14. Geometry

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CMU Putnam Seminar, Fall 2025

1 Famous results

Erdős-Szekeres. For every integer n, there is some finite N such that whenever there are N distinct points in the plane, no 3 collinear, some n of them are in convex position.

2 Problems

1. When and where is the Putnam?

2. Given two points P and Q on the same side of a line ℓ, find the point $X \in \ell$ which minimizes the sum of the distances $XP + XQ$.

3. Suppose that n points have been given in the plane, no 3 of which are collinear. Prove that they can be labeled $P_1, P_2, P_3, \ldots, P_n$ so that $P_1 P_2 \ldots P_n$ is a simple closed polygon (the only intersections between edges are at the common endpoints of consecutive edges). Note that the polygon is not required to be convex.

4. A convex polygon E lies in the closed unit square (its vertices are permitted to be on the boundary of the unit square). Prove that the sum of the squares of the side lengths of E is at most 4.

5. Given two points P and Q on the same side of a line ℓ, find the point X which minimizes the sum of the distances from X to P, Q, and ℓ. To minimize cases, consider only the situation when ℓ is the x-axis, P has coordinates $(0,3)$, and Q has coordinates $(3\sqrt{3}, 4)$. It's sufficient to characterize X by explaining how you would construct it; you don't need to work out its actual coordinates.

6. (Happy Ending Problem.) Suppose that P_1, P_2, P_3, P_4, P_5 are distinct points in $\mathbb{R}^2$, and no 3 of them are collinear. Prove that some 4 of them are in *convex position*, i.e., they form the corners of a convex polygon.

7. Suppose that there are 4 distinct points in the plane, no three of which are collinear, and such that the four points do not all lie on a common circle. Show that one point lies inside the circle through the other three.

8. An ellipsoid is a 3-dimensional body consisting of points of the form $\{(x, y, z) : (x/a)^2 + (y/b)^2 + (z/c)^2 = 1\}$, where a, b, c are the lengths of its semi-axes. If $a < b < c$, determine the radius of the largest circle which exists on the surface of the ellipsoid.

9. Prove that no matter how 3 points are placed in the closed unit square (the set of all (x, y) satisfying $0 \leq x \leq 1$ and $0 \leq y \leq 1$), some two of them are distance $\leq \sqrt{6} - \sqrt{2}$ apart.

10. Suppose that a triangle in the plane has vertices which are all lattice points. Show that its circumcircle has diameter less than or equal to the product of its side lengths.

3   Homework

Go to Baker A51 on Saturday, December 4, at 9:40am.