10. Combinatorics

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1 Classical results

Erdős-Ko-Rado. Let $\mathcal{F}$ be a family of k-element subsets of $\{1, 2, \ldots, n\}$, with the property that every pair of members of $\mathcal{F}$ has nonempty intersection, and $n \geq 2k$. Then the size of $\mathcal{F}$ is at most $\binom{n-1}{k-1}$.

Lucas. Let n and k be non-negative integers, with base-p expansions $n = (n_t n_{t-1} \ldots n_0)_{(p)}$ and $k = (k_t k_{t-1} \ldots k_0)_{(p)}$, respectively. Then

$$\binom{n}{k} \equiv \binom{n_t}{k_t} \times \binom{n_{t-1}}{k_{t-1}} \times \cdots \times \binom{n_0}{k_0} \pmod{p}.$$

2 Problems

1. Let X be a subset of $\{1, 2, 3, \ldots, 2n\}$ with $n + 1$ elements. Show that we can find $a, b \in X$ with a dividing b.

2. Given any five points in the interior of a square side 1, show that two of the points are a distance apart less than $k = \frac{1}{\sqrt{2}}$. Is this result true for a smaller k?

3. Let S be a finite set, and suppose that a collection $\mathcal{F}$ of subsets of S has the property that any two members of $\mathcal{F}$ have at least one element in common, but $\mathcal{F}$ cannot be extended (while keeping this property). Prove that $\mathcal{F}$ contains exactly half of the subsets of S.

4. Show that the number of ways of representing n as an ordered sum of 1's and 2's equals the number of ways of representing $n+2$ as an ordered sum of integers greater than 1. For example: $4 = 1+1+1+1 = 2+2 = 2+1+1 = 1+2+1 = 1+1+2$ (5 ways) and $6 = 4+2 = 2+4 = 3+3 = 2+2+2$ (5 ways).

5. Show that for any given positive integer n, the number of odd $\binom{n}{m}$ with $0 \leq m \leq n$ is a power of 2.

6. A graph has n vertices $\{1, 2, \ldots, n\}$ and a complete set of edges. Each edge is oriented, as either $i \to j$ or $j \to i$. Show that we can find a permutation of the vertices a_i so that $a_1 \to a_2 \to a_3 \to \cdots \to a_n$.

7. Let $a_1, a_2, \ldots, a_n$ be a permutation of the integers $1, \ldots, n$. Call a_i a "big" integer if $a_i > a_j$ for all $j > i$. Find the mean number of "big" integers over all permutations on the first n integers.

8. In a tournament of n players, every pair of players plays once. There are no draws. Player i wins w_i games and loses l_i games. Which of these is always true?

   (a) $\sum w_i = \sum l_i$

   (b) $\sum w_i^2 = \sum l_i^2$

   (c) $\sum w_i^3 = \sum l_i^3$

9. Let G_1, G_2, and G_3 be three trees on the same vertex set. Let G be the superposition of all three graphs, which means that G has the same vertex set, and a pair of vertices is adjacent in G if that pair is adjacent in at least one of the G_i. Prove that G is 6-colorable.

10. In a tournament of n players, every pair of players plays once. There are no draws. Player i wins w_i games. Prove that we can find three players i, j, k such that i beats j, j beats k and k beats i iff $\sum_{t=1}^{n} w_t^2 < \frac{(n-1)n(2n-1)}{6}$.

11. Let n be a positive integer. Suppose we have an infinite sequence of 0's and 1's is such that it only contains at most n different blocks of n consecutive terms. Show that it is eventually periodic.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.