

11. Integer polynomials

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1 Famous results

Divisibility. If a and b are integers, and $p(x)$ is a polynomial with integer coefficients, then $p(a) - p(b)$ is always divisible by $a - b$.

Chinese remainder theorem. Let m_1, m_2, \dots, m_k be positive integers which are pairwise relatively prime, and let a_1, \dots, a_k be arbitrary integers. Then, the following system has integer solutions for x :

$$\begin{aligned}x &\equiv a_1 \pmod{m_1} \\x &\equiv a_2 \pmod{m_2} \\&\vdots \\x &\equiv a_k \pmod{m_k},\end{aligned}$$

and all solutions x have the same residue modulo the product $m_1 m_2 \cdots m_k$.

Gauss's lemma. Non-constant integer polynomials which are irreducible over \mathbb{Z} are also irreducible over \mathbb{Q} .

Eisenstein's criterion. Suppose that $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, and there is a prime number p such that (i) p divides each of a_0, a_1, \dots, a_{n-1} , (ii) p does not divide a_n , and (iii) p^2 does not divide a_0 . Then, $f(x)$ is irreducible over \mathbb{Q} .

2 Problems

1. Let p , q , and s be nonconstant integer polynomials such that $p(x) = q(x)s(x)$. Suppose that the polynomial $p(x) - 2008$ has at least 81 distinct integer roots. Prove that the degree of q must be greater than 5.
2. Let p be a quadratic polynomial with integer coefficients. Suppose that $p(z)$ is divisible by 5 for every integer z . Prove that all coefficients of p are divisible by 5.
3. Albert Einstein and Homer Simpson are playing a game in which they are creating a polynomial

$$p(x) = x^{2012} + a_{2011}x^{2011} + \cdots + a_1x + a_0.$$

They take turns choosing one of the coefficients a_0, \dots, a_{2011} , assigning a real value to it (even though the topic of this week is integer polynomials). Once a value is assigned to a coefficient, it cannot be overwritten in a future turn, and the game ends when all coefficients have been assigned. Albert moves first. Homer's goal is to make $p(x)$ divisible by a fixed polynomial $m(x)$, and Albert's goal is to prevent this.

- (a) Which of the players has a winning strategy if $m(x) = x - 2012$?

- (b) What if $m(x) = x^2 + 1$?
4. Let x, y, z be integers such that $x^4 + y^4 + z^4$ is divisible by 29. Prove that $x^4 + y^4 + z^4$ is actually divisible by 29^4 .
 5. Let p be a polynomial with integer coefficients, and let a_1, \dots, a_k be distinct integers. Prove that there always exists an $a \in \mathbb{Z}$ such that $p(a)$ is an integer multiple of $p(a_i)$ for all i .
 6. Let $f(x)$ be a rational function, i.e., there are polynomials p and q such that $f(x) = p(x)/q(x)$ for all x . Prove that if $f(n)$ is an integer for infinitely many integers n , then f is actually a polynomial.
 7. Let a, b be integers. Show that the set $\{ax^2 + by^2 : x, y \in \mathbb{Z}\}$ misses infinitely many integers.
 8. Let a, b be integers. Show that the set $\{ax^5 + by^5 : x, y \in \mathbb{Z}\}$ misses infinitely many integers.
 9. Let a, b, n be integers (n positive) for which the set $\{ax^n + by^n : x, y \in \mathbb{Z}\}$ includes all but finitely many integers. Prove that $n = 1$.
 10. Let p be a polynomial with real coefficients and degree n . Suppose that $\frac{p(b)-p(a)}{b-a}$ is an integer for all $0 \leq a < b \leq n$. Prove that $\frac{p(b)-p(a)}{b-a}$ is an integer for all pairs of distinct integers $a < b$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.