2. Polynomials

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CMU Putnam Seminar, Fall 2023

1 Classical results

Algebra. If r is a root of the polynomial $P(x)$, then P factors as $(x - r)Q(x)$ for some polynomial Q.

Algebra. Every polynomial of degree n has at most n distinct roots.

Lagrange Interpolation. Show that there is a degree-4 polynomial which takes values $P(0) = 0$, $P(1) = 0$, $P(2) = 0$, $P(3) = 1$, and $P(4) = 1$.

Reed-Solomon codes. Automatic spell checkers know to correct "teh" to "the". More abstractly, an *error-correcting code* with minimum distance d is a collection of strings of length n from an alphabet A, with the property that any two strings differ by at least d pointwise edits. It turns out that there are nice error-correcting codes with minimum distance d over alphabets of size q, for prime powers q, and these are based on polynomials!

Multiple roots. If r is a real root of the polynomial $P(x)$, and r has multiplicity greater than 1, then both $P(r) = 0$ and $P'(r) = 0$.

Gauss-Lucas. The zeros of the derivative $P'(z)$ of any polynomial lie in the convex hull of the zeros of the polynomial $P(z)$.

2 Problems

1. Find a polynomial with integer coefficients that has the zero $\sqrt{2} + \sqrt[3]{3}$.
2. There is no polynomial which has the property that $P(k) = 2^k$ for all positive integers k.
3. Let $a_1, \ldots, a_n$ be positive real numbers. Prove that the polynomial $P(x) = x^n - a_1 x^{n-1} - a_2 x^{n-2} - \cdots - a_n$ has a unique positive zero.
4. Solve the system
$$\begin{aligned} x + y + z &= 1 \\ xyz &= 1 \end{aligned}$$
knowing that x, y, z are complex numbers of absolute value equal to 1.
5. Let $P(z)$ and $Q(z)$ be polynomials with complex coefficients of degree greater than or equal to 1 with the property that $P(z) = 0$ if and only if $Q(z) = 0$ and $P(z) = 1$ if and only if $Q(z) = 1$. Prove that the polynomials are equal.

6. Let $P(x)$ and $Q(x)$ be arbitrary polynomials with real coefficients, and let d be the degree of $P(x)$. Assume that $P(x)$ is not the zero polynomial. Prove that there exist polynomials $A(x)$ and $B(x)$ with real coefficients, such that:

   (i) both A and B have degree at most $d/2$, and

   (ii) at most one of A and B is the zero polynomial, and

   (iii) $\frac{A(x)+Q(x)B(x)}{P(x)}$ is a polynomial with real coefficients. That is, there is some polynomial $C(x)$ with real coefficients such that $A(x) + Q(x)B(x) = P(x)C(x)$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.