2. Polynomials

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CMU Putnam Seminar, Fall 2025

1 Famous results

Single-variable. Suppose that the polynomial $P(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0$ has $d+1$ distinct zeros. Then $P(z)$ is the zero polynomial, i.e., all $a_k = 0$. This works over any field.

Multi-variable. Let $P(x, y) = \sum_{i=0}^{d} \sum_{j=0}^{d} a_{i,j} x^i y^j$ be a polynomial, and let A_x, A_y be two (not necessarily distinct) sets of size $d+1$, such that $P(x, y) = 0$ for every $x \in A_x$, $y \in A_y$. Then $P(x, y)$ is the zero polynomial, i.e., all $a_{i_j} = 0$. This works over any field, and it generalizes to more than two variables.

Zero multiplicity. If a polynomial $p(z)$ has a root of multiplicity exactly m at $z = r$, then the $(m-1)$-st derivative of p at $z = r$ is 0, the m-th derivative is nonzero, and $p'(z)$ has a root of multiplicity exactly $m - 1$ at $z = r$.

2 Problems

1. Find all real polynomials $p(z)$ with the following property: for every real polynomial $q(z)$, the two polynomials $p(q(z))$ and $q(p(z))$ are equal.

2. Find all polynomials $p(z)$ which satisfy both $p(0) = 0$ and $p(z^2 + 1) = p(z)^2 + 1$.

3. I have a polynomial p of degree at most 100, whose coefficients are all positive integers. You can provide me with a number x, and ask me to tell you $p(x)$. You are to devise a strategy to figure out the coefficients of p. What is the fewest number of questions you can ask, after which you are guaranteed to know all of the coefficients of p?

4. Let $p(z)$ be a degree-n polynomial over $\mathbb{C}$, with $n \geq 1$. Prove that there are at least $n+1$ distinct complex numbers $z \in \mathbb{C}$ for which $p(z) \in \{0, 1\}$.

5. (Binomial theorem for falling factorials.) For any positive integer n and any real number x, let the *falling factorial* $(x)_n$ be the product of n numbers $x(x-1)(x-2)\cdots(x-n+1)$. Prove that

$$(x+y)_n = \sum_{k=0}^{n} \binom{n}{k} (x)_k (y)_{n-k} \, .$$

   This also holds for rising factorials $x^{(n)} = x(x+1)\cdots(x+n-1)$.

6. A weather station measures the temperature T continuously. Meteorologists discover that every day, the temperature T follows some polynomial curve $p(t)$ with degree ≤ 3. (The particular polynomial may change from day to day.) Show that we can find times $t_1 < t_2$, which are independent of the polynomial p, such that the average temperature over the period 9am to 3pm is $\frac{1}{2}(p(t_1) + p(t_2))$, with $t_1 \approx$ 10:16am and $t_2 \approx$ 1:44pm.

7. Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for $n = 1, 2, 3, \ldots$ the polynomial

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

has exactly n distinct real roots?

8. Let $p(z)$ be a degree-n polynomial with real coefficients, all of whose roots are real. Prove that

$$(n-1)p'(z)^2 \geq np(z)p''(z)$$

for all z, and determine all polynomials $p(z)$ for which

$$(n-1)p'(z)^2 = np(z)p''(z)\,.$$

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.