8. Recursions

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CMU Putnam Seminar, Fall 2023

1 Classical results

Tilings. Determine the number of ways to tile a 1×10 strip using only 1×1 or 1×2 tiles.

Catalan numbers. Find a closed-form expression for the number of valid sequences containing n pairs of parantheses. For example, when $n = 2$, there are 2 valid sequences: ()() and (()). The sequence ())(is not valid.

Fibonacci formula. For all positive integers n, the n-th Fibonacci number is the closest integer to $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$.

2 Problems

1. Prove that for any $n \geq 1$, a $2^n \times 2^n$ checkerboard with any 1×1 square removed can be tiled by L-shaped triominoes.

2. Prove or disprove: the formula $\lceil e^{(n-1)/2} \rceil$ gives the n-th Fibonacci number.

3. Determine the limit of the ratios of consecutive Fibonacci numbers F_{n+1}/F_n.

4. How many sequences of 1's and 3's sum to 16? (Examples of such sequences are $\{1, 3, 3, 3, 3, 3\}$ and $\{1, 3, 1, 3, 1, 3, 1, 3\}$.)

5. A sequence is defined by $a_0 = -1$, $a_1 = 0$, and
$$a_{n+1} = a_n^2 - (n+1)^2 a_{n-1} - 1$$
for all positive integers n. Find a_{100}.

6. A type 1 sequence is a sequence with each term 0 or 1 which does not have 0, 1, 0 as consecutive terms. A type 2 sequence is a sequence with each term 0 or 1 which does not have 0, 0, 1, 1 or 1, 1, 0, 0 as consecutive terms. Show that there are twice as many type 2 sequences of length $n + 1$ as type 1 sequences of length n.

7. Every positive integer can be uniquely represented as the sum of one or more distinct Fibonacci numbers, where no two are consecutive Fibonacci numbers.

8. Let F_n be the Fibonacci sequence with $F_0 = F_1 = 1$. Evaluate
$$\sum_{n=1}^{\infty} \frac{1}{F_{n-1}F_{n+1}}.$$

9. For n a positive integer, define $f_1(n) = n$, and then for each i, let $f_{i+1}(n) = f_i(n)^{f_i(n)}$. Determine $f_{100}(75)$ mod 17.

10. Prove that N is a Fibonacci number if and only if $5N^2 + 4$ or $5N^2 - 4$ is a square.

11. Define the function $f : (0,1) \to (0,1)$ by

$$f(x) = \begin{cases} x + \frac{1}{2} & \text{if } x < \frac{1}{2}, \\ x^2 & \text{if } x \geq \frac{1}{2}. \end{cases}$$

Let a and b be two real numbers such that $0 < a < b < 1$. We define the sequences a_n and b_n by $a_0 = a$, $b_0 = b$, and $a_n = f(a_{n-1})$, $b_n = f(b_{n-1})$ for $n > 0$. Show that there exists a positive integer n such that

$$(a_n - a_{n-1})(b_n - b_{n-1}) < 0.$$

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.