14. Probability

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1 Problems and well-known statements

1. Consider a 2-player game, between Even and Odd. Each player simultaneously decides whether to stick one finger out, or two fingers out. If the total number of fingers shown is even, then Odd pays Even a number of dollars equal to the total number of fingers shown. Otherwise, Even pays Odd a number of dollars equal to the total number of fingers shown. Is this a fair game?

2. You start 98% sure that a coin is fair, with a 1% belief that it has heads on both sides, and a 1% belief that it has tails on both sides. You then observe someone toss the coin 4 times, and get 4 heads in a row. Now what is your estimate of the probability that the coin has heads on both sides?

3. A line of 100 airline passengers is waiting to board a plane. They each hold a ticket to one of the 100 seats on that flight. (For convenience, suppose the n-th passenger in line has a ticket for seat number n.) Unfortunately, the first person in line is crazy, and will ignore the seat number on their ticket, picking a random seat to occupy. All of the other passengers are quite normal, and will go to their proper seat unless it is already occupied. If it is occupied, they will then find a free seat to sit in, at random. What is the probability that the last (100th) person to board the plane will sit in their proper seat (#100)?

4. You are conducting surveys by phone. You call a family, and ask how many children they have. The family says that they have two children. You ask whether at least one of them is a girl who happened to be born on a Tuesday. The family says yes. Given all of this information, what is the probability that both of the children are girls? (Hint: it is neither $\frac{1}{2}$ nor $\frac{2}{3}$.)

5. Two teams are playing in the World Series, which is a first-to-4-wins tournament, in which games are played one after another, but the series ends as soon as one team wins 4 games. Suppose that in each game, the team which starts at-bat has a 51% chance of winning that game, and suppose that the teams alternate between which team starts at-bat in consecutive games. Does the team which starts the first game at-bat have a higher or lower than 50% chance of winning the series?

6. If you play a game that is unfair (in your favor), in which you have a 51% chance of gaining \$1 on each turn and a 49% chance of losing \$1, what is the expected value of the maximum amount you are ever behind? (You are allowed to keep playing even if you are in debt, because you'll eventually win your way out.)

7. A magic urn starts with one red ball and one blue ball. It will always contain a perfect-square number a^2 of red balls and a (possibly different) perfect-square number b^2 of blue balls. You pick out a ball, with all balls in the urn being equally likely, and put it back into the urn, together with more new balls of that same color, so that the number of balls of that color rises from k^2 to $(k+1)^2$. Prove that with probability 1, there exists a time after which every picked ball is the same color.

8. A gardener plants n flowers. Each flower takes root with probability $\frac{1}{2}$. The next day, those which did not take root are replanted. The process continues until all n take root. What is the asymptotic expected number of planting days?

2 No Homework

There is no homework. Good luck with final exams. Have a terrific winter break!