply through by $\sin 1°$. Then use the sine-product formula to get the form:

$$(\cos 1 - \cos 3) + 2(\cos 3 - \cos 5) + 3(\cos 5 - \cos 7) + \cdots$$

Next bunch the $(\cos 1 - \cos 3)$ together with the $89(\cos 177 - \cos 179)$ because the 90 term is zero. But they add to get $90(\cos 1 - \cos 3)$. Proceed in this way and get telescope, except that at the end, we have:

$$\cdots + 44(\cos 87 - \cos 89) + 45(\cos 89 - \cos 91) + 46(\cos 91 - \cos 93) + \cdots$$

Here, the 44 and 46 combine as desired, leaving a residue of $-90 \cos 89$. Yet this is cancelled by the 45 term. Thus after telescoping, we only have $90 \cos 1$ left, which is what we wanted.

4. Evaluate:

$$\sum_{k=1}^{n} \cot^{-1}(2k^2) = \cot^{-1}(1 + 1/n)$$

Solution: Use the fact that the summand is $\cot^{-1}(1 + 1/k)n - \cot^{-1}(1 + 1/(k-1))$. Therefore the answer is $\cot^{-1}(1 + 1/n)$.

5. Evaluate, for x not a multiple of 2π,

$$\sum_{k=1}^{n} \cos kx.$$

Solution: Multiply it by $\sin(x/2)$, and use the expansion formula for $\sin\theta\cos\theta$. The answer is $(\sin(n+1/2)x)/(\sin x/2)$ - 1.

6. Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{a}{2^n}$$

where a is not an integer multiple of π.

7. Prove:

$$\sum_{n=1}^{\infty} 3^{n-1} \sin^3 \frac{a}{3^n} = \frac{1}{4}(a - \sin a).$$

2 No Trig (not entirely stolen from Titu97)

1. Get a nice formula for $\sum_{k=1}^{n} k!(k^2+k+1)$.

   Solution: Summand is $(k+1)(k+1)! - (k)(k)!$. So we get $(n+1)(n+1)! - 1$.

2. Let $\{a_k\}$ be an arithmetic progression with common difference d. Compute

$$\sum_{k=1}^{n} \frac{1}{a_k a_{k+1} a_{k+2}}$$

Solution: Summand is:

$$\frac{1}{2d}\left(\frac{1}{a_{k+1}a_k} - \frac{1}{a_{k+2}a_{k+1}}\right)$$

So the answer is

$$\frac{1}{2d}\left(\frac{1}{a_1 a_2} - \frac{1}{a_{n+1}a_{n+2}}\right)$$

3. (IMO 2001 Shortlist) Let $x_1, x_2, \ldots, x_n$ be arbitrary real numbers. Prove:

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+\cdots+x_n^2} < \sqrt{n}$$

Solution: Shortlist A3.

4. Let F_n be the Fibonacci sequence with $F_0 = F_1 = 1$. Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{F_{n-1}F_{n+1}}$$

Solution: Summand is $1/(F_{n-1}F_n) - 1/(F_nF_{n+1})$, so the sum is 1.

5. Prove:

$$2(\sqrt{n+1}-\sqrt{m}) < \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{m+1}} + \cdots + \frac{1}{\sqrt{n}} < 2(\sqrt{n}-\sqrt{m-1})$$

Solution: $1/(2\sqrt{m}) > 1/(\sqrt{m}+\sqrt{m+1})$, then prove that $\sqrt{n+1}-\sqrt{m} < 1/(2\sqrt{m})+\cdots+1/(2\sqrt{n})$.

6. (IMO 2001 Shortlist) Let $a_0, a_1, a_2, \ldots$ be an arbitrary infinite sequence of positive numbers. Show that the inequality $1 + a_n > a_{n-1}\sqrt[n]{2}$ holds for infinitely many positive integers n.

   Solution: Shortlist A2.

7. (Titu97) Compute the sum:

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{n-1}+\sqrt{n}}.$$

   Solution: It's $\sqrt{n}-1$ by rationalizing denominators.

8. (Titu97) Prove:

$$\frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{5}+\sqrt{7}} + \cdots + \frac{1}{\sqrt{9997}+\sqrt{9999}} \geq 24.$$

   Solution: Substitute to $1/(\sqrt{1}+\sqrt{3}) > 1/(\sqrt{1}+\sqrt{5})$, etc and:

$$\frac{1}{\sqrt{k}+\sqrt{k+4}} = \frac{\sqrt{k+4}-\sqrt{k}}{4}.$$

   That sums to something greater than 24.75.