

# Putnam $\Sigma.3$

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## 1 Problems

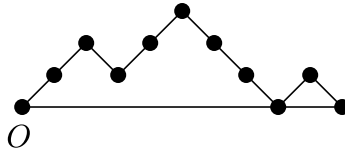
**Putnam 2003/A4.** Suppose that  $a, b, c, A, B, C$  are real numbers,  $a \neq 0$  and  $A \neq 0$ , such that

$$|ax^2 + bx + c| \leq |Ax^2 + Bx + C|$$

for all real numbers  $x$ . Show that

$$|b^2 - 4ac| \leq |B^2 - 4AC|.$$

**Putnam 2003/A5.** A Dyck  $n$ -path is a lattice path of  $n$  upsteps  $(1, 1)$  and  $n$  downsteps  $(1, -1)$  that starts at the origin  $O$  and never dips below the  $x$ -axis. A return is a maximal sequence of contiguous downsteps that terminates on the  $x$ -axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.



Show that there is a one-to-one correspondence between the Dyck  $n$ -paths with no return of even length and the Dyck  $(n - 1)$ -paths.

**Putnam 2003/A6.** For a set  $S$  of non-negative integers, let  $r_S(n)$  denote the number of ordered pairs  $(s_1, s_2)$  such that  $s_1 \in S$ ,  $s_2 \in S$ ,  $s_1 \neq s_2$ , and  $s_1 + s_2 = n$ . Is it possible to partition the non-negative integers into two sets  $A$  and  $B$  in such a way that  $r_A(n) = r_B(n)$  for all  $n$ ?