7. Convergence

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1 Famous results

Continued root. $\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}} = 3$.

Supremum/infimum. The *supremum* of a set S of real numbers is defined to be the smallest real number y such that all $s \in S$ are less than or equal to y. The *infimum* is the largest real number x such that all $s \in S$ are greater than or equal to x. A bounded sequence of monotonically increasing real numbers always converges to its supremum.

Limit superior/inferior.

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right); \qquad \limsup_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right).$$

The lim inf of a sequence is always less than or equal to its lim sup, and if they are equal, then the sequence has a limit.

Sub-additivity. Let $x_1, x_2, \ldots$ be a sequence of real numbers such that $x_{i+j} \leq x_i + x_j$ for all (not necessarily distinct) positive integers i and j. Then $\lim_{n\to\infty} \frac{1}{n} x_n$ always exists, and is either a real number or $-\infty$.

2 Problems

1. Let $a_1, a_2, \ldots$ be a sequence of non-negative real numbers such that $a_{m+n} \leq a_m a_n$ for all $m, n \in \mathbb{Z}^+$. (This even includes cases when $m = n$.) Show that the sequence $a_n^{1/n}$ converges.

2. For each n, let $f(n)$ denote the largest integer such that $2^{f(n)}$ divides n. For example, $f(3) = 0$ since 3 is odd, and $f(24) = 3$ since 2^3 is the highest power of 2 which divides 24. Let $g(n) = f(1) + f(2) + \cdots + f(n)$. Prove that
$$\sum_{n=1}^{\infty} e^{-g(n)}$$
converges.

3. Let $a_1, a_2, \ldots$ be a sequence of real numbers such that the sequence $a_1 + 2a_2$, $a_2 + 2a_3$, $a_3 + 2a_4$, $\ldots$ converges. Prove that the sequence $a_1, a_2, \ldots$ must then also converge.

4. What if we are told that the sequence $a_1 + a_2, a_2 + a_3, a_3 + a_4, \ldots$ converges? Does that imply that $a_1, a_2, \ldots$ converges? For a third variant, must $a_1, a_2, \ldots$ converge if we are told that $2a_1 + a_2$, $2a_2 + a_3$, $\ldots$ converges?

5. Let $a_1, a_2, \ldots$ be a sequence of non-negative real numbers, for which $\sum a_i$ converges. Suppose also that $a_j \leq 100 a_i$ for all $i \leq j \leq 2i$. Show that $\lim_{n\to\infty} n a_n = 0$.

6. Let $a_1, a_2, \ldots$ be a strictly increasing sequence of positive real numbers, such that $\sum a_i^{-1}$ converges. For each positive real x, let $f(x)$ be the largest integer i for which $a_i < x$. Prove that

$$\lim_{x\to\infty} \frac{f(x)}{x} = 0\,.$$

7. Let

$$a_n = \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{\cdots+(n-3)\sqrt{1+(n-2)\sqrt{1+(n-1)\sqrt{1+(n)}}}}}}}\,.$$

   Prove that $\lim_{n\to\infty} a_n = 3$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.