# 9. Linear Algebra

Po-Shen Loh

CMU Putnam Seminar, Fall 2023

## 1 Classical results

**Vandermonde.** The determinant of the matrix

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}$$

is

$$\prod_{1 \le i < j \le n} (x_j - x_i).$$

**Parallelogram.** Let $v$ and $w$ be vectors in $\mathbb{R}^n$, and let $\|v\|$ denote the length of $v$. Prove that:

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2.$$

**Isometry via Polarization.** Let the real $n \times n$ matrix $A$ be an *isometry*, i.e., so that for all vectors $x \in \mathbb{R}^n$:

$$\|Ax\| = \|x\|.$$

Prove that this is equivalent to the statement that

$$\langle Ax, Ay \rangle = \langle x, y \rangle$$

for all $x, y \in \mathbb{R}^n$.

## 2 Problems

1. Let $P$ be a square matrix, with the property that $P^2 = P$. (This is called a projection matrix.) If $c \neq 1$, compute $(I - cP)^{-1}$.

2. Find the determinant of the $n \times n$ matrix whose entries are all 1's, except that all entries on the main diagonal are 0's. (The 0's are the entries in the top left corner, the 2nd column of the 2nd row, the 3rd column of the 3rd row, etc.)

3. Calculate the value of the determinant of the $3 \times 3$ complex matrix $X$, provided that $\mathrm{tr}(X) = 1$, $\mathrm{tr}(X^2) = -3$, and $\mathrm{tr}(X^3) = 4$. Here, $\mathrm{tr}(A)$ denotes the *trace*, that is, the sum of the diagonal entries of the matrix $A$.

4. Prove that for any integers $x_1, x_2, \ldots, x_n$ and positive integers $k_1, k_2, \ldots, k_n$, the determinant of the matrix
$$\begin{pmatrix} x_1^{k_1} & x_1^{k_2} & \cdots & x_1^{k_n} \\ x_2^{k_1} & x_2^{k_2} & \cdots & x_2^{k_n} \\ \vdots & \vdots & \vdots & \vdots \\ x_n^{k_1} & x_n^{k_2} & \cdots & x_n^{k_n} \end{pmatrix}$$
is divisible by $(n!)$.

5. Let $n$ be a positive integer and let $x_1, \ldots, x_n$ be $n$ nonzero points in the plane. Suppose $\langle x_i, x_j \rangle$ (scalar or dot product) is a rational number for all $i, j$. Let $S$ denote all points of the plane of the form $\sum_{i=1}^{n} a_i x_i$ where the $a_i$ are integers. A closed disk of radius $R$ and center $P$ is the set of points at distance at most $R$ from $P$ (includes the points distance $R$ from $P$). Prove that there exists a positive number $R$ and closed disks $D_1, D_2, \ldots$ of radius $R$ such that (a) Each disk contains exactly two points of $S$; (b) Every point of $S$ lies in at least one disk; (c) Two distinct disks intersect in at most one point.

6. Let $A$ and $B$ be $2 \times 2$ matrices with integer entries, such that $AB = BA$ and $\det B = 1$. Prove that if $\det(A^3 + B^3) = 1$, then $A^2$ is the zero matrix.

7. Let $A$ be an $n \times n$ matrix. Prove that there exists an $n \times n$ matrix $B$ such that $ABA = A$.

8. For an $n \times n$ matrix $A$, denote by $\phi_k(A)$ the $k$-th symmetric polynomial in the eigenvalues $\lambda_1$, $\lambda_2$, $\ldots$, $\lambda_n$ of $A$,
$$\phi_k(A) = \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_k}.$$
For example, $\phi_1(A)$ is the trace and $\phi_n(A)$ is the determinant. Prove that for two $n \times n$ matrices $A$ and $B$, $\phi_k(AB) = \phi_k(BA)$ for all $k = 1, 2, \ldots, n$.

# 3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.