

6. Inequalities

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CMU Putnam Seminar, Fall 2025

1 Famous results

Cauchy-Schwarz. Let v and w be vectors in an inner product space. Then

$$|\langle v, w \rangle|^2 \leq \langle v, v \rangle \cdot \langle w, w \rangle,$$

with equality only if v and w are proportional.

Jensen. If f is a convex function, then $f(\text{average of } x\text{'s}) \leq \text{average of } f(x)\text{'s}$. This implies, for example, that $x^p y^{1-p} \leq px + (1-p)y$.

Bieberbach, via Steiner symmetrization. Every compact set $K \subset \mathbb{R}^n$ satisfies

$$\text{vol}(K) \leq \text{vol}(B_n) \cdot \left(\frac{\text{diam}(K)}{2} \right)^n,$$

where B_n is the unit ball in n dimensions, and $\text{diam}(K) = \max\{\text{dist}(x, y) : x, y \in K\}$ is the diameter of set K .

Beats. Superpositions of sine waves can form “beats”:

$$\frac{1}{2} \sin a\theta + \frac{1}{2} \sin b\theta = \sin \left(\frac{a+b}{2} \theta \right) \cos \left(\frac{a-b}{2} \theta \right).$$

2 Problems

1. In terms of n , determine the maximum possible value of the sum

$$\sum_{1 \leq i < j \leq n} |x_i - x_j|$$

where $x_1, \dots, x_n$ are (not necessarily distinct) real numbers in $[0, 1]$.

2. Determine the minimum possible value of the sum

$$\sum_{1 \leq i < j \leq n} |x_i - x_j|$$

where $x_1, \dots, x_n$ are (not necessarily distinct) real numbers in $[0, 1]$, and one of them is 0 and one of them is 1.

3. Find all pairs of real numbers (α, β) for which there is a constant C such that for all positive reals x and y ,

$$x^\alpha y^\beta < C(x + y).$$

4. Let α be a real number. Are there any continuous real-valued functions $f : [0, 1] \rightarrow \mathbb{R}^+$ such that

$$\int_0^1 f(x)dx = 1, \quad \int_0^1 xf(x)dx = \alpha, \text{ and } \quad \int_0^1 x^2 f(x)dx = \alpha^2?$$

5. Suppose that $a_1, \dots, a_n$ are real numbers such that

$$\left| \sum_{k=1}^n a_k \sin kx \right| \leq |\sin x|$$

for all real x . Prove that

$$\left| \sum_{k=1}^n ka_k \right| \leq 1.$$

6. Let K be a convex set in the plane with area at least π , whose boundary is a finite collection of line segments. Prove that there are points $X, Y \in K$ such that the distance between X and Y is at least 2.
7. Let $P_1, P_2, \dots, P_n$ be points on the surface of the unit sphere in $\mathbb{R}^3$, i.e., with coordinates satisfying $x^2 + y^2 + z^2 = 1$. Prove that

$$\sum_{1 \leq i < j \leq n} d_2(P_i, P_j)^2 \leq n^2,$$

where $d_2(X, Y)$ is the ordinary Euclidean distance between points X and Y .

8. Show that for every curve in $\mathbb{R}^2$ of length 1, there is a closed rectangle of area $\frac{1}{4}$ which covers it completely.
9. Show that a circle inscribed in a square has a larger perimeter than any other ellipse inscribed in a square.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.