5. Functional equations

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1 Famous results

Cauchy. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function that satisfies $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$. Then there must be a real number c such that $f(x) = cx$ for all $x \in \mathbb{R}$.

Cauchy-Schwarz. Let v and w be vectors in an inner product space. Then

$$|\langle v, w\rangle|^2 \leq \langle v, v\rangle \cdot \langle w, w\rangle ,$$

with equality only if v and w are proportional.

Triple iterate. Let $f(x) = 1 - \frac{1}{x}$. Then $f(f(f(x))) = x$.

2 Problems

1. An even function is one which satisfies the equation $f(-x) = f(x)$ for all x. Prove that if $P(x)$ is a polynomial that is an even function, then all of its nonzero terms have even powers of x. An odd function is one which satisfies $f(-x) = -f(x)$ for all x. Prove that if $P(x)$ is a polynomial that is an odd function, then all of its nonzero terms have odd powers of x.

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies
$$x + f(x) = f(f(x))$$
for every $x \in \mathbb{R}$. Find all a which satisfy $f(f(a)) = 0$.

3. Let $X = \mathbb{R} \setminus \{0, 1\}$. Find all functions $f : X \to \mathbb{R}$ that satisfy
$$f(x) + f\left(1 - \frac{1}{x}\right) = 1 + x$$
for all $x \in X$.

4. Find all strictly increasing functions $f : \mathbb{R} \to \mathbb{R}$ such that for all x:
$$f(x) + f^{-1}(x) = 2x.$$

5. Let α be a real number. Are there any continuous real-valued functions $f : [0, 1] \to \mathbb{R}^+$ such that
$$\int_0^1 f(x)dx = 1 , \qquad \int_0^1 xf(x)dx = \alpha \text{ , and} \qquad \int_0^1 x^2 f(x)dx = \alpha^2 \text{ ?}$$

6. Let $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ be strictly monotone increasing, meaning that $f(x) < f(y)$ for all $x < y$. Suppose that $f(2) = 2$, and for every positive integers x, y with $\gcd(x, y) = 1$, we have $f(xy) = f(x)f(y)$. Prove that $f(x) = x$ for all x.

7. Find all continuously differentiable functions from $\mathbb{R} \to \mathbb{R}^+$, if any, which satisfy $f'(x) = f(x)$ for all x. Then, find all continuously differentiable functions from $\mathbb{R} \to \mathbb{R}^+$, if any, which satisfy $f'(x) = f(f(x))$ for all x. What if the range is allowed to be all of $\mathbb{R}$?

8. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any $x, y \in \mathbb{R}$:

$$f(f(x)f(y)) + f(x + y) = f(xy).$$

9. Find all twice differentiable functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy $f(x)^2 - f(y)^2 = f(x + y)f(x - y)$ for all $x, y \in \mathbb{R}$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.