3. Number theory

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1 Famous results

Fermat's Little Theorem. For every prime p and any integer a which is not divisible by p, we have $a^{p-1} \equiv 1 \pmod{p}$.

Euler. Let $\varphi(n)$ denote the number of positive integers in $\{1, 2, \ldots, n\}$ which are relatively prime to n. Then, for any integer a which is relatively prime to n,

$$a^{\varphi(n)} \equiv 1 \pmod{n}.$$

Frobenius coin problem. Suppose that a country has two types of coins, worth a and b, where a and b are relatively prime. Then, the largest integer value which *cannot* be obtained through the coins is $ab - a - b$. However, if the country has three types of coins, worth a, b, and c, then there is no explicit formula known for the largest unattainable integer value.

2 Problems

1. Prove that for every integer $n > 1$, n does not divide $2^n - 1$.

2. Suppose that an infinite arithmetic progression of positive integers contains a perfect n-th power (some a^n for an integer a). Show that it must then contain infinitely many perfect n-th powers.

3. For positive integers n, define the function $f(n)$ as follows: write n as a product of (not necessarily distinct) primes $p_1 p_2 \cdots p_t$, and let $f(n) = (-1)^t$. For example, $f(24) = (-1)^4$ because $24 = 2 \times 2 \times 2 \times 3$. Define

$$F(n) = \sum_{d|n} f(d).$$

   Prove that for all positive integers n, the value of $F(n)$ is either 0 or 1, and characterize the n for which $F(n) = 1$.

4. McDonalds sells Chicken McNuggets in boxes of size a and b, where a and b are positive integers (of course). If you are hungry but picky, and would like to order exactly n McNuggets, the only way to do this is to order some combination of full boxes of the two available sizes. A sharp kid observes that it is not possible to obtain exactly 58 McNuggets in this way, but that there are exactly 35 impossible integer values. Find a and b.

5. Show that if a positive integer n is a multiple of 24, then if one adds up all of the positive divisors of $n - 1$ (including 1 and $n - 1$), the total is also divisible by 24.

6. Suppose that for some real number α, all of $1^\alpha, 2^\alpha, 3^\alpha, \ldots$ are integers. Prove that α is a nonnegative integer.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.